Pre-Calculus

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MIKE LEPINE

St. Clair College AA&T
Windsor, ON

Pre-Calculus is an adaptation from the following 4 books:

1. Algebra and Trigonometry Open Stax by Jay Abramson, is licensed under a Creative Commons Attribution 4.0 International (CC BY) license
2. Intermediate Algebra by Terrance Burg, is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA) License
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4. Douglas College Physics 1104 Custom Textbook – Winter and Summer 2020 by OpenStax is licensed under a Creative Commons Attribution 4.0 International (CC BY) License

This adaptation has seen substantial reordering and reformatting of the original texts, minor wording adjustments, the addition of new content, replacement of images, and deletions.

The revisions made to the original books are listed below.

Chapter 1

- Chapter 1.1 – adapted from chapter 3 in Algebra and Trigonometry by Jay Abramson
- Chapter 1.2 – adapted from chapter 4 in Algebra and Trigonometry by Jay Abramson
- Chapter 1.3 – adapted from chapter 32 in Algebra and Trigonometry by Jay Abramson
- Chapter 1.4 – adapted from chapter 3.4 in Intermediate Algebra II by Pooja Gupta
- Chapter 1.5 – adapted from chapter 3.5 in Intermediate Algebra II by Pooja Gupta
- Chapter 1.6 – adapted from chapter 3.6 in Intermediate Algebra II by Pooja Gupta
- Chapter 1.7 – adapted from chapter 3.7 in Intermediate Algebra II by Pooja Gupta

Chapter 2

- Chapter 2.1 – adapted from chapter 3.3 in Intermediate Algebra by Terrance Berg

- Chapter 2.2 – adapted from chapter 3.4 in Intermediate Algebra by Terrance Berg
- Chapter 2.3 – adapted from chapter 3.6 in Intermediate Algebra by Terrance Berg
- Chapter 2.4 – adapted from chapter 70 in Algebra and Trigonometry by Jay Abramson
- Chapter 2.5 – adapted from chapter 71 in Algebra and Trigonometry by Jay Abramson
- Chapter 2.6 – adapted from chapter 73 in Algebra and Trigonometry by Jay Abramson
- Chapter 2.7 – adapted from chapter 78 in Algebra and Trigonometry by Jay Abramson

Chapter 3

- Chapter 3.1 – adapted from chapter 46 in Algebra and Trigonometry by Jay Abramson
- Chapter 3.2 – adapted from chapter 47 in Algebra and Trigonometry by Jay Abramson
- Chapter 3.3 – adapted from chapter 48 in Algebra and Trigonometry by Jay Abramson
- Chapter 3.4 – adapted from chapter 49 in Algebra and Trigonometry by Jay Abramson
- Chapter 3.5 – adapted from chapter 50 in Algebra and Trigonometry by Jay Abramson

Chapter 4

- Chapter 4.1 – adapted from chapter 61 in Algebra and Trigonometry by Jay Abramson
- Chapter 4.2 – adapted from chapter 62 in Algebra and Trigonometry by Jay Abramson
- Chapter 4.3 – adapted from chapter 63 in Algebra and Trigonometry by Jay Abramson
- Chapter 4.4 – adapted from chapter 64 in Algebra and Trigonometry by Jay Abramson
- Chapter 4.5 – adapted from chapter 65 in Algebra and Trigonometry by Jay Abramson
- Chapter 4.6 – adapted from chapter 66 in Algebra and Trigonometry by Jay Abramson
- Chapter 4.7 – adapted from chapter 67 in Algebra and Trigonometry by Jay Abramson
- Chapter 4.8 – adapted from chapter 68 in Algebra and Trigonometry by Jay Abramson
- Chapter 4.9 – adapted from chapter 69 in Algebra and Trigonometry by Jay Abramson
- Chapter 4.10 – adapted from chapter 3.3 in Douglas College Physics 1104 Custom Textbook – Winter and Summer 2020 by Physics Department of Douglas College and OpenStax

Chapter 5

- Chapter 5.1 – adapted from chapter 7.1 in Intermediate Algebra by Terrance Berg
- Chapter 5.2 – adapted from chapter 7.2 in Intermediate Algebra by Terrance Berg
- Chapter 5.3 – adapted from chapter 7.3 in Intermediate Algebra by Terrance Berg
- Chapter 5.4 – adapted from chapter 7.4 in Intermediate Algebra by Terrance Berg
- Chapter 5.5 – adapted from chapter 7.5 in Intermediate Algebra by Terrance Berg
- Chapter 5.6 – adapted from chapter 7.6 in Intermediate Algebra by Terrance Berg
- Chapter 5.7 – adapted from chapter 7.7 in Intermediate Algebra by Terrance Berg

- Chapter 5.8 – adapted from chapter 7.8 in Intermediate Algebra by Terrance Berg
- Chapter 5.9 – adapted from chapter 7.9 in Intermediate Algebra by Terrance Berg
- Chapter 5.10 – adapted from chapter 10.3 in Intermediate Algebra by Terrance Berg
- Chapter 5.11 – adapted from chapter 10.4 in Intermediate Algebra by Terrance Berg
- Chapter 5.12 – adapted from chapter 10.5 in Intermediate Algebra by Terrance Berg
- Chapter 513 – adapted from chapter 10.6 in Intermediate Algebra by Terrance Berg

Chapter 6

- Chapter 6.1 – adapted from chapter 8.1 in Intermediate Algebra by Terrance Berg
- Chapter 6.2 – adapted from chapter 8.2 in Intermediate Algebra by Terrance Berg
- Chapter 6.3 – adapted from chapter 8.3 in Intermediate Algebra by Terrance Berg
- Chapter 6.4 – adapted from chapter 8.4 in Intermediate Algebra by Terrance Berg
- Chapter 6.5 – adapted from chapter 8.5 in Intermediate Algebra by Terrance Berg
- Chapter 6.6 – adapted from chapter 8.6 in Intermediate Algebra by Terrance Berg
- Chapter 6.7 – adapted from chapter 8.7 in Intermediate Algebra by Terrance Berg
- Chapter 6.8 – adapted from chapter 8.8 in Intermediate Algebra by Terrance Berg
- Chapter 6.9 – adapted from chapter 34 in Algebra and Trigonometry by Jay Abramson

Chapter 7

- Chapter 7.1 – adapted from chapter 51 in Algebra and Trigonometry by Jay Abramson
- Chapter 7.2 – adapted from chapter 52 in Algebra and Trigonometry by Jay Abramson
- Chapter 7.3 – adapted from chapter 53 in Algebra and Trigonometry by Jay Abramson
- Chapter 7.4 – adapted from chapter 54 in Algebra and Trigonometry by Jay Abramson

Chapter 8

- Chapter 8.1 – adapted from chapter 9.1 in Intermediate Algebra by Terrance Berg
- Chapter 8.2 – adapted from chapter 9.2 in Intermediate Algebra by Terrance Berg
- Chapter 8.3- adapted from chapter 9.3 in Intermediate Algebra by Terrance Berg
- Chapter 8.4 – adapted from chapter 9.4 in Intermediate Algebra by Terrance Berg
- Chapter 8.5 – adapted from chapter 9.5 in Intermediate Algebra by Terrance Berg
- Chapter 8.6 – adapted from chapter 9.6 in Intermediate Algebra by Terrance Berg
- Chapter 8.7 – adapted from chapter 9.7 in Intermediate Algebra by Terrance Berg
- Chapter 8.8 – adapted from chapter 9.8 in Intermediate Algebra by Terrance Berg
- Chapter 8.9 – adapted from chapter 10.1 in Intermediate Algebra by Terrance Berg

Chapter 9

- Chapter 9.1 – adapted from chapter 37 in Algebra and Trigonometry by Jay Abramson
- Chapter 9.2 – adapted from chapter 38 in Algebra and Trigonometry by Jay Abramson
- Chapter 9.3 – adapted from chapter 39 in Algebra and Trigonometry by Jay Abramson
- Chapter 9.4 – adapted from chapter 40 in Algebra and Trigonometry by Jay Abramson
- Chapter 9.5 – adapted from chapter 41 in Algebra and Trigonometry by Jay Abramson
- Chapter 9.6 – adapted from chapter 42 in Algebra and Trigonometry by Jay Abramson
- Chapter 9.7 – adapted from chapter 43 in Algebra and Trigonometry by Jay Abramson
- Chapter 9.8 – adapted from chapter 44 in Algebra and Trigonometry by Jay Abramson
- Chapter 9.9 – adapted from chapter 45 in Algebra and Trigonometry by Jay Abramson

Chapter 10

- Chapter 10.1 – adapted from chapter 12 in Algebra and Trigonometry by Jay Abramson
- Chapter 10.2 – adapted from chapter 66 in Algebra and Trigonometry by Jay Abramson

Reference section- adapted from Reference Section in Intermediate Algebra by Terrance Berg

CONTENTS

Chapter 4: Further Applications of Trigonometry

Chapter 5: Factoring and Quadratics

Chapter 6: Rational Expressions and Equations

Chapter 7: Periodic Functions

Chapter 8: Radical Expressions and Equations

CHAPTER 1: ALGEBRA REVIEW

CHAPTER 1.1: EXPONENTS AND SCIENTIFIC NOTATION

Learning Objectives

In this section students will:

- Use the product rule of exponents.
- Use the quotient rule of exponents.
- Use the power rule of exponents.
- Use the zero exponent rule of exponents.
- Use the negative rule of exponents.
- Find the power of a product and a quotient.
- Simplify exponential expressions.
- Use scientific notation.

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is 2,048 pixels by 1,536 pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour (3,600-second) digital film is then an extremely large number.

Using a calculator, we enter $2,048 \times 1,536 \times 48 \times 24 \times 3,600$ and press ENTER. The calculator displays 1.304596316E13. What does this mean? The "E13" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.310^{13} bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \cdot x^4$. Both terms have the same base, x, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$\begin{aligned} x^3 \cdot x^4 &= \overbrace{x \cdot x \cdot x}^{3 \text{ factors}} \cdot \overbrace{x \cdot x \cdot x \cdot x}^{4 \text{ factors}} \\ &= \overbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}^{7 \text{ factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \cdot x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the *product rule of exponents.*

$$a^m \cdot a^n = a^{m+n}$$

Now consider an example with real numbers.

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product $8 \cdot 16$ equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

The Product Rule of Exponents

For any real number a and natural numbers m and n, the product rule of exponents states that

$$a^m \cdot a^n = a^{m+n}$$

Using the Product Rule

Write each of the following products with a single base. Do not simplify further.

a. $t^5 \cdot t^3$

b. $(-3)^5 \cdot (-3)$
c. $x^2 \cdot x^5 \cdot x^3$

Show Solution

Use the product rule to simplify each expression.

a. $t^5 \cdot t^3 = t^{5+3} = t^8$
b. $(-3)^5 \cdot (-3) = (-3)^5 \cdot (-3)^1 = (-3)^{5+1} = (-3)^6$
c. $x^2 \cdot x^5 \cdot x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \cdot x^5 \cdot x^3 = \left(x^2 \cdot x^5\right) \cdot x^3 = \left(x^{2+5}\right) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$$

Try It

Write each of the following products with a single base. Do not simplify further.

a. $k^6 \cdot k^9$
b. $\left(\frac{2}{y}\right)^4 \cdot \left(\frac{2}{y}\right)$
c. $t^3 \cdot t^6 \cdot t^5$

Show Solution

a. k^{15}

b. $\left(\frac{2}{y}\right)^5$
c. t^{14}

Using the Quotient Rule of Exponents

The *quotient rule of exponents* allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$, where $m > n$. Consider the example $\frac{y^9}{y^5}$. Perform the division by canceling common factors.

$$\begin{aligned} \frac{y^9}{y^5} &= \frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\ &= \frac{y \cdot y \cdot y \cdot y}{1} \\ &= y^4 \end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

$$\frac{a^m}{a^n} = a^{m-n}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m > n$. Otherwise, the difference $m - n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

The Quotient Rule of Exponents

For any real number a and natural numbers m and n, such that $m > n$, the quotient rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n}$$

Using the Quotient Rule

Write each of the following products with a single base. Do not simplify further.

a. $\frac{(-2)^{14}}{(-2)^9}$
b. $\frac{t^{23}}{t^{15}}$
c. $\frac{\left(z\sqrt{2}\right)^5}{z\sqrt{2}}$

Show Solution

Use the quotient rule to simplify each expression.

a. $\frac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$
b. $\frac{t^{23}}{t^{15}} = t^{23-15} = t^8$
c. $\frac{\left(z\sqrt{2}\right)^5}{z\sqrt{2}} = \left(z\sqrt{2}\right)^{5-1} = \left(z\sqrt{2}\right)^4$

Try It

Write each of the following products with a single base. Do not simplify further.

a. $\frac{s^{75}}{s^{68}}$
b. $\frac{(-3)^6}{-3}$
c. $\frac{\left(ef^2\right)^5}{(ef^2)^3}$

Show Solution

a. s^7
b. $(-3)^5$
c. $\left(ef^2\right)^2$

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the *power rule of exponents*. Consider the expression $\left(x^2\right)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

$$\begin{aligned}\left(x^2\right)^3 &= \overbrace{\left(x^2\right)\cdot\left(x^2\right)\cdot\left(x^2\right)}^{3\text{ factors}} \\ &= \overbrace{\left(\overbrace{x\cdot x}^{2\text{ factors}}\right)\cdot\left(\overbrace{x\cdot x}^{2\text{ factors}}\right)\cdot\left(\overbrace{x\cdot x}^{2\text{ factors}}\right)}^{3\text{ factors}} \\ &= x\cdot x\cdot x\cdot x\cdot x\cdot x \\ &= x^6\end{aligned}$$

The exponent of the answer is the product of the exponents: $\left(x^2\right)^3 = x^{2\cdot 3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

$$(a^m)^n = a^{m\cdot n}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

$$\begin{array}{lllllllll} & \text{Product Rule} & & & & & \text{Power Rule} & & \\ 5^3 \cdot 5^4 & = & 5^{3+4} & = & 5^7 & \text{but} & (5^3)^4 & = & 5^{3\cdot 4} & = & 5^{12} \\ x^5 \cdot x^2 & = & x^{5+2} & = & x^7 & \text{but} & (x^5)^2 & = & x^{5\cdot 2} & = & x^{10} \\ (3a)^7 \cdot (3a)^{10} & = & (3a)^{7+10} & = & (3a)^{17} & \text{but} & \left((3a)^7\right)^{10} & = & (3a)^{7\cdot 10} & = & (3a)^{70} \end{array}$$

The Power Rule of Exponents

For any real number a and positive integers m and n, the power rule of exponents states that

$(a^m)^n = a^{m\cdot n}$

Using the Power Rule

Write each of the following products with a single base. Do not simplify further.

a. $(x^2)^7$
b. $\left((2t)^5\right)^3$
c. $\left((-3)^5\right)^{11}$

Show Solution
Use the power rule to simplify each expression.

a. $(x^2)^7 = x^{2\cdot 7} = x^{14}$
b. $\left((2t)^5\right)^3 = (2t)^{5\cdot 3} = (2t)^{15}$
c. $\left((-3)^5\right)^{11} = (-3)^{5\cdot 11} = (-3)^{55}$

Try It

Write each of the following products with a single base. Do not simplify further.

a. $\left((3y)^8\right)^3$
b. $\left(t^5\right)^7$
c. $\left((-g)^4\right)^4$

Show Solution

a. $(3y)^{24}$
b. t^{35}
c. $(-g)^{16}$

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m > n$ so that the difference $m - n$ would never be zero or negative. What would happen if $m = n$? In this case, we would use the *zero exponent rule of exponents* to simplify the expression to 1. To see how this is done, let us begin with an example.

$\frac{t^8}{t^8} = 1$

If we were to simplify the original expression using the quotient rule, we would have

$\frac{t^8}{t^8} = t^{8-8} = t^0$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number.

$a^0 = 1$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for now, we will consider the value to be undefined.

The Zero Exponent Rule of Exponents

For any nonzero real number a, the zero exponent rule of exponents states that

$a^0 = 1$

Using the Zero Exponent Rule

Simplify each expression using the zero exponent rule of exponents.

a. $\frac{c^3}{c^3}$
b. $\frac{-3x^5}{x^5}$
c. $\frac{\left(j^2k\right)^4}{(j^2k)\cdot(j^2k)^3}$
d. $\frac{5\left(rs^2\right)^2}{(rs^2)^2}$

Show Solution

Use the zero exponent and other rules to simplify each expression.

a. $$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3}\\ &= c^0\\ &= 1\end{aligned}$$

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b. $$\begin{aligned}\frac{-3x^5}{x^5} &= -3\cdot\frac{x^5}{x^5}\\ &= -3\cdot x^{5-5}\\ &= -3\cdot x^0\\ &= -3\cdot 1\\ &= -3\end{aligned}$$

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c.

$$\begin{aligned} \frac{\left(j^2k\right)^4}{(j^2k)\cdot(j^2k)^3} &= \frac{\left(j^2k\right)^4}{(j^2k)^{1+3}} && \text{Use the product rule in the denominator.} \\ &= \frac{\left(j^2k\right)^4}{(j^2k)^4} && \text{Simplify.} \\ &= \left(j^2k\right)^{4-4} && \text{Use the quotient rule.} \\ &= \left(j^2k\right)^0 && \text{Simplify.} \\ &= 1 \end{aligned}$$

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d.

$$\begin{aligned} \frac{5\left(rs^2\right)^2}{(rs^2)^2} &= 5\left(rs^2\right)^{2-2} && \text{Use the quotient rule.} \\ &= 5\left(rs^2\right)^0 && \text{Simplify.} \\ &= 5\cdot 1 && \text{Use the zero exponent rule.} \\ &= 5 && \text{Simplify.} \end{aligned}$$

Try It

Simplify each expression using the zero exponent rule of exponents.

a. $\frac{t^7}{t^7}$
b. $\frac{\left(de^2\right)^{11}}{2(de^2)^{11}}$
c. $\frac{w^4\cdot w^2}{w^6}$
d. $\frac{t^3\cdot t^4}{t^2\cdot t^5}$

Show Solution

a. 1
b. $\frac{1}{2}$
c. 1
d. 1

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m > n$ in the quotient rule even further. For example, can we simplify $\frac{h^3}{h^5}$? When $m < n$ —that is, where the difference $m - n$ is negative—we can use the *negative rule of exponents* to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{h^3}{h^5}$.

$$\begin{aligned} \frac{h^3}{h^5} &= \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} \\ &= \frac{1}{h \cdot h} \\ &= \frac{1}{h^2} \end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\begin{aligned} \frac{h^3}{h^5} &= h^{3-5} \\ &= h^{-2} \end{aligned}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}$. This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression a^n is defined when n is a natural number, 0, or the negative of a natural number. That means that a^n is defined for any integer n. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n.

The Negative Rule of Exponents

For any nonzero real number a and natural number n, the negative rule of exponents states that

$a^{-n} = \frac{1}{a^n}$

Using the Negative Exponent Rule

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{\theta^3}{\theta^{10}}$
b. $\frac{z^2 \cdot z}{z^4}$
c. $\frac{\left(-5t^3\right)^4}{(-5t^3)^8}$

Show Solution

a. $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$
b. $\frac{z^2 \cdot z}{z^4} = \frac{z^{2+1}}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$
c. $\frac{\left(-5t^3\right)^4}{(-5t^3)^8} = \left(-5t^3\right)^{4-8} = \left(-5t^3\right)^{-4} = \frac{1}{(-5t^3)^4}$

Try It

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{(-3t)^2}{(-3t)^8}$
b. $\frac{f^{47}}{f^{49} \cdot f}$
c. $\frac{2k^4}{5k^7}$

Show Solution

a. $\frac{1}{(-3t)^6}$
b. $\frac{1}{f^3}$
c. $\frac{2}{5k^3}$

Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $b^2 \cdot b^{-8}$
b. $(-x)^5 \cdot (-x)^{-5}$
c. $\frac{-7z}{(-7z)^5}$

Show Solution

a. $b^2 \cdot b^{-8} = b^{2-8} = b^{-6} = \frac{1}{b^6}$
b. $(-x)^5 \cdot (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$
c. $\frac{-7z}{(-7z)^5} = \frac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \frac{1}{(-7z)^4}$

Try It

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $t^{-11} \cdot t^{6}$
b. $\frac{25^{12}}{25^{13}}$

Show Solution

a. $t^{-5} = \frac{1}{t^5}$
b. $\frac{1}{25}$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the *power of a product rule of exponents,* which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$\begin{aligned} (pq)^3 &= \overbrace{(pq)\cdot(pq)\cdot(pq)}^{3 \text{ factors}} \\ &= p\cdot q\cdot p\cdot q\cdot p\cdot q \\ &= \overbrace{p\cdot p\cdot p}^{3 \text{ factors}}\cdot\overbrace{q\cdot q\cdot q}^{3 \text{ factors}} \\ &= p^3\cdot q^3 \end{aligned}$$

In other words, $(pq)^3 = p^3 \cdot q^3$.

The Power of a Product Rule of Exponents

For any real numbers a and b and any integer n, the power of a product rule of exponents states that

$$(ab)^n = a^n b^n$$

Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $\left(ab^2\right)^3$
b. $(2t)^{15}$
c. $\left(-2w^3\right)^3$
d. $\frac{1}{(-7z)^4}$
e. $\left(e^{-2}f^2\right)^7$

Show Solution

Use the product and quotient rules and the new definitions to simplify each expression.

a. $\left(ab^2\right)^3 = (a)^3 \cdot \left(b^2\right)^3 = a^{1\cdot 3} \cdot b^{2\cdot 3} = a^3 b^6$
b. $(2t)^{15} = (2)^{15} \cdot (t)^{15} = 2^{15} t^{15} = 32,768t^{15}$
c. $\left(-2w^3\right)^3 = (-2)^3 \cdot \left(w^3\right)^3 = -8 \cdot w^{3\cdot 3} = -8w^9$
d. $\frac{1}{(-7z)^4} = \frac{1}{(-7)^4 \cdot (z)^4} = \frac{1}{2{,}401z^4}$
e. $\left(e^{-2}f^2\right)^7 = \left(e^{-2}\right)^7 \cdot \left(f^2\right)^7 = e^{-2\cdot 7} \cdot f^{2\cdot 7} = e^{-14} f^{14} = \frac{f^{14}}{e^{14}}$

Try It

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $\left(g^2h^3\right)^5$
b. $(5t)^3$
c. $\left(-3y^5\right)^3$
d. $\frac{1}{(a^6b^7)^3}$
e. $\left(r^3s^{-2}\right)^4$

Show Solution

a. $g^{10}h^{15}$
b. $125t^3$
c. $-27y^{15}$
d. $\frac{1}{a^{18}b^{21}}$
e. $\frac{r^{12}}{s^8}$

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the *power of a quotient rule,* which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let's look at the following example.

$$\left(e^{-2}f^2\right)^7 = \frac{f^{14}}{e^{14}}$$

Let's rewrite the original problem differently and look at the result.

$$\begin{aligned}\left(e^{-2}f^2\right)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

$$\begin{aligned}\left(e^{-2}f^2\right)^7 &= \left(\frac{f^2}{e^2}\right)^7\\ &= \frac{\left(f^2\right)^7}{\left(e^2\right)^7}\\ &= \frac{f^{2\cdot 7}}{e^{2\cdot 7}}\\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

The Power of a Quotient Rule of Exponents

For any real numbers a and b and any integer n, the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Using the Power of a Quotient Rule

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{4}{z^{11}}\right)^3$
b. $\left(\frac{p}{q^3}\right)^6$
c. $\left(\frac{-1}{t^2}\right)^{27}$
d. $\left(j^3k^{-2}\right)^4$
e. $\left(m^{-2}n^{-2}\right)^3$

Show Solution

a. $\left(\frac{4}{z^{11}}\right)^3 = \frac{(4)^3}{\left(z^{11}\right)^3} = \frac{64}{z^{11\cdot 3}} = \frac{64}{z^{33}}$
b. $\left(\frac{p}{q^3}\right)^6 = \frac{(p)^6}{\left(q^3\right)^6} = \frac{p^{1\cdot 6}}{q^{3\cdot 6}} = \frac{p^6}{q^{18}}$
c. $\left(\frac{-1}{t^2}\right)^{27} = \frac{(-1)^{27}}{\left(t^2\right)^{27}} = \frac{-1}{t^{2\cdot 27}} = \frac{-1}{t^{54}} = -\frac{1}{t^{54}}$

d. $\left(j^3k^{-2}\right)^4 = \left(\frac{j^3}{k^2}\right)^4 = \frac{\left(j^3\right)^4}{\left(k^2\right)^4} = \frac{j^{3\cdot4}}{k^{2\cdot4}} = \frac{j^{12}}{k^8}$

e. $\left(m^{-2}n^{-2}\right)^3 = \left(\frac{1}{m^2n^2}\right)^3 = \frac{(1)^3}{\left(m^2n^2\right)^3} = \frac{1}{\left(m^2\right)^3\left(n^2\right)^3} = \frac{1}{m^{2\cdot3}\cdot n^{2\cdot3}} = \frac{1}{m^6n^6}$

Try It

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{b^5}{c}\right)^3$
b. $\left(\frac{5}{u^8}\right)^4$
c. $\left(\frac{-1}{w^3}\right)^{35}$
d. $\left(p^{-4}q^3\right)^8$
e. $\left(c^{-5}d^{-3}\right)^4$

Show Solution

a. $\frac{b^{15}}{c^3}$
b. $\frac{625}{u^{32}}$
c. $\frac{-1}{w^{105}}$
d. $\frac{q^{24}}{p^{32}}$
e. $\frac{1}{c^{20}d^{12}}$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

a. $\left(6m^2n^{-1}\right)^3$
b. $17^5 \cdot 17^{-4} \cdot 17^{-3}$
c. $\left(\frac{u^{-1}v}{v^{-1}}\right)^2$
d. $\left(-2a^3b^{-1}\right)\left(5a^{-2}b^2\right)$
e. $\left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4}$
f. $\frac{\left(3w^2\right)^5}{\left(6w^{-2}\right)^2}$

Show Solution

a.
$$\begin{aligned} \left(6m^2n^{-1}\right)^3 &= (6)^3\left(m^2\right)^3\left(n^{-1}\right)^3 && \text{The power of a product rule} \\ &= 6^3m^{2\cdot 3}n^{-1\cdot 3} && \text{The power rule} \\ &= 216m^6n^{-3} && \text{Simplify.} \\ &= \frac{216m^6}{n^3} && \text{The negative exponent rule} \end{aligned}$$

---

b.
$$\begin{aligned} 17^5 \cdot 17^{-4} \cdot 17^{-3} &= 17^{5-4-3} && \text{The product rule} \\ &= 17^{-2} && \text{Simplify.} \\ &= \frac{1}{17^2} \text{ or } \frac{1}{289} && \text{The negative exponent rule} \end{aligned}$$

---

c.
$$\begin{aligned} \left(\frac{u^{-1}v}{v^{-1}}\right)^2 &= \frac{\left(u^{-1}v\right)^2}{\left(v^{-1}\right)^2} && \text{The power of a quotient rule} \\ &= \frac{u^{-2}v^2}{v^{-2}} && \text{The power of a product rule} \\ &= u^{-2}v^{2-(-2)} && \text{The quotient rule} \\ &= u^{-2}v^4 && \text{Simplify.} \\ &= \frac{v^4}{u^2} && \text{The negative exponent rule} \end{aligned}$$

---

d.
$$\begin{aligned} \left(-2a^3b^{-1}\right)\left(5a^{-2}b^2\right) &= -2\cdot 5\cdot a^3\cdot a^{-2}\cdot b^{-1}\cdot b^2 && \text{Commutative and associative laws of multiplication} \\ &= -10\cdot a^{3-2}\cdot b^{-1+2} && \text{The product rule} \\ &= -10ab && \text{Simplify.} \end{aligned}$$

---

e.
$$\begin{aligned} \left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4} &= \left(x^2\sqrt{2}\right)^{4-4} && \text{The product rule} \\ &= \left(x^2\sqrt{2}\right)^0 && \text{Simplify.} \\ &= 1 && \text{The zero exponent rule} \end{aligned}$$

---

f.
$$\begin{aligned} \frac{\left(3w^2\right)^5}{\left(6w^{-2}\right)^2} &= \frac{(3)^5\cdot\left(w^2\right)^5}{(6)^2\cdot\left(w^{-2}\right)^2} && \text{The power of a product rule} \\ &= \frac{3^5w^{2\cdot 5}}{6^2w^{-2\cdot 2}} && \text{The power rule} \\ &= \frac{243w^{10}}{36w^{-4}} && \text{Simplify.} \\ &= \frac{27w^{10-(-4)}}{4} && \text{The quotient rule and reduce fraction} \\ &= \frac{27w^{14}}{4} && \text{Simplify.} \end{aligned}$$

Try It

Simplify each expression and write the answer with positive exponents only.

a. $\left(2uv^{-2}\right)^{-3}$
b. $x^8\cdot x^{-12}\cdot x$

c. $\left(\frac{e^2 f^{-3}}{f^{-1}}\right)^2$
d. $\left(9r^{-5}s^3\right)\left(3r^6s^{-4}\right)$
e. $\left(\frac{4}{9}tw^{-2}\right)^{-3}\left(\frac{4}{9}tw^{-2}\right)^3$
f. $\frac{\left(2h^2k\right)^4}{\left(7h^{-1}k^2\right)^2}$

Show Solution

a. $\frac{v^6}{8u^3}$
b. $\frac{1}{x^3}$
c. $\frac{e^4}{f^4}$
d. $\frac{27r}{s}$
e. 1
f. $\frac{16h^{10}}{49}$

Using Scientific Notation

Recall at the beginning of the section that we found the number 1.3×10^{13} when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about 0.00005 m, and the radius of an electron, which is about 0.00000000000047 m. How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called scientific notation, in which we express numbers in terms of exponents of 10. To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places n that you moved the decimal point. Multiply the decimal number by 10 raised to a power of n. If you moved the decimal left as in a very large number, n is positive. If you moved the decimal right as in a small large number, n is negative.

For example, consider the number 2,780,418. Move the decimal left until it is to the right of the first nonzero digit, which is 2.

$$2{,}780{,}418 \longrightarrow 2.\overbrace{780418}^{\text{6 places left}}$$

We obtain 2.780418 by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6, and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.
2.780418×10^6

Working with small numbers is similar. Take, for example, the radius of an electron, 0.00000000000047 m. Perform the same series of steps as above, except move the decimal point to the right.

$$0.00000000000047 \longrightarrow \overbrace{0000000000004}^{\text{13 places right}}.7$$

Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13. The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.
4.7×10^{-13}

Scientific Notation

A number is written in scientific notation if it is written in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer.

Converting Standard Notation to Scientific Notation

Write each number in scientific notation.

a. Distance to Andromeda Galaxy from Earth: 24,000,000,000,000,000,000,000 m
b. Diameter of Andromeda Galaxy: 1,300,000,000,000,000,000,000 m
c. Number of stars in Andromeda Galaxy: 1,000,000,000,000
d. Diameter of electron: 0.00000000000094 m
e. Probability of being struck by lightning in any single year: 0.00000143

Show Solution

a.
$$24,000,000,000,000,000,000,000 \text{ m}$$
$$24,000,000,000,000,000,000,000 \text{ m}$$
$$\text{22 places}$$
$$2.4 \times 10^{22} \text{ m}$$

---

b.
$$1,300,000,000,000,000,000,000 \text{ m}$$
$$1,300,000,000,000,000,000,000 \text{ m}$$
$$\text{21 places}$$
$$1.3 \times 10^{21} \text{ m}$$

---

c.
$$1,000,000,000,000$$
$$1,000,000,000,000$$
$$\text{12 places}$$
$$1 \times 10^{12}$$

---

d.
$$0.00000000000094 \text{ m}$$
$$0.00000000000094 \text{ m}$$
$$\rightarrow \text{13 places}$$
$$9.4 \times 10^{-13} \text{ m}$$

---

e.
$$0.00000143$$
$$0.00000143$$
$$\rightarrow \text{6 places}$$
$$1.43 \times 10^{-6}$$

Analysis

Observe that, if the given number is greater than 1, as in examples a–c, the exponent of 10 is positive; and if the number is less than 1, as in examples d–e, the exponent is negative.

Try It

Write each number in scientific notation.

a. U.S. national debt per taxpayer (April 2014): $152,000
b. World population (April 2014): 7,158,000,000
c. World gross national income (April 2014): $85,500,000,000,000
d. Time for light to travel 1 m: 0.00000000334 s
e. Probability of winning lottery (match 6 of 49 possible numbers): 0.0000000715

Show Solution

a. 1.52×10^{5}
b. 7.158×10^{9}
c. 8.55×10^{13}
d. 3.34×10^{-9}
e. 7.15×10^{-8}

Converting from Scientific to Standard Notation

To convert a number in scientific notation to standard notation, simply reverse the process. Move the decimal n places to the right if n is positive or n places to the left if n is negative and add zeros as needed. Remember, if n is positive, the value of the number is greater than 1, and if n is negative, the value of the number is less than one.

Converting Scientific Notation to Standard Notation

Convert each number in scientific notation to standard notation.

a. 3.547×10^{14}
b. -2×10^{6}
c. 7.91×10^{-7}
d. -8.05×10^{-12}

Show Solution

a.
$$\begin{array}{l} 3.547 \times 10^{14} \\ 3.54700000000000 \\ \quad\quad \xrightarrow{14 \text{ places}} \\ 354,700,000,000,000 \end{array}$$

---

b.
$$\begin{array}{l} -2 \times 10^{6} \\ -2.000000 \\ \quad \xrightarrow{6 \text{ places}} \\ -2,000,000 \end{array}$$

---

c.
$$\begin{array}{l} 7.91 \times 10^{-7} \\ 0000007.91 \\ \quad \xrightarrow{7 \text{ places}} \\ 0.000000791 \end{array}$$

---

d.
$$\begin{array}{l} -8.05 \times 10^{-12} \\ -000000000008.05 \\ \quad\quad \xrightarrow{12 \text{ places}} \\ -0.00000000000805 \end{array}$$

Try It

Convert each number in scientific notation to standard notation.

a. 7.03×10^5
b. -8.16×10^{11}
c. -3.9×10^{-13}
d. 8×10^{-6}

Show Solution

a. $703,000$
b. $-816,000,000,000$
c. -0.00000000000039
d. 0.000008

Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents, makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in 1 L of water. Each water molecule contains 3 atoms (2 hydrogen and 1 oxygen). The average drop of water contains around 1.32×10^{21} molecules of water and 1 L of water holds about 1.22×10^4 average drops. Therefore, there are approximately $3 \cdot \left(1.32 \times 10^{21}\right) \cdot \left(1.22 \times 10^4\right) \approx 4.83 \times 10^{25}$ atoms in 1 L of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $\left(7 \times 10^4\right) \cdot \left(5 \times 10^6\right) = 35 \times 10^{10}$. The answer is not in proper scientific notation because 35 is greater than 10. Consider 35 as $3.5 \times 10.$ That adds a ten to the exponent of the answer.

$(35) \times 10^{10} = (3.5 \times 10) \times 10^{10} = 3.5 \times \left(10 \times 10^{10}\right) = 3.5 \times 10^{11}$

Using Scientific Notation

Perform the operations and write the answer in scientific notation.

a. $\left(8.14 \times 10^{-7}\right)\left(6.5 \times 10^{10}\right)$
b. $\left(4 \times 10^{5}\right) \div \left(-1.52 \times 10^{9}\right)$
c. $\left(2.7 \times 10^{5}\right)\left(6.04 \times 10^{13}\right)$
d. $\left(1.2 \times 10^{8}\right) \div \left(9.6 \times 10^{5}\right)$
e. $\left(3.33 \times 10^{4}\right)\left(-1.05 \times 10^{7}\right)\left(5.62 \times 10^{5}\right)$

Show Solution

a.
$$\begin{array}{rcll} \left(8.14 \times 10^{-7}\right)\left(6.5 \times 10^{10}\right) & = & \left(8.14 \times 6.5\right)\left(10^{-7} \times 10^{10}\right) & \text{Commutative and associative properties of multiplication} \\ & = & \left(52.91\right)\left(10^{3}\right) & \text{Product rule of exponents} \\ & = & 5.291 \times 10^{4} & \text{Scientific notation} \end{array}$$

---

b.
$$\begin{array}{rcll} \left(4 \times 10^{5}\right) \div \left(-1.52 \times 10^{9}\right) & = & \left(\frac{4}{-1.52}\right)\left(\frac{10^{5}}{10^{9}}\right) & \text{Commutative and associative properties of multiplication} \\ & \approx & \left(-2.63\right)\left(10^{-4}\right) & \text{Quotient rule of exponents} \\ & = & -2.63 \times 10^{-4} & \text{Scientific notation} \end{array}$$

---

c.
$$\begin{array}{rcll} \left(2.7 \times 10^{5}\right)\left(6.04 \times 10^{13}\right) & = & \left(2.7 \times 6.04\right)\left(10^{5} \times 10^{13}\right) & \text{Commutative and associative properties of multiplication} \\ & = & \left(16.308\right)\left(10^{18}\right) & \text{Product rule of exponents} \\ & = & 1.6308 \times 10^{19} & \text{Scientific notation} \end{array}$$

---

d.
$$\begin{array}{rcll} \left(1.2 \times 10^{8}\right) \div \left(9.6 \times 10^{5}\right) & = & \left(\frac{1.2}{9.6}\right)\left(\frac{10^{8}}{10^{5}}\right) & \text{Commutative and associative properties of multiplication} \\ & = & \left(0.125\right)\left(10^{3}\right) & \text{Quotient rule of exponents} \\ & = & 1.25 \times 10^{2} & \text{Scientific notation} \end{array}$$

---

e. $$\begin{aligned}\left(3.33 \times 10^4\right)\left(-1.05 \times 10^7\right)\left(5.62 \times 10^5\right) &= [3.33 \times (-1.05) \times 5.62]\left(10^4 \times 10^7 \times 10^5\right) \\ &\approx (-19.65)\left(10^{16}\right) \\ &= -1.965 \times 10^{17}\end{aligned}$$

Try It

Perform the operations and write the answer in scientific notation.

a. $\left(-7.5 \times 10^8\right)\left(1.13 \times 10^{-2}\right)$
b. $\left(1.24 \times 10^{11}\right) \div \left(1.55 \times 10^{18}\right)$
c. $\left(3.72 \times 10^9\right)\left(8 \times 10^3\right)$
d. $\left(9.933 \times 10^{23}\right) \div \left(-2.31 \times 10^{17}\right)$
e. $\left(-6.04 \times 10^9\right)\left(7.3 \times 10^2\right)\left(-2.81 \times 10^2\right)$

Show Solution

a. -8.475×10^6
b. 8×10^{-8}
c. 2.976×10^{13}
d. -4.3×10^6
e. $\approx 1.24 \times 10^{15}$

Applying Scientific Notation to Solve Problems

In April 2014, the population of the United States was about 308,000,000 people. The national debt was about $17,547,000,000,000. Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

Show Solution

The population was $308,000,000 = 3.0810^8$.

The national debt was $\$17,547,000,000,000 \approx \$1.7510^{13}$.

To find the amount of debt per citizen, divide the national debt by the number of citizens.

$$\begin{aligned} \left(1.7510^{13}\right)\left(3.0810^{8}\right) &= \left(\frac{1.75}{3.08}\right)\cdot\left(\frac{10^{13}}{10^{8}}\right) \\ &\approx 0.5710^{5} \\ &= 5.710^{4} \end{aligned}$$

The debt per citizen at the time was about $\$5.7 \times 10^4$, or $57,000.

Try It

An average human body contains around 30,000,000,000,000 red blood cells. Each cell measures approximately 0.000008 m long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.

Show Solution

Number of cells: 310^{13}; length of a cell: 8×10^{-6} m; total length: 2.4×10^{8} m or $240,000,000$ m.

Access these online resources for additional instruction and practice with exponents and scientific notation.

- Exponential Notation
- Properties of Exponents
- Zero Exponent
- Simplify Exponent Expressions
- Quotient Rule for Exponents
- Scientific Notation
- Converting to Decimal Notation

Key Equations

Rules of Exponents< For nonzero real numbers a and b and integers m and n	
Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \frac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Key Concepts

- Products of exponential expressions with the same base can be simplified by adding exponents. See (Figure).
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents. See (Figure).
- Powers of exponential expressions with the same base can be simplified by multiplying exponents. See (Figure).
- An expression with exponent zero is defined as 1. See (Figure).
- An expression with a negative exponent is defined as a reciprocal. See (Figure) and (Figure).
- The power of a product of factors is the same as the product of the powers of the same factors. See (Figure).
- The power of a quotient of factors is the same as the quotient of the powers of the same factors. See (Figure).
- The rules for exponential expressions can be combined to simplify more complicated expressions. See (Figure).
- Scientific notation uses powers of 10 to simplify very large or very small numbers. See (Figure) and (Figure).
- Scientific notation may be used to simplify calculations with very large or very small numbers. See (Figure) and (Figure).

Section Exercises

Verbal

1. Is 2^3 the same as 3^2? Explain.

Show Solution
No, the two expressions are not the same. An exponent tells how many times you multiply the base. So 2^3 is the same as $2 \times 2 \times 2,$ which is 8. 3^2 is the same as $3 \times 3,$ which is 9.

2. When can you add two exponents?

3. What is the purpose of scientific notation?

Show Solution
It is a method of writing very small and very large numbers.

4. Explain what a negative exponent does.

Numeric

For the following exercises, simplify the given expression. Write answers with positive exponents.

5. 9^2

Show Solution
81

6. 15^{-2}

7. $3^2 \times 3^3$

Show Solution
243

8. $4^4 \div 4$

9. $\left(2^2\right)^{-2}$

Show Solution
$\frac{1}{16}$

10. $(5-8)^0$

11. $11^3 \div 11^4$

Show Solution
$\frac{1}{11}$

12. $6^5 \times 6^{-7}$

13. $\left(8^0\right)^2$

Show Solution
1

14. $5^{-2} \div 5^2$

For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents.

15. $4^2 \times 4^3 \div 4^{-4}$

Show Solution
4^9

16. $\frac{6^{12}}{6^9}$

17. $\left(12^3 \times 12\right)^{10}$

Show Solution
12^{40}

18. $10^6 \div \left(10^{10}\right)^{-2}$

19. $7^{-6} \times 7^{-3}$

Show Solution
$\frac{1}{7^9}$

20. $\left(3^3 3^4\right)^5$

For the following exercises, express the decimal in scientific notation.

21. 0.0000314

Show Solution
3.14×10^{-5}

22. 148,000,000

For the following exercises, convert each number in scientific notation to standard notation.

23. 1.6×10^{10}

Show Solution

16,000,000,000

24. 9.8×10^{-9}

Algebraic

For the following exercises, simplify the given expression. Write answers with positive exponents.

25. $\frac{a^3a^2}{a}$

Show Solution
a^4

26. $\frac{mn^2}{m^{-2}}$

27. $\left(b^3c^4\right)^2$

Show Solution
b^6c^8

28. $\left(\frac{x^{-3}}{y^2}\right)^{-5}$

29. $ab^2 \div d^{-3}$

Show Solution
ab^2d^3

30. $\left(w^0x^5\right)^{-1}$

31. $\frac{m^4}{n^0}$

Show Solution

m^4

32. $y^{-4}\left(y^2\right)^2$

33. $\frac{p^{-4}q^2}{p^2q^{-3}}$

Show Solution

$\frac{q^5}{p^6}$

34. $(l)^2$

35. $\left(y^7\right)^3 \div x^{14}$

Show Solution

$\frac{y^{21}}{x^{14}}$

36. $\left(\frac{a}{2^3}\right)^2$

37. $5^2m \div 5^0m$

Show Solution

25

38. $\frac{(16sqrtx)^2}{y^{-1}}$

39. $\frac{2^3}{(3a)^{-2}}$

Show Solution
$72a^2$

40. $\left(ma^6\right)^2\frac{1}{m^3a^2}$

41. $\left(b^{-3}c\right)^3$

Show Solution
$\frac{c^3}{b^9}$

42. $\left(x^2y^{13}\div y^0\right)^2$

43. $\left(9z^3\right)^{-2}y$

Show Solution
$\frac{y}{81z^6}$

Real-World Applications

44. To reach escape velocity, a rocket must travel at the rate of 2.2×10^6 ft/min. Rewrite the rate in standard notation.

45. A dime is the thinnest coin in U.S. currency. A dime's thickness measures 1.35×10^{-3} m. Rewrite the number in standard notation.

Show Solution
0.00135 m

46. The average distance between Earth and the Sun is 92,960,000 mi. Rewrite the distance using scientific notation.

47. A terabyte is made of approximately 1,099,500,000,000 bytes. Rewrite in scientific notation.

Show Solution
1.0995×10^{12}

48. The Gross Domestic Product (GDP) for the United States in the first quarter of 2014 was $\$1.71496 \times 10^{13}$. Rewrite the GDP in standard notation.

50. One picometer is approximately 3.397×10^{-11} in. Rewrite this length using standard notation.

Show Solution
0.00000000003397 in.

51. The value of the services sector of the U.S. economy in the first quarter of 2012 was $10,633.6 billion. Rewrite this amount in scientific notation.

Technology

52. For the following exercises, use a graphing calculator to simplify. Round the answers to the nearest hundredth.

53. $\left(\frac{12^3m^{33}}{4^{-3}}\right)^2$

Show Solution
12,230,590,464 m^{66}

54. $17^3 \div 15^2x^3$

Extensions

For the following exercises, simplify the given expression. Write answers with positive exponents.

55. $\left(\frac{3^2}{a^3}\right)^{-2}\left(\frac{a^4}{2^2}\right)^2$

Show Solution
$\frac{a^{14}}{1296}$

56. $\left(6^2-24\right)^2 \div \left(\frac{x}{y}\right)^{-5}$

57. $\frac{m^2n^3}{a^2c^{-3}} \cdot \frac{a^{-7}n^{-2}}{m^2c^4}$

Show Solution
$\frac{n}{a^9c}$

58. $\left(\frac{x^6y^3}{x^3y^{-3}}\cdot\frac{y^{-7}}{x^{-3}}\right)^{10}$

59. $\left(\frac{\left(ab^2c\right)^{-3}}{b^{-3}}\right)^2$

Show Solution
$\frac{1}{a^6b^6c^6}$

60. Avogadro's constant is used to calculate the number of particles in a mole. A mole is a basic unit in chemistry to measure the amount of a substance. The constant is 6.0221413×10^{23}. Write Avogadro's constant in standard notation.

61. Planck's constant is an important unit of measure in quantum physics. It describes the relationship between energy and frequency. The constant is written as 6.62606957×10^{-34}. Write Planck's constant in standard notation.

Show Solution
0.000000000000000000000000000000000662606957

Glossary

scientific notation
: a shorthand notation for writing very large or very small numbers in the form $a\times10^n$ where $1\leq|a|<10$ and n is an integer

CHAPTER 1.2: RADICALS AND RATIONAL EXPONENTS

Learning Objectives

In this section students will:

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

A hardware store sells 16-ft ladders and 24-ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in (Figure), and use the Pythagorean Theorem.

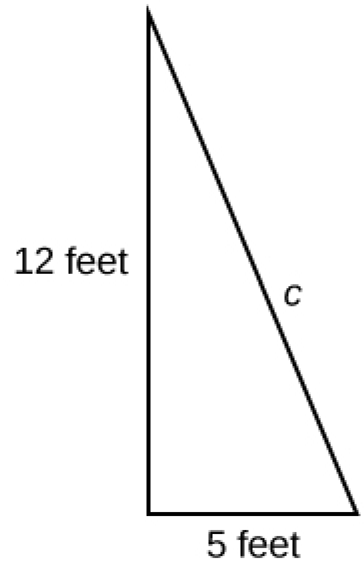


Figure 1.

$$\begin{aligned} a^2+b^2 &= c^2 \\ 5^2+12^2 &= c^2 \\ 169 &= c^2 \end{aligned}$$

Now, we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other words, we need to find a square root. In this section, we will investigate methods of finding solutions to problems such as this one.

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2 = 16$, the square root of 16 is 4. The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a. The square root could be positive or negative because multiplying two negative numbers gives a positive number. The principal square root is the nonnegative number that when multiplied by itself equals a. The square root obtained using a calculator is the principal square root.

The principal square root of a is written as $\sqrt{a}$. The symbol is called a radical, the term under the symbol is called the radicand, and the entire expression is called a radical expression.

Principal Square Root

The principal square root of a is the nonnegative number that, when multiplied by itself, equals a. It is written as a radical expression**,** with a symbol called a radical over the term called the radicand: $\sqrt{a}$.

Does $\sqrt{25} = 5$?

No. Although both 5^2 *and* $(-5)^2$ *are* 25, *the radical symbol implies only a nonnegative root, the principal square root. The principal square root of 25 is* $\sqrt{25} = 5$.

Evaluating Square Roots

Evaluate each expression.

a. $\sqrt{100}$
b. $\sqrt{\sqrt{16}}$
c. $\sqrt{25+144}$
d. $\sqrt{49} - \sqrt{81}$

Show Solution

a. $\sqrt{100} = 10$ because $10^2 = 100$
b. $\sqrt{\sqrt{16}} = \sqrt{4} = 2$ because $4^2 = 16$ and $2^2 = 4$
c. $\sqrt{25+144} = \sqrt{169} = 13$ because $13^2 = 169$
d. $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$ because $7^2 = 49$ and $9^2 = 81$

For $\sqrt{25+144}$, can we find the square roots before adding?

No. $\sqrt{25}+\sqrt{144}=5+12=17.$ *This is not equivalent to* $\sqrt{25+144}=13.$ *The order of operations requires us to add the terms in the radicand before finding the square root.*

Try It

Evaluate each expression.

a. $\sqrt{225}$
b. $\sqrt{\sqrt{81}}$
c. $\sqrt{25-9}$
d. $\sqrt{36}+\sqrt{121}$

Show Solution

a. 15
b. 3
c. 4
d. 17

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the *product rule for simplifying square roots,* which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3}\cdot\sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

The Product Rule for Simplifying Square Roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b.

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

How To

Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

Using the Product Rule to Simplify Square Roots

Simplify the radical expression.

a. $\sqrt{300}$
b. $\sqrt{162a^5b^4}$

Show Solution

a.
$$\begin{array}{ll} \sqrt{100 \cdot 3} & \text{Factor perfect square from radicand.} \\ \sqrt{100} \cdot \sqrt{3} & \text{Write radical expression as product of radical expressions.} \\ 10\sqrt{3} & \text{Simplify.} \end{array}$$

b.

$\sqrt{81a^4b^4 \cdot 2a}$	Factor perfect square from radicand.
$\sqrt{81a^4b^4} \cdot \sqrt{2a}$	Write radical expression as product of radical expressions.
$9a^2b^2\sqrt{2a}$	Simplify.

Try It

Simplify $\sqrt{50x^2y^3z}$.

Show Solution

$5|x||y|\sqrt{2yz}$. Notice the absolute value signs around *x* and *y*? That's because their value must be positive!

How To

Given the product of multiple radical expressions, use the product rule to combine them into one radical expression.

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

Using the Product Rule to Simplify the Product of Multiple Square Roots

Simplify the radical expression.

$\sqrt{12} \cdot \sqrt{3}$

Show Solution

$$\begin{array}{ll} \sqrt{12 \cdot 3} & \text{Express the product as a single radical expression.} \\ \sqrt{36} & \text{Simplify.} \\ 6 & \end{array}$$

Try It

Simplify $\sqrt{50x} \cdot \sqrt{2x}$ assuming $x > 0$.

Show Solution

$10|x|$

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the *quotient rule for simplifying square roots.* It can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately. We can rewrite $\sqrt{\frac{5}{2}}$ as $\frac{\sqrt{5}}{\sqrt{2}}$.

The Quotient Rule for Simplifying Square Roots

The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b, where $b \neq 0$.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

How To

Given a radical expression, use the quotient rule to simplify it.

1. Write the radical expression as the quotient of two radical expressions.
2. Simplify the numerator and denominator.

Using the Quotient Rule to Simplify Square Roots

Simplify the radical expression.

$$\sqrt{\frac{5}{36}}$$

Show Solution

$$\begin{array}{ll} \frac{\sqrt{5}}{\sqrt{36}} & \text{Write as quotient of two radical expressions.} \\ \frac{\sqrt{5}}{6} & \text{Simplify denominator.} \end{array}$$

Try It

Simplify $\sqrt{\frac{2x^2}{9y^4}}$.

Show Solution

$\frac{x\sqrt{2}}{3y^2}$. We do not need the absolute value signs for y^2 because that term will always be nonnegative.

Using the Quotient Rule to Simplify an Expression with Two Square Roots

Simplify the radical expression.

$$\frac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}}$$

Show Solution

$\sqrt{\frac{234x^{11}y}{26x^7y}}$	Combine numerator and denominator into one radical expression.
$\sqrt{9x^4}$	Simplify fraction.
$3x^2$	Simplify square root.

Try It

Simplify $\frac{\sqrt{9a^5b^{14}}}{\sqrt{3a^4b^5}}$.

Show Solution

$b^4\sqrt{3ab}$

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3\sqrt{2}$ is $4\sqrt{2}$. However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3\sqrt{2}$, so $\sqrt{2}+\sqrt{18}=\sqrt{2}+3\sqrt{2}=4\sqrt{2}$.

How To

Given a radical expression requiring addition or subtraction of square roots, solve.

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

Adding Square Roots

Add $5\sqrt{12}+2\sqrt{3}$.

Show Solution

We can rewrite $5\sqrt{12}$ as $5\sqrt{4\cdot 3}.$ According the product rule, this becomes $5\sqrt{4}\sqrt{3}.$ The square root of $\sqrt{4}$ is 2, so the expression becomes $5\left(2\right)\sqrt{3},$ which is $10\sqrt{3}.$ Now we can the terms have the same radicand so we can add.

$$10\sqrt{3}+2\sqrt{3}=12\sqrt{3}$$

Try It

Add $\sqrt{5}+6\sqrt{20}.$

Show Solution

$13\sqrt{5}$

Subtracting Square Roots

Subtract $20\sqrt{72{a}^{3}{b}^{4}c}\;-14\sqrt{8{a}^{3}{b}^{4}c}.$

Show Solution

Rewrite each term so they have equal radicands.

$$\begin{aligned} 20\sqrt{72a^3b^4c} &= 20\sqrt{9}\sqrt{4}\sqrt{2}\sqrt{a}\sqrt{a^2}\sqrt{\left(b^2\right)^2}\sqrt{c} \\ &= 20\left(3\right)\left(2\right)|a|b^2\sqrt{2ac} \\ &= 120|a|b^2\sqrt{2ac} \end{aligned}$$

$$\begin{aligned} 14\sqrt{8a^3b^4c} &= 14\sqrt{2}\sqrt{4}\sqrt{a}\sqrt{a^2}\sqrt{\left(b^2\right)^2}\sqrt{c} \\ &= 14\left(2\right)|a|b^2\sqrt{2ac} \\ &= 28|a|b^2\sqrt{2ac} \end{aligned}$$

Now the terms have the same radicand so we can subtract.

$$120|a|b^2\sqrt{2ac} - 28|a|b^2\sqrt{2ac} = 92|a|b^2\sqrt{2ac}$$

Try It

Subtract $3\sqrt{80x} - 4\sqrt{45x}$.

Show Solution

0

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called *rationalizing the denominator*.

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b\sqrt{c}$, multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

For a denominator containing the sum or difference of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a+b\sqrt{c}$, then the conjugate is $a-b\sqrt{c}$.

How To

Given an expression with a single square root radical term in the denominator, rationalize the denominator.

a. Multiply the numerator and denominator by the radical in the denominator.
b. Simplify.

Rationalizing a Denominator Containing a Single Term

Write $\frac{2\sqrt{3}}{3\sqrt{10}}$ in simplest form.

Show Solution

The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

$$\frac{2\sqrt{3}}{3\sqrt{10}}\cdot\frac{\sqrt{10}}{\sqrt{10}}$$
$$\frac{2\sqrt{30}}{30}$$
$$\frac{\sqrt{30}}{15}$$

Try It

Write $\frac{12\sqrt{3}}{\sqrt{2}}$ in simplest form.

Show Solution

$6\sqrt{6}$

How To

Given an expression with a radical term and a constant in the denominator, rationalize the denominator.

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

Rationalizing a Denominator Containing Two Terms

Write $\frac{4}{1+\sqrt{5}}$ in simplest form.

Show Solution

Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1+\sqrt{5}$ is $1-\sqrt{5}$. Then multiply the fraction by $\frac{1-\sqrt{5}}{1-\sqrt{5}}$.

$\frac{4}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}$

$\frac{4-4\sqrt{5}}{-4}$ Use the distributive property.

$\sqrt{5}-1$ Simplify.

Try It

Write $\frac{7}{2+\sqrt{3}}$ in simplest form.

Show Solution

$14-7\sqrt{3}$

Using Rational Roots

Although square roots are the most common rational roots, we can also find cube roots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number that, when raised to a certain power, gives a certain number.

Understanding *n*th Roots

Suppose we know that $a^3=8$. We want to find what number raised to the 3rd power is equal to 8. Since $2^3=8$, we say that 2 is the cube root of 8.

The *n*th root of a is a number that, when raised to the *n*th power, gives a. For example, -3 is the 5th root of

-243 because $(-3)^5 = -243$. If a is a real number with at least one *n*th root, then the principal *n*th root of a is the number with the same sign as a that, when raised to the *n*th power, equals a.

The principal *n*th root of a is written as $\sqrt[n]{a}$, where n is a positive integer greater than or equal to 2. In the radical expression, n is called the index of the radical.

Principal *n*th Root

If a is a real number with at least one *n*th root, then the principal *n*th root of a, written as $\sqrt[n]{a}$, is the number with the same sign as a that, when raised to the *n*th power, equals a. The index of the radical is n.

Simplifying *n*th Roots

Simplify each of the following:

a. $\sqrt[5]{-32}$
b. $\sqrt[4]{4} \cdot \sqrt[4]{1,024}$
c. $-\sqrt[3]{\frac{8x^6}{125}}$
d. $8\sqrt[4]{3} - \sqrt[4]{48}$

Show Solution

a. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$
b. First, express the product as a single radical expression. $\sqrt[4]{4,096} = 8$ because $8^4 = 4,096$
c.
$$\begin{array}{ll} \frac{-\sqrt[3]{8x^6}}{\sqrt[3]{125}} & \text{Write as quotient of two radical expressions.} \\ \frac{-2x^2}{5} & \text{Simplify.} \end{array}$$
d.
$$\begin{array}{ll} 8\sqrt[4]{3} - 2\sqrt[4]{3} & \text{Simplify to get equal radicands.} \\ 6\sqrt[4]{3} & \text{Add.} \end{array}$$

Try It

Simplify.

a. $\sqrt[3]{-216}$
b. $\frac{3\sqrt[4]{80}}{\sqrt[4]{5}}$
c. $6\sqrt[3]{9,000} + 7\sqrt[3]{576}$

Show Solution

a. -6
b. 6
c. $88\sqrt[3]{9}$

Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index n is even, then a cannot be negative.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an nth root. The numerator tells us the power and the denominator tells us the root.

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

Rational Exponents

Rational exponents are another way to express principal *n*th roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

How To

Given an expression with a rational exponent, write the expression as a radical.

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Writing Rational Exponents as Radicals

Write $343^{\frac{2}{3}}$ as a radical. Simplify.

Show Solution

The 2 tells us the power and the 3 tells us the root.

$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = \sqrt[3]{343^2}$

We know that $\sqrt[3]{343} = 7$ because $7^3 = 343.$ Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = 7^2 = 49$$

Try It

Write $9^{\frac{5}{2}}$ as a radical. Simplify.

Show Solution

$$\left(\sqrt{9}\right)^5 = 3^5 = 243$$

Writing Radicals as Rational Exponents

Write $\frac{4}{\sqrt[7]{a^2}}$ using a rational exponent.

Show Solution

The power is 2 and the root is 7, so the rational exponent will be $\frac{2}{7}$. We get $\frac{4}{a^{\frac{2}{7}}}$. Using properties of exponents, we get $\frac{4}{\sqrt[7]{a^2}} = 4a^{\frac{-2}{7}}$.

Try It

Write $x\sqrt{(5y)^9}$ using a rational exponent.

Show Solution

$x(5y)^{\frac{9}{2}}$

Simplifying Rational Exponents

Simplify:

a. $5\left(2x^{\frac{3}{4}}\right)\left(3x^{\frac{1}{5}}\right)$
b. $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

Show Solution

a.
$$\begin{array}{ll} 30x^{\frac{3}{4}}x^{\frac{1}{5}} & \text{Multiply the coefficients.} \\ 30x^{\frac{3}{4}+\frac{1}{5}} & \text{Use properties of exponents.} \\ 30x^{\frac{19}{20}} & \text{Simplify.} \end{array}$$

b.
$$\begin{array}{ll} \left(\frac{9}{16}\right)^{\frac{1}{2}} & \text{Use definition of negative exponents.} \\ \sqrt{\frac{9}{16}} & \text{Rewrite as a radical.} \\ \frac{\sqrt{9}}{\sqrt{16}} & \text{Use the quotient rule.} \\ \frac{3}{4} & \text{Simplify.} \end{array}$$

Try It

Simplify $\left[(8x)^{\frac{1}{3}}\right]\left(14x^{\frac{6}{5}}\right)$.

Show Solution

$28x^{\frac{23}{15}}$

Access these online resources for additional instruction and practice with radicals and rational exponents.

- Radicals
- Rational Exponents
- Simplify Radicals
- Rationalize Denominator

Key Concepts

- The principal square root of a number a is the nonnegative number that when multiplied by itself equals a. See (Figure).
- If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b See (Figure) and (Figure).
- If a and b are nonnegative, the square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b See (Figure) and (Figure).
- We can add and subtract radical expressions if they have the same radicand and the same index. See (Figure) and (Figure).
- Radical expressions written in simplest form do not contain a radical in the denominator. To eliminate the square root radical from the denominator, multiply both the numerator and the denominator by the conjugate of the denominator. See (Figure) and (Figure).
- The principal *n*th root of a is the number with the same sign as a that when raised to the

*n*th power equals a. These roots have the same properties as square roots. See (Figure).
- Radicals can be rewritten as rational exponents and rational exponents can be rewritten as radicals. See (Figure) and (Figure).
- The properties of exponents apply to rational exponents. See (Figure).

Section Exercises

Verbal

1. What does it mean when a radical does not have an index? Is the expression equal to the radicand? Explain.

Show Solution
When there is no index, it is assumed to be 2 or the square root. The expression would only be equal to the radicand if the index were 1.

2. Where would radicals come in the order of operations? Explain why.

3. Every number will have two square roots. What is the principal square root?

Show Solution
The principal square root is the nonnegative root of the number.

4. Can a radical with a negative radicand have a real square root? Why or why not?

Numeric

For the following exercises, simplify each expression.

5. $\sqrt{256}$

Show Solution
16

6. $\sqrt{\sqrt{256}}$

7. $\sqrt{4\left(9+16\right)}$

Show Solution
10

8. $\sqrt{289}-\sqrt{121}$

9. $\sqrt{196}$

Show Solution
14

10. $\sqrt{1}$

11. $\sqrt{98}$

Show Solution
$7\sqrt{2}$

12. $\sqrt{\frac{27}{64}}$

13. $\sqrt{\frac{81}{5}}$

Show Solution
$\frac{9\sqrt{5}}{5}$

14. $\sqrt{800}$

15. $\sqrt{169}+\sqrt{144}$

Show Solution
25

16. $\sqrt{\frac{8}{50}}$

17. $\frac{18}{\sqrt{162}}$

Show Solution
$\sqrt{2}$

18. $\sqrt{192}$

19. $14\sqrt{6}-6\sqrt{24}$

Show Solution
$2\sqrt{6}$

20. $15\sqrt{5}+7\sqrt{45}$

21. $\sqrt{150}$

Show Solution

$5\sqrt{6}$

22. $\sqrt{\frac{96}{100}}$

23. $\left(\sqrt{42}\right)\left(\sqrt{30}\right)$

Show Solution

$6\sqrt{35}$

24. $12\sqrt{3} - 4\sqrt{75}$

25. $\sqrt{\frac{4}{225}}$

Show Solution

$\frac{2}{15}$

26. $\sqrt{\frac{405}{324}}$

27. $\sqrt{\frac{360}{361}}$

Show Solution

$\frac{6\sqrt{10}}{19}$

28. $\frac{5}{1+\sqrt{3}}$

29. $\frac{8}{1-\sqrt{17}}$

Show Solution

$-\frac{1+\sqrt{17}}{2}$

30. $\sqrt[4]{16}$

31. $\sqrt[3]{128}+3\sqrt[3]{2}$

Show Solution

$7\sqrt[3]{2}$

32. $\sqrt[5]{\frac{-32}{243}}$

33. $\frac{15\sqrt[4]{125}}{\sqrt[4]{5}}$

Show Solution

$15\sqrt{5}$

34. $3\sqrt[3]{-432}+\sqrt[3]{16}$

Algebraic

For the following exercises, simplify each expression.

35. $\sqrt{400x^4}$

Show Solution

$20x^2$

36. $\sqrt{4y^2}$

37. $\sqrt{49p}$

Show Solution

$7\sqrt{p}$

38. $\left(144p^2q^6\right)^{\frac{1}{2}}$

39. $m^{\frac{5}{2}}\sqrt{289}$

Show Solution

$17m^2\sqrt{m}$

40. $9\sqrt{3m^2}+\sqrt{27}$

41. $3\sqrt{ab^2}-b\sqrt{a}$

Show Solution

$2b\sqrt{a}$

42. $\frac{4\sqrt{2n}}{\sqrt{16n^4}}$

43. $\sqrt{\frac{225x^3}{49x}}$

Show Solution
$\frac{15x}{7}$

44. $3\sqrt{44z}+\sqrt{99z}$

45. $\sqrt{50y^8}$

Show Solution
$5y^4\sqrt{2}$

46. $\sqrt{490bc^2}$

47. $\sqrt{\frac{32}{14d}}$

Show Solution
$\frac{4\sqrt{7d}}{7d}$

48. $q^{\frac{3}{2}}\sqrt{63p}$

49. $\frac{\sqrt{8}}{1-\sqrt{3x}}$

Show Solution
$\frac{2\sqrt{2}+2\sqrt{6x}}{1-3x}$

50. $\sqrt{\frac{20}{121d^4}}$

51. $w^{\frac{3}{2}}\sqrt{32} - w^{\frac{3}{2}}\sqrt{50}$

Show Solution
$-w\sqrt{2w}$

52. $\sqrt{108x^4} + \sqrt{27x^4}$

53. $\frac{\sqrt{12x}}{2+2\sqrt{3}}$

Show Solution
$\frac{3\sqrt{x}-\sqrt{3x}}{2}$

54. $\sqrt{147k^3}$

55. $\sqrt{125n^{10}}$

Show Solution
$5n^5\sqrt{5}$

56. $\sqrt{\frac{42q}{36q^3}}$

57. $\sqrt{\frac{81m}{361m^2}}$

Show Solution
$\frac{9\sqrt{m}}{19m}$

58. $\sqrt{72c} - 2\sqrt{2c}$

59. $\sqrt{\frac{144}{324d^2}}$

Show Solution
$\frac{2}{3d}$

60. $\sqrt[3]{24x^6} + \sqrt[3]{81x^6}$

61. $\sqrt[4]{\frac{162x^6}{16x^4}}$

Show Solution
$\frac{3\sqrt[4]{2x^2}}{2}$

62. $\sqrt[3]{64y}$

63. $\sqrt[3]{128z^3} - \sqrt[3]{-16z^3}$

Show Solution
$6z\sqrt[3]{2}$

64. $\sqrt[5]{1,024c^{10}}$

Real-World Applications

65. A guy wire for a suspension bridge runs from the ground diagonally to the top of the closest pylon to make a triangle. We can use the Pythagorean Theorem to find the length of guy wire needed. The square of the distance between the wire on the ground and the pylon on the ground is

90,000 feet. The square of the height of the pylon is 160,000 feet. So the length of the guy wire can be found by evaluating $\sqrt{90,000+160,000}$. What is the length of the guy wire?

Show Solution
500 feet

66. A car accelerates at a rate of $6-\frac{\sqrt{4}}{\sqrt{t}}\ \mathrm{m/s^2}$ where t is the time in seconds after the car moves from rest. Simplify the expression.

Extensions

For the following exercises, simplify each expression.

67. $\frac{\sqrt{8}-\sqrt{16}}{4-\sqrt{2}}-2^{\frac{1}{2}}$

Show Solution
$\frac{-5\sqrt{2}-6}{7}$

68. $\frac{4^{\frac{3}{2}}-16^{\frac{3}{2}}}{8^{\frac{1}{3}}}$

69. $\frac{\sqrt{mn^3}}{a^2\sqrt{c^{-3}}}\cdot\frac{a^{-7}n^{-2}}{\sqrt{m^2c^4}}$

Show Solution
$\frac{\sqrt{mnc}}{a^9cmn}$

70. $\frac{a}{a-\sqrt{c}}$

71. $\frac{x\sqrt{64y}+4\sqrt{y}}{\sqrt{128y}}$

Show Solution

$\frac{2\sqrt{2}x+\sqrt{2}}{4}$

72. $\left(\frac{\sqrt{250x^2}}{\sqrt{100b^3}}\right)\left(\frac{7\sqrt{b}}{\sqrt{125x}}\right)$

73. $\sqrt{\frac{\sqrt[3]{64}+\sqrt[4]{256}}{\sqrt{64}+\sqrt{256}}}$

Show Solution

$\frac{\sqrt{3}}{3}$

Glossary

index

the number above the radical sign indicating the *n*th root

principal *n*th root

the number with the same sign as a that when raised to the *n*th power equals a

principal square root

the nonnegative square root of a number a that, when multiplied by itself, equals a

radical

the symbol used to indicate a root

radical expression

an expression containing a radical symbol

radicand

the number under the radical symbol

CHAPTER 1.3: DIVIDING POLYNOMIALS

Learning Objectives

In this section, you will:

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

Figure 1. Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m. [1] We can easily find the volume using elementary geometry.

1. National Park Service. "Lincoln Memorial Building Statistics." http://www.nps.gov/linc/historyculture/lincoln-memorial-building-statistics.htm. Accessed 4/3/2014

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73,800 \end{aligned}$$

So the volume is 73,800 cubic meters (m) .

Suppose we knew the volume, length, and width. We could divide to find the height.

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73{,}800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any, or all, of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$.

The length of the solid is given by $3x$;

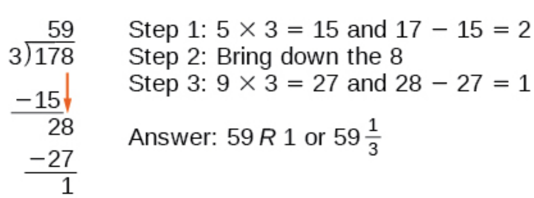
the width is given by $x - 2$.

To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

We are familiar with the long division algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

Long Division



Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor} \cdot \text{quotient}) + \text{remainder} \\ 178 &= (3 \cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder.

The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3-3x^2+4x+5$ by $x+2$
using the long division algorithm, it would look like this:

$$x+2\,\overline{)\,2x^3-3x^2+4x+5}$$

Set up the division problem.

$$\begin{array}{r l} & \,2x^2 \\ x+2 & \overline{)\,2x^3-3x^2+4x+5} \end{array}$$

$2x^3$ divided by x is $2x^2$.

$$\begin{array}{r l} & \,2x^2 \\ x+2 & \overline{)\,2x^3-3x^2+4x+5} \\ & \underline{-(2x^3+4x^2)} \\ & \qquad -7x^2+4x \end{array}$$

Multiply $x+2$ by $2x^2$.
Subtract.
Bring down the next term.

$$\begin{array}{r l} & \,2x^2-7x \\ x+2 & \overline{)\,2x^3-3x^2+4x+5} \\ & \underline{-(2x^3+4x^2)} \\ & \qquad -7x^2+4x \\ & \quad \underline{-(-7x^2-14x)} \\ & \qquad\qquad 18x+5 \end{array}$$

$-7x^2$ divided by x is $-7x$.
Multiply $x+2$ by $-7x$.
Subtract. Bring down the next term.

$$\begin{array}{r l} & \,2x^2-7x+18 \\ x+2 & \overline{)\,2x^3-3x^2+4x+5} \\ & \underline{-(2x^3+4x^2)} \\ & \qquad -7x^2+4x \\ & \quad \underline{-(-7x^2-14x)} \\ & \qquad\qquad 18x+5 \\ & \qquad\quad \underline{-(18x+36)} \\ & \qquad\qquad\quad -31 \end{array}$$

$18x$ divided by x is 18.
Multiply $x+2$ by 18.
Subtract.

We have found
$\frac{2x^3-3x^2+4x+5}{x+2}=2x^2-7x+18-\frac{31}{x+2}$

or
$\frac{2x^3-3x^2+4x+5}{x+2}=(x+2)\left(2x^2-7x+18\right)-31$

We can identify the dividend, the divisor, the quotient, and the remainder.

$$\underbrace{2x^3-3x^2+4x+5}_{\text{Dividend}}=\underbrace{(x+2)}_{\text{Divisor}}\underbrace{(2x^2-7x+18)}_{\text{Quotient}}+\underbrace{(-31)}_{\text{Remainder}}$$

Writing the result in this manner illustrates the Division Algorithm.

The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f\left(x\right)$ and a non-zero polynomial divisor $d\left(x\right)$ where the degree of $d\left(x\right)$ is less than or equal to the degree of $f\left(x\right)$, there exist unique polynomials $q\left(x\right)$ and $r\left(x\right)$ such that $f\left(x\right)=d\left(x\right)q\left(x\right)+r\left(x\right)$ $q\left(x\right)$ is the quotient and $r\left(x\right)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d\left(x\right)$. If $r\left(x\right)=0,$ then $d\left(x\right)$ divides evenly into $f\left(x\right)$. This means that, in this case, both
$d\left(x\right)$
and $q\left(x\right)$ are factors of $f\left(x\right)$.

How To

Given a polynomial and a binomial, use long division to divide the polynomial by the binomial.

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom binomial from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Using Long Division to Divide a Second-Degree Polynomial

Divide $5{x}^{2}+3x-2$ by $x+1.$

Show Solution

The quotient is $5x-2.$ The remainder is 0. We write the result as

$\frac{5x^2+3x-2}{x+1}=5x-2$

or

$$5x^2+3x-2=(x+1)(5x-2)$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3+11x^2-31x+15$ by $3x-2.$

Show Solution

$2x^2+5x-7$	
$3x-2\,\overline{)\,6x^3+11x^2-31x+15}$	$6x^3$ divided by $3x$ is $2x^2$.
$-(6x^3-4x^2)$	Multiply $3x-2$ by $2x^2$.
$15x^2-31x$	Subtract. Bring down the next term. $15x^2$ divided by $3x$ is $5x$.
$-(15x^2-10x)$	Multiply $3x-2$ by $5x$.
$-21x+15$	Subtract. Bring down the next term. $-21x$ divided by $3x$ is -7.
$-(-21x+14)$	Multiply $3x-2$ by -7.
1	Subtract. The remainder is 1.

There is a remainder of 1. We can express the result as:

$\frac{6x^3+11x^2-31x+15}{3x-2}=2x^2+5x-7+\frac{1}{3x-2}$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$(3x-2)\left(2x^2+5x-7\right)+1=6x^3+11x^2-31x+15$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

Try It

Divide $16x^3-12x^2+20x-3$ by $4x+5$.

Show Solution

$4x^2-8x+15-\frac{78}{4x+5}$

Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm.

The final form of the process looked like this:

$$
\begin{array}{r}
2x^2-x+18 \\
x+2\,\overline{)\,2x^3-3x^2+4x+5} \\
\underline{-(2x^3+4x^2)} \\
-7x^2+4x \\
\underline{-(-7x^2-14x)} \\
18x+5 \\
\underline{-(18x+36)} \\
-31
\end{array}
$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$\begin{array}{r|rrrr}
2 & 2 & -3 & 4 & 5 \\ \hline
 & -2 & -4 & & \\
 & & -7 & 14 & \\
 & & & 18 & -36 \\
 & & & & -31
\end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr}
-2 & 2 & -3 & 4 & 5 \\
 & & -4 & 14 & -36 \\ \hline
 & 2 & -7 & 18 & -31
\end{array}$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2 - 7x + 18$ and the remainder is -31. The process will be made more clear in (Figure).

Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k$ where k is a real number.

In synthetic division, only the coefficients are used in the division process.

How To

Given two polynomials, use synthetic division to divide.

1. Write k for the divisor.

2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by $k.$ Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by $k.$ Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3.$

Show Solution

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & & \\ \hline & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by $k.$

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by $k.$ Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x + 12.$ The remainder is 0. So $x - 3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x-3)\left(5x+12\right)+0=5x^2-3x-36$$

Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3+10x^2-6x-20$ by $x+2$.

Show Solution

The binomial divisor is $x+2$ so $k=-2$.

Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The result is $4x^2+2x-10$. The remainder is 0. Thus, $x+2$ is a factor of $4x^3+10x^2-6x-20$.

Analysis

The graph of the polynomial function $f\left(x\right)=4x^3+10x^2-6x-20$ in (Figure) shows a zero at $x=k=-2$. This confirms that $x+2$ is a factor of $4x^3+10x^2-6x-20$.

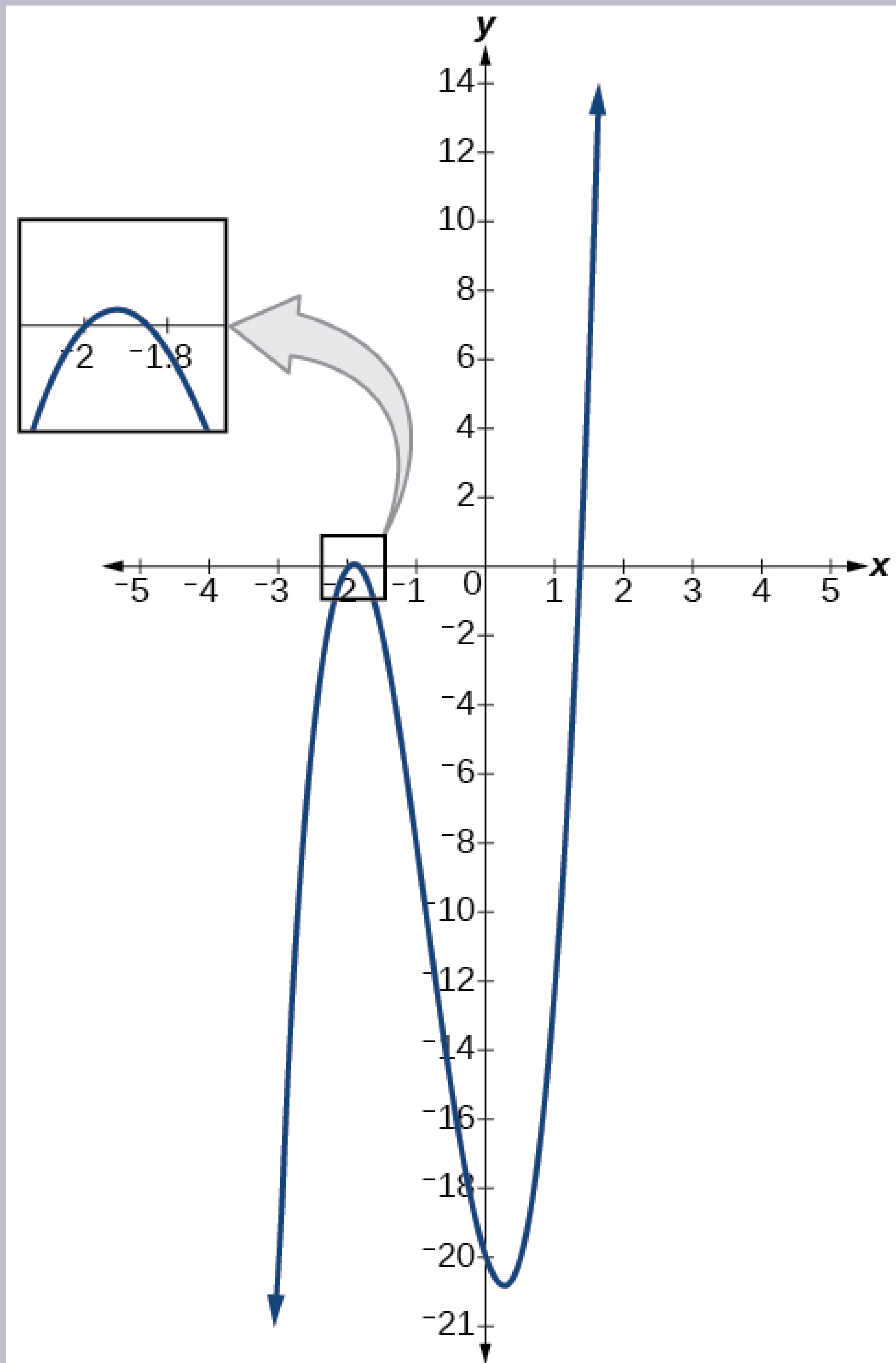


Figure 2.

Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Show Solution

Notice there is no *x*-term. We will use a zero as the coefficient for that term.

1	−9	10	7	0	−6
		−9	1	8	8
	−9	1	8	8	2

The result is $-9x^3 + x^2 + 8x + 8 + \frac{2}{x-1}$.

Try It

Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

Show Solution

$3x^3 - 3x^2 + 21x - 150 + \frac{1{,}090}{x+7}$

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$.
Find the height, h, of the solid.

Show Solution

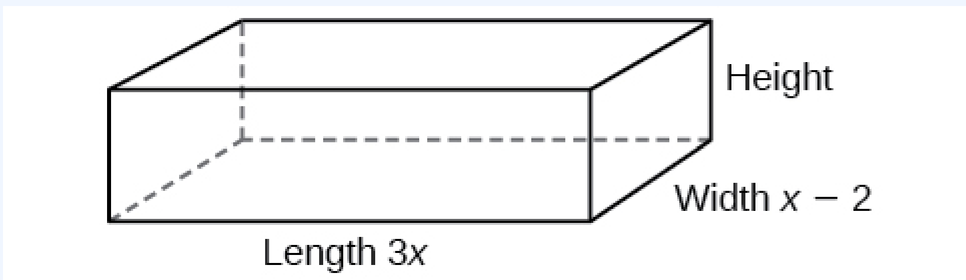


Figure 3.

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in (Figure).

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$\begin{aligned} V &= l \cdot w \cdot h \\ 3x^4 - 3x^3 - 33x^2 + 54x &= 3x \cdot (x - 2) \cdot h \end{aligned}$$

To solve for h, first divide both sides by $3x$.

$$\begin{aligned} \frac{3x \cdot (x-2) \cdot h}{3x} &= \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x} \\ (x - 2)\, h &= x^3 - x^2 - 11x + 18 \end{aligned}$$

Now solve for h using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2}$$

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

Try It

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$.
Find an expression for the length of the rectangle.

Show Solution
$3x^2 - 4x + 1$

Access these online resources for additional instruction and practice with polynomial division.

- Dividing a Trinomial by a Binomial Using Long Division
- Dividing a Polynomial by a Binomial Using Long Division
- Ex 2: Dividing a Polynomial by a Binomial Using Synthetic Division
- Ex 4: Dividing a Polynomial by a Binomial Using Synthetic Division

Key Equations

Division Algorithm	$f(x) = d(x)\,q(x) + r(x)$ where $q(x) \neq 0$

Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree. See (Figure) and (Figure).
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x - k$.

See (Figure), (Figure), and (Figure).

- Polynomial division can be used to solve application problems, including area and volume. See (Figure).

Section Exercises

Verbal

1. If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

Show Solution

The binomial is a factor of the polynomial.

2. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

Algebraic

For the following exercises, use long division to divide. Specify the quotient and the remainder.

3. $\left(x^2+5x-1\right)\div(x-1)$

Show Solution

$x+6+\frac{5}{x-1}\text{,quotient:}x+6\text{,remainder:}5$

4. $\left(2x^2-9x-5\right)\div(x-5)$

5. $\left(3x^2+23x+14\right)\div(x+7)$

Show Solution

$3x+2, \text{quotient: } 3x+2, \text{remainder: } 0$

6. $\left(4x^2-10x+6\right) \div (4x+2)$

7. $\left(6x^2-25x-25\right) \div (6x+5)$

Show Solution

$x-5, \text{quotient:} x-5, \text{remainder:} 0$

8. $\left(-x^2-1\right) \div (x+1)$

9. $\left(2x^2-3x+2\right) \div (x+2)$

Show Solution

$2x-7+\frac{16}{x+2}, \text{quotient:} 2x-7, \text{remainder:} 16$

10. $\left(x^3-126\right) \div (x-5)$

11. $\left(3x^2-5x+4\right) \div (3x+1)$

Show Solution

$x-2+\frac{6}{3x+1}, \text{quotient:} x-2, \text{remainder:} 6$

12. $\left(x^3-3x^2+5x-6\right) \div (x-2)$

13. $\left(2x^3+3x^2-4x+15\right) \div (x+3)$

Show Solution

$2x^2 - 3x + 5$,quotient:$2x^2 - 3x + 5$,remainder:0

For the following exercises, use synthetic division to find the quotient. Ensure the equation is in the form required by synthetic division. (Hint: divide the dividend and divisor by the coefficient of the linear term in the divisor.)

14. $\left(3x^3 - 2x^2 + x - 4\right) \div (x + 3)$

15. $\left(2x^3 - 6x^2 - 7x + 6\right) \div (x - 4)$

Show Solution

$2x^2 + 2x + 1 + \frac{10}{x-4}$

16. $\left(6x^3 - 10x^2 - 7x - 15\right) \div (x + 1)$

17. $\left(4x^3 - 12x^2 - 5x - 1\right) \div (2x + 1)$

Show Solution

$2x^2 - 7x + 1 - \frac{2}{2x+1}$

18. $\left(9x^3 - 9x^2 + 18x + 5\right) \div (3x - 1)$

19. $\left(3x^3 - 2x^2 + x - 4\right) \div (x + 3)$

Show Solution

$3x^2 - 11x + 34 - \frac{106}{x+3}$

20. $\left(-6x^3 + x^2 - 4\right) \div (2x - 3)$

21. $\left(2x^3 + 7x^2 - 13x - 3\right) \div (2x - 3)$

Show Solution
$x^2 + 5x + 1$

22. $\left(3x^3 - 5x^2 + 2x + 3\right) \div (x + 2)$

23. $\left(4x^3 - 5x^2 + 13\right) \div (x + 4)$

Show Solution
$4x^2 - 21x + 84 - \frac{323}{x+4}$

24. $\left(x^3 - 3x + 2\right) \div (x + 2)$

25. $\left(x^3 - 21x^2 + 147x - 343\right) \div (x - 7)$

Show Solution
$x^2 - 14x + 49$

26. $\left(x^3 - 15x^2 + 75x - 125\right) \div (x - 5)$

27. $\left(9x^3 - x + 2\right) \div (3x - 1)$

Show Solution
$3x^2 + x + \frac{2}{3x-1}$

28. $\left(6x^3 - x^2 + 5x + 2\right) \div (3x + 1)$

29. $\left(x^4 + x^3 - 3x^2 - 2x + 1\right) \div (x + 1)$

Show Solution
$x^3 - 3x + 1$

30. $\left(x^4 - 3x^2 + 1\right) \div (x - 1)$

31. $\left(x^4 + 2x^3 - 3x^2 + 2x + 6\right) \div (x + 3)$

Show Solution
$x^3 - x^2 + 2$

32. $\left(x^4 - 10x^3 + 37x^2 - 60x + 36\right) \div (x - 2)$

33. $\left(x^4 - 8x^3 + 24x^2 - 32x + 16\right) \div (x - 2)$

Show Solution
$x^3 - 6x^2 + 12x - 8$

34. $\left(x^4 + 5x^3 - 3x^2 - 13x + 10\right) \div (x + 5)$

35. $\left(x^4 - 12x^3 + 54x^2 - 108x + 81\right) \div (x - 3)$

Show Solution
$x^3 - 9x^2 + 27x - 27$

36. $\left(4x^4 - 2x^3 - 4x + 2\right) \div (2x - 1)$

37. $\left(4x^4 + 2x^3 - 4x^2 + 2x + 2\right) \div (2x + 1)$

Show Solution
$2x^3 - 2x + 2$

For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

38. $x - 2, 4x^3 - 3x^2 - 8x + 4$

39. $x - 2, 3x^4 - 6x^3 - 5x + 10$

Show Solution
Yes $(x - 2)\left(3x^3 - 5\right)$

40. $x + 3, -4x^3 + 5x^2 + 8$

41. $x - 2, 4x^4 - 15x^2 - 4$

Show Solution
Yes $(x - 2)\left(4x^3 + 8x^2 + x + 2\right)$

42. $x - \frac{1}{2}, 2x^4 - x^3 + 2x - 1$

43. $x + \frac{1}{3}, 3x^4 + x^3 - 3x + 1$

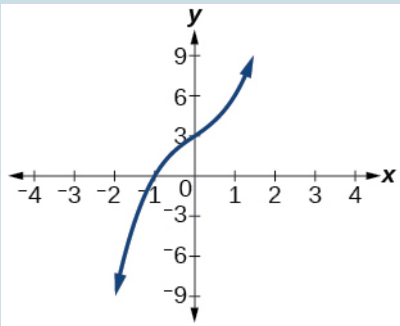
Show Solution

No

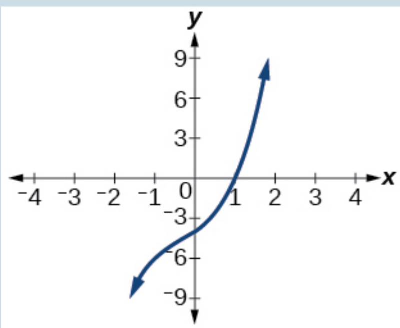
Graphical

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

44. Factor is $x^2 - x + 3$



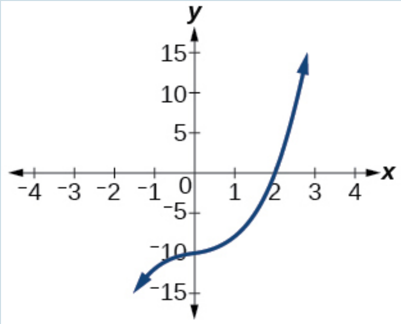
45. Factor is $\left(x^2 + 2x + 4\right)$



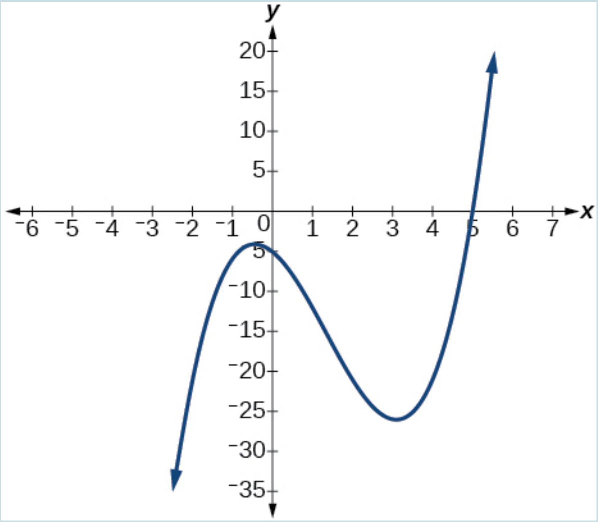
Show Solution

$(x - 1)\left(x^2 + 2x + 4\right)$

46. Factor is $x^2 + 2x + 5$



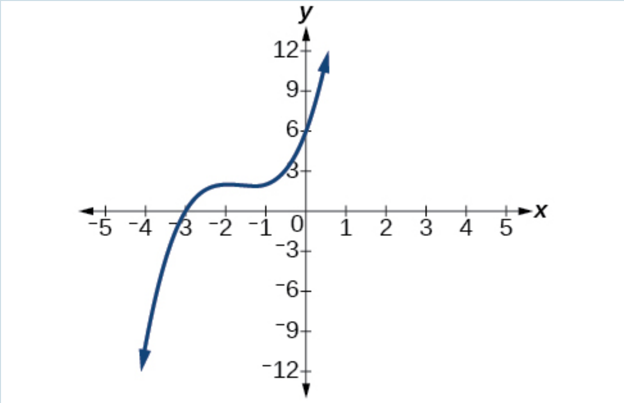
47. Factor is x^2+x+1



Show Solution

$\left(x-5\right)\left(x^2+x+1\right)$

48. Factor is x^2+2x+2



For the following exercises, use synthetic division to find the quotient and remainder.

49. $\frac{4x^3-33}{x-2}$

Show Solution

$\text{Quotient:}4x^2+8x+16,\text{remainder:}-1$

50. $\frac{2x^3+25}{x+3}$

51. $\frac{3x^3+2x-5}{x-1}$

Show Solution

$\text{Quotient:}3x^2+3x+5,\text{remainder:}0$

52. $\frac{-4x^3-x^2-12}{x+4}$

53. $\frac{x^4-22}{x+2}$

Show Solution

$\text{Quotient:}x^3-2x^2+4x-8,\text{remainder:}-6$

Technology

For the following exercises, use a calculator with CAS to answer the questions.

54. Consider $\frac{x^k-1}{x-1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

55. Consider $\frac{x^k+1}{x+1}$ for $k=1,3,5$. What do you expect the result to be if $k=7$?

Show Solution
$x^6-x^5+x^4-x^3+x^2-x+1$

56. Consider $\frac{x^4-k^4}{x-k}$ for $k=1,2,3$. What do you expect the result to be if $k=4$?

57. Consider $\frac{x^k}{x+1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

Show Solution
$x^3-x^2+x-1+\frac{1}{x+1}$

58. Consider $\frac{x^k}{x-1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

Extensions

For the following exercises, use synthetic division to determine the quotient involving a complex number.

59. $\frac{x+1}{x-i}$

Show Solution
$1+\frac{1+i}{x-i}$

60. $\frac{x^2+1}{x-i}$

61. $\frac{x+1}{x+i}$

Show Solution
$1+\frac{1-i}{x+i}$

62. $\frac{x^2+1}{x+i}$

63. $\frac{x^3+1}{x-i}$

Show Solution
$x^2-ix-1+\frac{1-i}{x-i}$

Real-World Applications

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

64. Length is $x+5$, area is $2x^2+9x-5$.

65. Length is $2x\ +\ 5$, area is $4x^3+10x^2+6x+15$

Show Solution
$2x^2+3$

66. Length is $3x-4$, area is $6x^4-8x^3+9x^2-9x-4$

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

67. Volume is $12x^3 + 20x^2 - 21x - 36$, length is $2x + 3$, width is $3x - 4$.

Show Solution
$2x + 3$

68. Volume is $18x^3 - 21x^2 - 40x + 48$, length is $3x - 4$, width is $3x - 4$.

69. Volume is $10x^3 + 27x^2 + 2x - 24$, length is $5x - 4$, width is $2x + 3$.

Show Solution
$x + 2$

70. Volume is $10x^3 + 30x^2 - 8x - 24$, length is 2, width is $x + 3$.

71. For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

72. Volume is $\pi\left(25x^3 - 65x^2 - 29x - 3\right)$, radius is $5x + 1$.

Show Solution
$x - 3$

73. Volume is $\pi\left(4x^3 + 12x^2 - 15x - 50\right)$, radius is $2x + 5$.

74. Volume is $\pi\left(3x^4 + 24x^3 + 46x^2 - 16x - 32\right)$, radius is $x + 4$.

Show Solution
$3x^2 - 2$

Glossary

Division Algorithm
: given a polynomial dividend $f\left(x\right)$ and a non-zero polynomial divisor $d\left(x\right)$ where the degree of $d\left(x\right)$ is less than or equal to the degree of $f\left(x\right)$, there exist unique polynomials $q\left(x\right)$ and $r\left(x\right)$ such that $f\left(x\right)=d\left(x\right)q\left(x\right)+r\left(x\right)$ where $q\left(x\right)$ is the quotient and $r\left(x\right)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d\left(x\right)$.

synthetic division
: a shortcut method that can be used to divide a polynomial by a binomial of the form $x-k$

CHAPTER 1.4: SOLVE EQUATIONS WITH FRACTION OR DECIMAL COEFFICIENTS

Learning Objectives

By the end of this section, you will be able to:

- Solve equations with fraction coefficients
- Solve equations with decimal coefficients

Solve Equations with Fraction Coefficients

Let's use the General Strategy for Solving Linear Equations introduced earlier to solve the equation $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$
To isolate the x term, subtract $\frac{1}{2}$ from both sides.	$\frac{1}{8}x + \frac{1}{2} - \frac{1}{2} = \frac{1}{4} - \frac{1}{2}$
Simplify the left side.	$\frac{1}{8}x = \frac{1}{4} - \frac{1}{2}$
Change the constants to equivalent fractions with the LCD.	$\frac{1}{8}x = \frac{1}{4} - \frac{2}{4}$
Subtract.	$\frac{1}{8}x = -\frac{1}{4}$
Multiply both sides by the reciprocal of $\frac{1}{8}$.	$\frac{8}{1} \cdot \frac{1}{8}x = \frac{8}{1}\left(-\frac{1}{4}\right)$
Simplify.	$x = -2$

This method worked fine, but many students don't feel very confident when they see all those fractions. So we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator of *all* the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but with no fractions. This process is called *clearing the equation of fractions*. Let's solve the same equation again, but this time use the method that clears the fractions.

EXAMPLE 1

Solve: $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

Solution

Find the least common denominator of *all* the fractions in the equation.	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4} \quad \text{LCD} = 8$
Multiply both sides of the equation by that LCD, 8. This clears the fractions.	$8\left(\frac{1}{8}x + \frac{1}{2}\right) = 8\left(\frac{1}{4}\right)$
Use the Distributive Property.	$8 \cdot \frac{1}{8}x + 8 \cdot \frac{1}{2} = 8 \cdot \frac{1}{4}$
Simplify — and notice, no more fractions!	$x + 4 = 2$
Solve using the General Strategy for Solving Linear Equations.	$x + 4 - 4 = 2 - 4$
Simplify.	$x = -2$
Check: Let $x = -2$	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$ $\frac{1}{8}(-2) + \frac{1}{2} \stackrel{?}{=} \frac{1}{4}$ $-\frac{2}{8} + \frac{1}{2} \stackrel{?}{=} \frac{1}{4}$ $-\frac{2}{8} + \frac{4}{8} \stackrel{?}{=} \frac{1}{4}$ $\frac{2}{4} \stackrel{?}{=} \frac{1}{4}$ $\frac{1}{4} = \frac{1}{4} \checkmark$

TRY IT 1.1

Solve: $\frac{1}{4}x+\frac{1}{2}=\frac{5}{8}$.

Show answer

$x=\frac{1}{2}$

TRY IT 1.2

Solve: $\frac{1}{6}y-\frac{1}{3}=\frac{1}{6}$.

Show answer

y = 3

Notice in (Figure) that once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve! We then used the General Strategy for Solving Linear Equations.

HOW TO: Solve Equations with Fraction Coefficients by Clearing the Fractions

1. Find the least common denominator of *all* the fractions in the equation.
2. Multiply both sides of the equation by that LCD. This clears the fractions.
3. Solve using the General Strategy for Solving Linear Equations.

EXAMPLE 2

Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$.

Solution

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the least common denominator of *all* the fractions in the equation.	$7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x \quad \text{LCD} = 12$
Multiply both sides of the equation by 12.	$12(7) = 12 \cdot \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$
Distribute.	$12(7) = 12 \cdot \frac{1}{2}x + 12 \cdot \frac{3}{4}x - 12 \cdot \frac{2}{3}x$
Simplify — and notice, no more fractions!	$84 = 6x + 9x - 8x$
Combine like terms.	$84 = 7x$
Divide by 7.	$\frac{84}{7} = \frac{7x}{7}$
Simplify.	$12 = x$
Check: Let $x = 12$.	$7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$ $7 \stackrel{?}{=} \frac{1}{2}(12) + \frac{3}{4}(12) - \frac{2}{3}(12)$ $7 \stackrel{?}{=} 6 + 9 - 8$ $7 = 7\,\checkmark$

TRY IT 2.1

Solve: $6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$.

Show answer

$v = 40$

TRY IT 2.2

Solve: $-1 = \frac{1}{2}u + \frac{1}{4}u - \frac{2}{3}u.$

Show answer

$u = -12$

In the next example, we'll have variables and fractions on both sides of the equation.

EXAMPLE 3

Solve: $x + \frac{1}{3} = \frac{1}{6}x - \frac{1}{2}.$

Solution

Find the LCD of all the fractions in the equation.	$x+\frac{1}{3}=\frac{1}{6}x-\frac{1}{2}$, LCD = 6
Multiply both sides by the LCD.	$6\left(x+\frac{1}{3}\right)=6\left(\frac{1}{6}x-\frac{1}{2}\right)$
Distribute.	$6\cdot x+6\cdot\frac{1}{3}=6\cdot\frac{1}{6}x-6\cdot\frac{1}{2}$
Simplify — no more fractions!	$6x+2=x-3$
Subtract x from both sides.	$6x-x+2=x-x-3$
Simplify.	$5x+2=-3$
Subtract 2 from both sides.	$5x+2-2=-3-2$
Simplify.	$5x=-5$
Divide by 5.	$\frac{5x}{5}=\frac{-5}{5}$
Simplify.	$x=-1$
Check: Substitute $x=-1$.	$x+\frac{1}{3}=\frac{1}{6}x-\frac{1}{2}$ $(-1)+\frac{1}{3}\stackrel{?}{=}\frac{1}{6}(-1)-\frac{1}{2}$ $(-1)+\frac{1}{3}\stackrel{?}{=}-\frac{1}{6}-\frac{1}{2}$ $-\frac{3}{3}+\frac{1}{3}\stackrel{?}{=}-\frac{1}{6}-\frac{3}{6}$ $-\frac{2}{3}\stackrel{?}{=}-\frac{4}{6}$ $-\frac{2}{3}=-\frac{2}{3}$ ✓

TRY IT 3.1

Solve: $a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}$.

Show answer

a = −2

TRY IT 3.2

Solve: $c+\frac{3}{4}=\frac{1}{2}c-\frac{1}{4}$.

Show answer

c = −2

In (Figure), we'll start by using the Distributive Property. This step will clear the fractions right away!

EXAMPLE 4

Solve: $1=\frac{1}{2}\left(4x+2\right)$.

Solution

	$1 = \frac{1}{2}(4x+2)$
Distribute.	$1 = \frac{1}{2} \cdot 4x + \frac{1}{2} \cdot 2$
Simplify. Now there are no fractions to clear!	$1 = 2x + 1$
Subtract 1 from both sides.	$1 - 1 = 2x + 1 - 1$
Simplify.	$0 = 2x$
Divide by 2.	$\frac{0}{2} = \frac{2x}{2}$
Simplify.	$0 = x$
Check: Let $x = 0$.	$1 = \frac{1}{2}(4x+2)$ $1 \stackrel{?}{=} \frac{1}{2}(4(0)+2)$ $1 \stackrel{?}{=} \frac{1}{2}(2)$ $1 \stackrel{?}{=} \frac{2}{2}$ $1 = 1$ ✓

TRY IT 4.1

Solve: $-11 = \frac{1}{2}(6p+2)$.

Show answer

$p = -4$

TRY IT 4.2

Solve: $8=\frac{1}{3}\left(9q+6\right)$.

Show answer

q = 2

Many times, there will still be fractions, even after distributing.

EXAMPLE 5

Solve: $\frac{1}{2}\left(y-5\right)=\frac{1}{4}\left(y-1\right)$.

Solution

	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2}\cdot y-\frac{1}{2}\cdot 5=\frac{1}{4}\cdot y-\frac{1}{4}\cdot 1$
Simplify.	$\frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$
Multiply by the LCD, 4.	$4\left(\frac{1}{2}y-\frac{5}{2}\right)=4\left(\frac{1}{4}y-\frac{1}{4}\right)$
Distribute.	$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.	$2y-10=y-1$
Collect the y terms to the left.	$2y-10-y=y-1-y$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$
Check: Substitute 9 for y.	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$ $\frac{1}{2}(9-5)\stackrel{?}{=}\frac{1}{4}(9-1)$ $\frac{1}{2}(4)\stackrel{?}{=}\frac{1}{4}(8)$ $2=2\checkmark$

TRY IT 5.1

Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$.

Show answer

$n=2$

TRY IT 5.2

Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$.

Show answer

m = −1

Solve Equations with Decimal Coefficients

Some equations have decimals in them. This kind of equation will occur when we solve problems dealing with money and percent. But decimals are really another way to represent fractions. For example, $0.3=\frac{3}{10}$ and $0.17=\frac{17}{100}$. So, when we have an equation with decimals, we can use the same process we used to clear fractions—multiply both sides of the equation by the least common denominator.

EXAMPLE 6

Solve: $0.8x-5=7$.

Solution

The only decimal in the equation is 0.8. Since $0.8=\frac{8}{10}$, the LCD is 10. We can multiply both sides by 10 to clear the decimal.

	$0.8x - 5 = 7$
Multiply both sides by the LCD.	$10(0.8x - 5) = 10(7)$
Distribute.	$10(0.8x) - 10(5) = 10(7)$
Multiply, and notice, no more decimals!	$8x - 50 = 70$
Add 50 to get all constants to the right.	$8x - 50 + 50 = 70 + 50$
Simplify.	$8x = 120$
Divide both sides by 8.	$\frac{8x}{8} = \frac{120}{8}$
Simplify.	$x = 15$
Check: Let $x = 15$.	$0.8(15) - 5 \stackrel{?}{=} 7$ $12 - 5 \stackrel{?}{=} 7$ $7 = 7$ ✓

TRY IT 6.1

Solve: $0.6x - 1 = 11$.

Show answer

$x = 20$

TRY IT 6.2

Solve: $1.2x - 3 = 9$.

Show answer

$x = 10$

EXAMPLE 7

Solve: $0.06x + 0.02 = 0.25x - 1.5$.

Solution

Look at the decimals and think of the equivalent fractions.

$0.06 = \frac{6}{100}, \quad 0.02 = \frac{2}{100}, \quad 0.25 = \frac{25}{100}, \quad 1.5 = 1\frac{5}{10}$

Notice, the LCD is 100.

By multiplying by the LCD we will clear the decimals.

	$0.06x + 0.02 = 0.25x - 1.5$
Multiply both sides by 100.	$100(0.06x + 0.02) = 100(0.25x - 1.5)$
Distribute.	$100(0.06x) + 100(0.02) = 100(0.25x) - 100(1.5)$
Multiply, and now no more decimals.	$6x + 2 = 25x - 150$
Collect the variables to the right.	$6x - 6x + 2 = 25x - 6x - 150$
Simplify.	$2 = 19x - 150$
Collect the constants to the left.	$2 + 150 = 19x - 150 + 150$
Simplify.	$152 = 19x$
Divide by 19.	$\frac{152}{19} = \frac{19x}{19}$
Simplify.	$8 = x$

Check: Let $x = 8$.

$0.06(8) + 0.02 = 0.25(8) - 1.5$

$0.48 + 0.02 = 2.00 - 1.5$

$0.50 = 0.50 \checkmark$

TRY IT 7.1

Solve: $0.14h + 0.12 = 0.35h - 2.4$.

Show answer

h = 12

TRY IT 7.2

Solve: $0.65k - 0.1 = 0.4k - 0.35$.

Show answer

k = −1

The next example uses an equation that is typical of the ones we will see in the money applications in the next chapter. Notice that we will distribute the decimal first before we clear all decimals in the equation.

EXAMPLE 8

Solve: $0.25x + 0.05\left(x + 3\right) = 2.85$.

Solution

	$0.25x + 0.05(x+3) = 2.85$
Distribute first.	$0.25x + 0.05x + 0.15 = 2.85$
Combine like terms.	$0.30x + 0.15 = 2.85$
To clear decimals, multiply by 100.	$100(0.30x + 0.15) = 100(2.85)$
Distribute.	$30x + 15 = 285$
Subtract 15 from both sides.	$30x + 15 - 15 = 285 - 15$
Simplify.	$30x = 270$
Divide by 30.	$\frac{30x}{30} = \frac{270}{30}$
Simplify.	$x = 9$
Check: Let $x = 9$.	$0.25x + 0.05(x+3) = 2.85$ $0.25(9) + 0.05(9+3) \stackrel{?}{=} 2.85$ $2.25 + 0.05(12) \stackrel{?}{=} 2.85$ $2.25 + 0.60 \stackrel{?}{=} 2.85$ $2.85 = 2.85$ ✓

TRY IT 8.1

Solve: $0.25n + 0.05\left(n+5\right) = 2.95$.

Show answer

$n = 9$

TRY IT 8.2

Solve: $0.10d + 0.05\left(d - 5\right) = 2.15$.

Show answer

d = 16

Key Concepts

- **Solve equations with fraction coefficients by clearing the fractions.**

    1. Find the least common denominator of *all* the fractions in the equation.
    2. Multiply both sides of the equation by that LCD. This clears the fractions.
    3. Solve using the General Strategy for Solving Linear Equations.

Practices Makes Perfect

Solve equations with fraction coefficients

In the following exercises, solve the equation by clearing the fractions.

1. $\frac{1}{4}x-\frac{1}{2}=-\frac{3}{4}$	2. $\frac{3}{4}x-\frac{1}{2}=\frac{1}{4}$
3. $\frac{5}{6}y-\frac{2}{3}=-\frac{3}{2}$	4. $\frac{5}{6}y-\frac{1}{3}=-\frac{7}{6}$
5. $\frac{1}{2}a+\frac{3}{8}=\frac{3}{4}$	6. $\frac{5}{8}b+\frac{1}{2}=-\frac{3}{4}$
7. $2=\frac{1}{3}x-\frac{1}{2}x+\frac{2}{3}x$	8. $2=\frac{3}{5}x-\frac{1}{3}x+\frac{2}{5}x$
9. $\frac{1}{4}m-\frac{4}{5}m+\frac{1}{2}m=-1$	10. $\frac{5}{6}n-\frac{1}{4}n-\frac{1}{2}n=-2$
11. $x+\frac{1}{2}=\frac{2}{3}x-\frac{1}{2}$	12. $x+\frac{3}{4}=\frac{1}{2}x-\frac{5}{4}$
13. $\frac{1}{3}w+\frac{5}{4}=w-\frac{1}{4}$	14. $\frac{3}{2}z+\frac{1}{3}=z-\frac{2}{3}$
15. $\frac{1}{2}x-\frac{1}{4}=\frac{1}{12}x+\frac{1}{6}$	16. $\frac{1}{2}a-\frac{1}{4}=\frac{1}{6}a+\frac{1}{12}$
17. $\frac{1}{3}b+\frac{1}{5}=\frac{2}{5}b-\frac{3}{5}$	18. $\frac{1}{3}x+\frac{2}{5}=\frac{1}{5}x-\frac{2}{5}$
19. $1=\frac{1}{6}(12x-6)$	20. $1=\frac{1}{5}(15x-10)$
21. $\frac{1}{4}(p-7)=\frac{1}{3}(p+5)$	22. $\frac{1}{5}(q+3)=\frac{1}{2}(q-3)$
23. $\frac{1}{2}(x+4)=\frac{3}{4}$	24. $\frac{1}{3}(x+5)=\frac{5}{6}$

Solve Equations with Decimal Coefficients

In the following exercises, solve the equation by clearing the decimals.

25. $0.6y+3=9$	26. $0.4y-4=2$
27. $3.6j-2=5.2$	28. $2.1k+3=7.2$
29. $0.4x+0.6=0.5x-1.2$	30. $0.7x+0.4=0.6x+2.4$
31. $0.23x+1.47=0.37x-1.05$	32. $0.48x+1.56=0.58x-0.64$
33. $0.9x-1.25=0.75x+1.75$	34. $1.2x-0.91=0.8x+2.29$
35. $0.05n+0.10(n+8)=2.15$	36. $0.05n+0.10(n+7)=3.55$
37. $0.10d+0.25(d+5)=4.05$	38. $0.10d+0.25(d+7)=5.25$
39. $0.05(q-5)+0.25q=3.05$	40. $0.05(q-8)+0.25q=4.10$

Everyday Math

Coins 41. Taylor has $\$2.00$ in dimes and pennies. The number of pennies is 2 more than the number of dimes. Solve the equation $0.10d + 0.01\left(d + 2\right) = 2$ for d, the number of dimes.	**Stamps 42.** Travis bought $\$9.45$ worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was 5 less than the number of 49-cent stamps. Solve the equation $0.49s + 0.21\left(s - 5\right) = 9.45$ for s, to find the number of 49-cent stamps Travis bought.

Writing Exercises

43. Explain how to find the least common denominator of $\frac{3}{8},\ \frac{1}{6},\ \text{and}\ \frac{2}{3}$.	44. If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?
45. If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?	46. In the equation $0.35x + 2.1 = 3.85$, what is the LCD? How do you know?

Answers

1. x = -1	3. y = -1	5. $a = \frac{3}{4}$
7. x = 4	9. m = 20	11. x = -3
13. $w = \frac{9}{4}$	15. x = 1	17. b = 12
19. x = 1	21. p = -41	23. $x = -\frac{5}{2}$
25. y = 10	27. j = 2	29. x = 18
31. x = 18	33. x = 20	35. n = 9
37. d = 8	39. q = 11	41 d = 18
43. Answers will vary.	45.Answers will vary.	

Attributions

CHAPTER 1.5: USE A GENERAL STRATEGY TO SOLVE LINEAR EQUATIONS

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using a general strategy
- Classify equations

Solve Equations Using the General Strategy

Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

Beginning by simplifying each side of the equation makes the remaining steps easier.

EXAMPLE 1. How to Solve Linear Equations Using the General Strategy

Solve: $-6\left(x+3\right)=24$.

Solution

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is simplified as much as possible.	$-6(x+3)=24$ $-6x-18=24$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do – all *x*'s are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 18 to each side. Simplify.	$-6x-18+18=24+18$ $-6x=42$
Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by −6. Simplify.	$\frac{-6x}{-6}=\frac{42}{-6}$ $x=-7$
Step 5. Check the solution.	Let $x=-7$ Simplify. Multiply.	**Check:** $-6(x+3)=24$ $-6(-7+3)\stackrel{?}{=}24$ $-6(-4)\stackrel{?}{=}24$ $24=24\checkmark$

TRY IT 1.1

Solve: $5\left(x+3\right)=35$.

Show answer

$x = 4$

TRY IT 1.2

Solve: $6\left(y - 4\right) = -18$.

Show answer

$y = 1$

General strategy for solving linear equations.

1. **Simplify each side of the equation as much as possible.**
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. **Collect all the variable terms on one side of the equation.**
   Use the Addition or Subtraction Property of Equality.
3. **Collect all the constant terms on the other side of the equation.**
   Use the Addition or Subtraction Property of Equality.
4. **Make the coefficient of the variable term to equal to 1.**
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. **Check the solution.** Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 2

Solve: $-(y+9)=8$.

Solution

	$-(y+9)=8$
Simplify each side of the equation as much as possible by distributing.	$-y-9=8$
The only y term is on the left side, so all variable terms are on the left side of the equation.	
Add 9 to both sides to get all constant terms on the right side of the equation.	$-y-9+9=8+9$
Simplify.	$-y=17$
Rewrite $-y$ as $-1y$.	$-1y=17$
Make the coefficient of the variable term to equal to 1 by dividing both sides by -1.	$\frac{-1y}{-1}=\frac{17}{-1}$
Simplify.	$y=-17$
Check: Let $y=-17$.	$-(y+9)=8$ $-(-17+9)\stackrel{?}{=}8$ $-(-8)\stackrel{?}{=}8$ $8=8\checkmark$

TRY IT 2.1

Solve: $-(y+8)=-2$.

Show answer

$y = -6$

TRY IT 2.2

Solve: $-(z+4) = -12$.

Show answer

$z = 8$

EXAMPLE 3

Solve: $5(a-3) + 5 = -10$.

Solution

	$5(a-3)+5=-10$
Simplify each side of the equation as much as possible.	
Distribute.	$5a-15+5=-10$
Combine like terms.	$5a-10=-10$
The only a term is on the left side, so all variable terms are on one side of the equation.	
Add 10 to both sides to get all constant terms on the other side of the equation.	$5a-10+10=-10+10$
Simplify.	$5a=0$
Make the coefficient of the variable term to equal to 1 by dividing both sides by 5.	$\frac{5a}{5}=\frac{0}{5}$
Simplify.	$a=0$
Check:	$5(a-3)+5=-10$
Let $a=0$.	$5(0-3)+5\stackrel{?}{=}-10$
	$5(-3)+5\stackrel{?}{=}-10$
	$-15+5\stackrel{?}{=}-10$
	$-10=-10$ ✓

TRY IT 3.1

Solve: $2\left(m-4\right)+3=-1$.

Show answer

$m=2$

TRY IT 3.2

Solve: $7\left(n-3\right)-8=-15$.

Show answer

$n=2$

EXAMPLE 4

Solve: $\frac{2}{3}\left(6m-3\right)=8-m$.

Solution

	$\frac{2}{3}(6m-3)=8-m$
Distribute.	$4m-2=8-m$
Add m to get the variables only to the left.	$4m+m-2=8-m+m$
Simplify.	$5m-2=8$
Add 2 to get constants only on the right.	$5m-2+2=8+2$
Simplify.	$5m=10$
Divide by 5.	$\frac{5m}{5}=\frac{10}{5}$
Simplify.	$m=2$
Check:	$\frac{2}{3}(6m-3)=8-m$
Let $m=2$.	$\frac{2}{3}(6\cdot 2-3)\stackrel{?}{=}8-2$
	$\frac{2}{3}(12-3)\stackrel{?}{=}6$
	$\frac{2}{3}(9)\stackrel{?}{=}6$
	$6=6\checkmark$

TRY IT 4.1

Solve: $\frac{1}{3}\left(6u+3\right)=7-u$.

Show answer

$u=2$

TRY IT 4.2

Solve: $\frac{2}{3}\left(9x-12\right)=8+2x.$

Show answer

$x=4$

EXAMPLE 5

Solve: $8-2\left(3y+5\right)=0.$

Solution

	$8 - 2(3y + 5) = 0$
Simplify—use the Distributive Property.	$8 - 6y - 10 = 0$
Combine like terms.	$-6y - 2 = 0$
Add 2 to both sides to collect constants on the right.	$-6y - 2 + 2 = 0 + 2$
Simplify.	$-6y = 2$
Divide both sides by -6.	$\frac{-6y}{-6} = \frac{2}{-6}$
Simplify.	$y = -\frac{1}{3}$
Check: Let $y = -\frac{1}{3}$.	$8 - 2(3y + 5) = 0$ $8 - 2\left[3\left(-\frac{1}{3}\right) + 5\right] = 0$ $8 - 2(-1 + 5) \stackrel{?}{=} 0$ $8 - 2(4) \stackrel{?}{=} 0$ $8 - 8 \stackrel{?}{=} 0$ $0 = 0$ ✓

TRY IT 5.1

Solve: $12 - 3\left(4j + 3\right) = -17$.

Show answer

$j = \frac{5}{3}$

TRY IT 5.2

Solve: $-6 - 8\left(k - 2\right) = -10$.

Show answer

$k = \frac{5}{2}$

EXAMPLE 6

Solve: $4\left(x - 1\right) - 2 = 5\left(2x + 3\right) + 6$.

Solution

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract $4x$ to get the variables only on the right side since $10>4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$
Divide by 6.	$\frac{-27}{6}=\frac{6x}{6}$
Simplify.	$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$
Let $x=-\frac{9}{2}$.	$4\left(-\frac{9}{2}-1\right)-2\stackrel{?}{=}5\left[2\left(-\frac{9}{2}\right)+3\right]+6$
	$4\left(-\frac{11}{2}\right)-2\stackrel{?}{=}5(-9+3)+6$
	$-22-2\stackrel{?}{=}5(-6)+6$
	$-24\stackrel{?}{=}-30+6$
	$-24=-24\ \checkmark$

TRY IT 6.1

Solve: $6(p-3)-7=5(4p+3)-12$.

Show answer

$p=-2$

TRY IT 6.2

Solve: $8\left(q+1\right)-5=3\left(2q-4\right)-1$.

Show answer

$q=-8$

EXAMPLE 7

Solve: $10\left[3-8\left(2s-5\right)\right]=15\left(40-5s\right)$.

Solution

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add $160s$ to get the s's to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$
Subtract 600 to get the constants to the left.	$430-600=600+85s-600$
Simplify.	$-170=85s$
Divide.	$\frac{-170}{85}=\frac{85s}{85}$
Simplify.	$-2=s$
Check:	$10[3-8(2s-5)]=15(40-5s)$
Substitute $s=-2$.	$10[3-8(2(-2)-5)]\stackrel{?}{=}15(40-5(-2))$
	$10[3-8(-4-5)]\stackrel{?}{=}15(40+10)$
	$10[3-8(-9)]\stackrel{?}{=}15(50)$
	$10[3+72]\stackrel{?}{=}750$
	$10[75]\stackrel{?}{=}750$
	$750=750$ ✓

TRY IT 7.1

Solve: $6\left[4-2\left(7y-1\right)\right]=8\left(13-8y\right)$.

Show answer

$y = -\frac{17}{5}$

TRY IT 7.2

Solve: $12\left[1-5\left(4z-1\right)\right]=3\left(24+11z\right)$.

Show answer

$z=0$

EXAMPLE 8

Solve: $0.36\left(100n+5\right)=0.6\left(30n+15\right)$.

Solution

	$0.36(100n+5)=0.6(30n+15)$
Distribute.	$36n+1.8=18n+9$
Subtract $18n$ to get the variables to the left.	$36n-18n+1.8=18n-18n+9$
Simplify.	$18n+1.8=9$
Subtract 1.8 to get the constants to the right.	$18n+1.8-1.8=9-1.8$
Simplify.	$18n=7.2$
Divide.	$\frac{18n}{18}=\frac{7.2}{18}$
Simplify.	$n=0.4$
Check:	$0.36(100n+5)=0.6(30n+15)$
Let $n=0.4$.	$0.36(100(0.4)+5)\stackrel{?}{=}0.6(30(0.4)+15)$
	$0.36(40+5)\stackrel{?}{=}0.6(12+15)$
	$0.36(45)\stackrel{?}{=}0.6(27)$
	$16.2=16.2\checkmark$

TRY IT 8.1

Solve: $0.55\left(100n+8\right)=0.6\left(85n+14\right)$.

Show answer

$n=1$

TRY IT 8.2

Solve: $0.15\left(40m-120\right)=0.5\left(60m+12\right)$.

Show answer

$m=-1$

Classify Equations

Consider the equation we solved at the start of the last section, $7x+8=-13$. The solution we found was $x=-3$. This means the equation $7x+8=-13$ is true when we replace the variable, *x*, with the value -3. We showed this when we checked the solution $x=-3$ and evaluated $7x+8=-13$ for $x=-3$.

$$7(-3)+8\stackrel{?}{=}-13$$
$$-21+8\stackrel{?}{=}-13$$
$$-13=-13\checkmark$$

If we evaluate $7x+8$ for a different value of *x*, the left side will not be -13.

The equation $7x+8=-13$ is true when we replace the variable, *x*, with the value -3, but not true when we replace *x* with any other value. Whether or not the equation $7x+8=-13$ is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are conditional equations.

Conditional equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

Now let's consider the equation $2y + 6 = 2\left(y + 3\right)$. Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for y.

	$2y + 6 = 2(y + 3)$
Distribute.	$2y + 6 = 2y + 6$
Subtract $2y$ to get the y's to one side.	$2y - 2y + 6 = 2y - 2y + 6$
Simplify—the y's are gone!	$6 = 6$

But $6 = 6$ is true.

This means that the equation $2y + 6 = 2\left(y + 3\right)$ is true for any value of y. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable like this is called an identity.

Identity

An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is every real number.

What happens when we solve the equation $5z = 5z - 1$?

	$5z = 5z - 1$
Subtract $5z$ to get the constant alone on the right.	$5z - 5z = 5z - 5z - 1$
Simplify—the z's are gone!	$0 \neq -1$

But $0 \neq 1$.

Solving the equation $5z = 5z - 1$ led to the false statement $0 = -1$. The equation $5z = 5z - 1$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a contradiction.

Contradiction

An equation that is false for all values of the variable is called a contradiction.

A contradiction has no solution.

EXAMPLE 9

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$6\,(2n-1)+3=2n-8+5\,(2n+1)$

Solution

	$6(2n-1)+3=2n-8+5(2n+1)$
Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract $12n$ to get the n's to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$
This is a true statement.	The equation is an identity. The solution is every real number.

TRY IT 9.1

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$4+9\left(3x-7\right)=-42x-13+23\left(3x-2\right)$$

Show answer

identity; all real numbers

TRY IT 9.2

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$8\left(1-3x\right)+15\left(2x+7\right)=2\left(x+50\right)+4\left(x+3\right)+1$$

Show answer

identity; all real numbers

EXAMPLE 10

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$$10+4\left(p-5\right)=0$$

Solution

	$10 + 4(p - 5) = 0$
Distribute.	$10 + 4p - 20 = 0$
Combine like terms.	$4p - 10 = 0$
Add 10 to both sides.	$4p - 10 + 10 = 0 + 10$
Simplify.	$4p = 10$
Divide.	$\frac{4p}{4} = \frac{10}{4}$
Simplify.	$p = \frac{5}{2}$
The equation is true when $p = \frac{5}{2}$.	This is a conditional equation. The solution is $p = \frac{5}{2}$.

TRY IT 10.1

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11\left(q+3\right)-5=19$

Show answer

conditional equation; $q=\frac{9}{11}$

TRY IT 10.2

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14\left(k-8\right)=95$

Show answer

conditional equation; $k=\frac{193}{14}$

EXAMPLE 11

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$5m+3\left(9+3m\right)=2\left(7m-11\right)$

Solution

	$5m+3(9+3m)=2(7m-11)$
Distribute.	$5m+27+9m=14m-22$
Combine like terms.	$14m+27=14m-22$
Subtract $14m$ from both sides.	$14m+27-14m=14m-22-14m$
Simplify.	$27\neq-22$
But $27\neq-22$.	The equation is a contradiction. It has no solution.

TRY IT 11.1

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$12c + 5\left(5 + 3c\right) = 3\left(9c - 4\right)$$

Show answer

contradiction; no solution

TRY IT 11.2

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$4\left(7d + 18\right) = 13\left(3d - 2\right) - 11d$$

Show answer

contradiction; no solution

Type of equation – Solution

Type of equation	**What happens when you solve it?**	**Solution**
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**

  1. Simplify each side of the equation as much as possible.
     Use the Distributive Property to remove any parentheses.
     Combine like terms.
  2. Collect all the variable terms on one side of the equation.
     Use the Addition or Subtraction Property of Equality.
  3. Collect all the constant terms on the other side of the equation.
     Use the Addition or Subtraction Property of Equality.
  4. Make the coefficient of the variable term to equal to 1.
     Use the Multiplication or Division Property of Equality.
     State the solution to the equation.
  5. Check the solution.
     Substitute the solution into the original equation.

Glossary

conditional equation
: An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction
: An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity
: An equation that is true for any value of the variable is called an identity. The solution of an identity is all real numbers.

Practice Makes Perfect

Solve Equations Using the General Strategy for Solving

Linear Equations

In the following exercises, solve each linear equation.

1. $21\left(y-5\right)=-42$
2. $15\left(y-9\right)=-60$
3. $-16\left(3n+4\right)=32$
4. $-9\left(2n+1\right)=36$
5. $5\left(8+6p\right)=0$
6. $8\left(22+11r\right)=0$
7. $-\left(t-19\right)=28$
8. $-\left(w-12\right)=30$
9. $21+2\left(m-4\right)=25$
10. $32+3\left(z+4\right)=41$
11. $-6+6\left(5-k\right)=15$
12. $51+5\left(4-q\right)=56$
13. $8\left(6t-5\right)-35=-27$
14. $2\left(9s-6\right)-62=16$
15. $-2\left(11-7x\right)+54=4$
16. $3\left(10-2x\right)+54=0$
17. $\frac{3}{5}\left(10x-5\right)=27$
18. $\frac{2}{3}\left(9c-3\right)=22$
19. $\frac{1}{4}\left(20d+12\right)=d+7$
20. $\frac{1}{5}\left(15c+10\right)=c+7$
21. $15-\left(3r+8\right)=28$
22. $18-\left(9r+7\right)=-16$
23. $-3-\left(m-1\right)=13$
24. $5-\left(n-1\right)=19$
25. $18-2\left(y-3\right)=32$
26. $11-4\left(y-8\right)=43$
27. $35-5\left(2w+8\right)=-10$
28. $24-8\left(3v+6\right)=0$
29. $-2\left(a-6\right)=4\left(a-3\right)$
30. $4\left(a-12\right)=3\left(a+5\right)$
31. $5\left(8-r\right)=-2\left(2r-16\right)$
32. $2\left(5-u\right)=-3\left(2u+6\right)$
33. $9\left(2m-3\right)-8=4m+7$
34. $3\left(4n-1\right)-2=8n+3$
35. $-15+4\left(2-5y\right)=-7\left(y-4\right)+4$
36. $12+2\left(5-3y\right)=-9\left(y-1\right)-2$
37. $5\left(x-4\right)-4x=14$
38. $8\left(x-4\right)-7x=14$
39. $-12+8\left(x-5\right)=-4+3\left(5x-2\right)$
40. $5+6\left(3s-5\right)=-3+2\left(8s-1\right)$
41. $7\left(2n-5\right)=8\left(4n-1\right)-9$
42. $4\left(u-1\right)-8=6\left(3u-2\right)-7$
43. $3\left(a-2\right)-\left(a+6\right)=4\left(a-1\right)$
44. $4\left(p-4\right)-\left(p+7\right)=5\left(p-3\right)$
45. $-\left(7m+4\right)-\left(2m-5\right)$ $=14-\left(5m-3\right)$
46. $-\left(9y+5\right)-\left(3y-7\right)=16-\left(4y-2\right)$
47. $5\left[9-2\left(6d-1\right)\right]=11\left(4-10d\right)-139$
48. $4\left[5-8\left(4c-3\right)\right]=12\left(1-13c\right)-8$
49. $3\left[-14+2\left(15k-6\right)\right]=8\left(3-5k\right)-24$
50. $3\left[-9+8\left(4h-3\right)\right]=2\left(5-12h\right)-19$
51. $10\left[5\left(n+1\right)+4\left(n-1\right)\right]$ $=11\left[7\left(5+n\right)-\left(25-3n\right)\right]$
52. $5\left[2\left(m+4\right)+8\left(m-7\right)\right]$ $=2\left[3\left(5+m\right)-\left(21-3m\right)\right]$
53. $4\left(2.5v-0.6\right)=7.6$
54. $5\left(1.2u-4.8\right)=-12$

55. $0.2\,(p-6)=0.4\,(p+14)$

56. $0.25\,(q-6)=0.1\,(q+18)$

57. $0.5\,(16m+34)=-15$

58. $0.2\,(30n+50)=28$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

59. $15y+32=2\,(10y-7)-5y+46$

60. $23z+19=3\,(5z-9)+8z+46$

61. $9\,(a-4)+3\,(2a+5)=7\,(3a-4)-6a+7$

62. $5\,(b-9)+4\,(3b+9)=6\,(4b-5)-7b+21$

63. $24\,(3d-4)+100=52$

64. $18\,(5j-1)+29=47$

65. $30\,(2n-1)=5\,(10n+8)$

66. $22\,(3m-4)=8\,(2m+9)$

67. $18u-51=9\,(4u+5)-6\,(3u-10)$

68. $7v+42=11\,(3v+8)-2\,(13v-1)$

69. $5\,(p+4)+8\,(2p-1)=9\,(3p-5)-6\,(p-2)$

70. $3\,(6q-9)+7\,(q+4)=5\,(6q+8)-5\,(q+1)$

71. $9\,(4k-7)=11\,(3k+1)+4$

72. $12\,(6h-1)=8\,(8h+5)-4$

73. $60\,(2x-1)=15\,(8x+5)$

74. $45\,(3y-2)=9\,(15y-6)$

75. $36\,(4m+5)=12\,(12m+15)$

76. $16\,(6n+15)=48\,(2n+5)$

77. $11\,(8c+5)-8c=2\,(40c+25)+5$

78. $9\,(14d+9)+4d=13\,(10d+6)+3$

Everyday Math

79. **Coins.** Rhonda has \$1.90 in nickels and dimes. The number of dimes is one less than twice the number of nickels. Find the number of nickels, n, by solving the equation $0.05n+0.10\,(2n-1)=1.90$.

80. **Fencing.** Micah has 44 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, L, by solving the equation $2L+2\,(L-2.5)=44$.

Writing Exercises

81. Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.	82. Using your own words, list the steps in the general strategy for solving linear equations.
83. Solve the equation $\frac{1}{4}\left(8x+20\right)=3x-4$ explaining all the steps of your solution as in the examples in this section.	84. What is the first step you take when solving the equation $3-7\left(y-4\right)=38$? Why is this your first step?

Answers

1. $y=3$	3. $n=-2$	5. $p=-\frac{4}{3}$
7. $t=-9$	9. $m=6$	11. $k=\frac{3}{2}$
13. $t=1$	15. $x=-2$	17. $x=5$
19. $d=1$	21. $r=-7$	23. $m=-15$
25. $y=-4$	27. $w=\frac{1}{2}$	29. $a=4$
31. $r=8$	33. $m=3$	35. $y=-3$
37. $x=34$	39. $x=-6$	41. $n=-1$
43. $a=-4$	45. $m=-4$	47. $d=-3$
49. $k=\frac{3}{5}$	51. $n=-5$	53. $v=1$
55. $p=-34$	57. $m=-4$	59. identity; all real numbers
61. identity; all real numbers	63. conditional equation; $d=\frac{2}{3}$	65. conditional equation; $n=7$
67. contradiction; no solution	69. contradiction; no solution	71. conditional equation; $k=26$
73. contradiction; no solution	75. identity; all real numbers	77. identity; all real numbers
79. 8 nickels	81. Answers will vary.	83. Answers will vary.

Attributions

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CHAPTER 1.6: SOLVE A FORMULA FOR A SPECIFIC VARIABLE

Learning Objectives

By the end of this section, you will be able to:

- Use the Distance, Rate, and Time formula
- Solve a formula for a specific variable

Use the Distance, Rate, and Time Formula

One formula you will use often in algebra and in everyday life is the formula for distance traveled by an object moving at a constant rate. Rate is an equivalent word for "speed." The basic idea of rate may already familiar to you. Do you know what distance you travel if you drive at a steady rate of 60 miles per hour for 2 hours? (This might happen if you use your car's cruise control while driving on the highway.) If you said 120 miles, you already know how to use this formula!

Distance, Rate, and Time

For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula:

$$d = rt \qquad \text{where} \qquad \begin{aligned} d &= \text{distance} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

We will use the Strategy for Solving Applications that we used earlier in this chapter. When our problem requires a formula, we change Step 4. In place of writing a sentence, we write the appropriate formula. We write the revised steps here for reference.

HOW TO: Solve an application (with a formula).

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

You may want to create a mini-chart to summarize the information in the problem. See the chart in this first example.

EXAMPLE 1

Jamal rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. What distance has he traveled?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	distance traveled
Step 3. Name. Choose a variable to represent it.	Let *d* = distance.
Step 4. Translate: Write the appropriate formula.	$d = rt$
	$d = ?$ $r = 12$ mph $t = 3\frac{1}{2}$ hours
Substitute in the given information.	$d = 12 \cdot 3\frac{1}{2}$
Step 5. Solve the equation.	$d = 42$ miles
Step 6. Check	
Does 42 miles make sense?	
Jamal rides:	
	12 miles in 1 hour, 24 miles in 2 hours, 36 miles in 3 hours, 42 miles in $3\frac{1}{2}$ hours is reasonable 48 miles in 4 hours.
Step 7. Answer the question with a complete sentence.	Jamal rode 42 miles.

TRY IT 1.1

Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Show answer

330 miles

TRY IT 1.2

Trinh walked for $2\frac{1}{3}$ hours at 3 miles per hour. How far did she walk?

Show answer

7 miles

EXAMPLE 2

Rey is planning to drive from his house in Saskatoon to visit his grandmother in Winnipeg, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	How many hours (time)
Step 3. Name. Choose a variable to represent it.	Let t = time.
	$d = 520$ miles $r = 65$ mph $t = ?$ hours d = 600 km r = 75 km/h t = ? hours
Step 4. Translate. Write the appropriate formula.	$d = rt$
Substitute in the given information.	$520 = 65t$
Step 5. Solve the equation.	$t = 8$
Step 6. Check. Substitute the numbers into the formula and make sure the result is a true statement.	$\begin{aligned} d &= rt \\ 520 &\stackrel{?}{=} 65 \cdot 8 \\ 520 &= 520 \end{aligned}$
Step 7. Answer the question with a complete sentence. Rey's trip will take 8 hours.	

TRY IT 2.1

Lee wants to drive from Kamloops to his brother's apartment in Banff, a distance of 495 km. If he drives at a steady rate of 90 km/h, how many hours will the trip take?

Show answer

5 1/2 hours

TRY IT 2.2

Yesenia is 168 km from Toronto. If she needs to be in Toronto in 2 hours, at what rate does she need to drive?

Show answer
84 km/h

Solve a Formula for a Specific Variable

You are probably familiar with some geometry formulas. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be familiar with formulas and be able to manipulate them easily.

In (Example 1) and (Example 2), we used the formula $d = rt$. This formula gives the value of d, distance, when you substitute in the values of $r \text{ and } t$, the rate and time. But in (Example 2), we had to find the value of t. We substituted in values of $d \text{ and } r$ and then used algebra to solve for t. If you had to do this often, you might wonder why there is not a formula that gives the value of t when you substitute in the values of $d \text{ and } r$. We can make a formula like this by solving the formula $d = rt$ for t.

To solve a formula for a specific variable means to isolate that variable on one side of the equals sign with a coefficient of 1. All other variables and constants are on the other side of the equals sign. To see how to solve a formula for a specific variable, we will start with the distance, rate and time formula.

EXAMPLE 3

Solve the formula $d = rt$ for t:

a. when $d = 520$ and $r = 65$
b. in general

Solution

We will write the solutions side-by-side to demonstrate that solving a formula in general uses the same steps as when we have numbers to substitute.

a) when $d = 520$ and $r = 65$		b) in general	
Write the formula.	$d = rt$	Write the formula.	$d = rt$
Substitute.	$520 = 65t$		
Divide, to isolate t.	$\frac{520}{65} = \frac{65t}{65}$	Divide, to isolate t.	$\frac{d}{r} = \frac{rt}{r}$
Simplify.	$8 = t$	Simplify.	$\frac{d}{r} = t$

We say the formula $t = \frac{d}{r}$ is solved for t.

TRY IT 3.1

Solve the formula $d = rt$ for r:

a) when $d = 180 \text{ and } t = 4$ b) in general

Show answer

a) $r = 45$ b) $r = \frac{d}{t}$

TRY IT 3.2

Solve the formula $d = rt$ for r:

a) when $d = 780 \text{ and } t = 12$ b) in general

Show answer

a) $r = 65$ b) $r = \frac{d}{t}$

EXAMPLE 4

Solve the formula $A = \frac{1}{2}bh$ for h:

a) when $A = 90$ and $b = 15$ b) in general

Solution

a) when $A = 90$ and $b = 15$		b) in general	
Write the formula.	$A = \frac{1}{2}bh$	Write the formula.	$A = \frac{1}{2}bh$
Substitute.	$90 = \frac{1}{2} \cdot 15 \cdot h$		
Clear the fractions.	$2 \cdot 90 = 2 \cdot \frac{1}{2}15h$	Clear the fractions.	$2 \cdot A = 2 \cdot \frac{1}{2}bh$
Simplify.	$180 = 15h$	Simplify.	$2A = bh$
Solve for h.	$12 = h$	Solve for h.	$\frac{2A}{b} = h$

We can now find the height of a triangle, if we know the area and the base, by using the formula $h = \frac{2A}{b}$.

TRY IT 4.1

Use the formula $A = \frac{1}{2}bh$ to solve for h:

a) when $A = 170$ and $b = 17$ b) in general

Show answer

a) $h = 20$ b) $h = \frac{2A}{b}$

TRY IT 4.2

Use the formula $A = \frac{1}{2}bh$ to solve for b:

a) when $A = 62$ and $h = 31$ b) in general

Show answer

a) $b = 4$ b) $b = \frac{2A}{h}$

The formula $I = Prt$ is used to calculate simple interest, *I*, for a principal, *P*, invested at rate, *r*, for *t* years.

EXAMPLE 5

Solve the formula $I = Prt$ to find the principal, P:

a) when $I = \$5,600$, $r = 4\%$, $t = 7\ years$ b) in general

Solution

a) $I = \$5,600, r = 4\%, t = 7 years$		b) in general	
Write the formula.	$I = Prt$	Write the formula.	$I = Prt$
Substitute.	$5600 = P(0.04)(7)$		
Simplify.	$5600 = P(0.28)$	Simplify.	$I = P(rt)$
Divide, to isolate P.	$\frac{5600}{0.28} = \frac{P(0.28)}{0.28}$	Divide, to isolate P.	$\frac{I}{rt} = \frac{P(rt)}{rt}$
Simplify.	$20,000 = P$	Simplify.	$\frac{I}{rt} = P$
The principal is	$20,000		$P = \frac{I}{rt}$

TRY IT 5.1

Use the formula $I = Prt$ to find the principal, P:

a) when $I = \$2,160, r = 6\%, t = 3\ years$ b) in general

Show answer

a) $12,000   b) $P = \frac{I}{rt}$

TRY IT 5.2

Use the formula $I = Prt$ to find the principal, P:

a) when $I = \$5,400, r = 12\%, t = 5\ years$ b) in general

Show answer

a) $9,000 b) $P = \frac{I}{rt}$

Later in this class, and in future algebra classes, you'll encounter equations that relate two variables, usually *x* and *y*. You might be given an equation that is solved for *y* and need to solve it for *x*, or vice versa. In the following example, we're given an equation with both *x* and *y* on the same side and we'll solve it for *y*.

EXAMPLE 6

Solve the formula $3x + 2y = 18$ for *y*.

a) when $x = 4$ b) in general

Solution

a) when $x = 4$		b) in general	
	$3x + 2y = 18$		$3x + 2y = 18$
Substitute.	$3(4) + 2y = 18$		
Subtract to isolate the y-term.	$12 - 12 + 2y = 18 - 12$	Subtract to isolate the y-term.	$3x - 3x + 2y = 18 - 3x$
Divide.	$\frac{2y}{2} = \frac{6}{2}$	Divide.	$\frac{2y}{2} = \frac{18}{2} - \frac{3x}{2}$
Simplify.	$y = 3$	Simplify.	$y = -\frac{3x}{2} + 9$

TRY IT 6.1

Solve the formula $3x + 4y = 10$ for y:

a) when $x = \frac{14}{3}$ b) in general

Show answer

a)$y = 1$ b)$y = \frac{10-3x}{4}$

TY IT 6.2

Solve the formula $5x + 2y = 18$ for y:

a) when $x = 4$ b) in general

Show answer

a)$y = -1$ b)$y = \frac{18-5x}{2}$

Now we will solve a formula in general without using numbers as a guide.

EXAMPLE 7

Solve the formula $P = a + b + c$ for a.

Solution

We will isolate a on one side of the equation.	$P = a + b + c$
Both b and c are added to a, so we subtract them from both sides of the equation.	$P - b - c = a + b + c - b - c$
Simplify.	$P - b - c = a$ $a = P - b - c$

TRY IT 7.1

Solve the formula $P = a + b + c$ for *b*.

Show answer

$b = P - a - c$

TRY IT 7.2

Solve the formula $P = a + b + c$ for *c*.

Show answer

$c = P - a - b$

EXAMPLE 8

Solve the formula $6x + 5y = 13$ for *y*.

Solution

	$6x + 5y = 13$
Subtract $6x$ from both sides to isolate the term with y.	$6x - 6x + 5y = 13 - 6x$
Simplify.	$5y = 13 - 6x$
Divide by 5 to make the coefficient 1.	$\frac{5y}{5} = \frac{13 - 6x}{5}$
Simplify.	$y = \frac{13 - 6x}{5}$

The fraction is simplified. We cannot divide $13 - 6x$ by 5

TRY IT 8.1

Solve the formula $4x + 7y = 9$ for *y*.

Show answer

$y = \frac{9-4x}{7}$

TRY IT 8.2

Solve the formula $5x + 8y = 1$ for *y*.

Show answer

$y = \frac{1-5x}{8}$

Key Concepts

- **To Solve an Application (with a formula)**

  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

- **Distance, Rate and Time**
  For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula: $d = rt$ where d = distance, r = rate, t = time.
- **To solve a formula for a specific variable** means to get that variable by itself with a coefficient of 1 on one side of the equation and all other variables and constants on the other side.

Practice Makes Perfect

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

1. Socorro drove for $4\frac{5}{6}$ hours at 60 miles per hour. How much distance did she travel?	2. Steve drove for $8\frac{1}{2}$ hours at 72 miles per hour. How much distance did he travel?
3. Francie rode her bike for $2\frac{1}{2}$ hours at 12 miles per hour. How far did she ride?	4. Yuki walked for $1\frac{3}{4}$ hours at 4 miles per hour. How far did she walk?
5. Marta is taking the bus from Abbotsford to Cranbrook. The distance is 774 km and the bus travels at a steady rate of 86 miles per hour. How long will the bus ride be?	6. Connor wants to drive from Vancouver to the Nakusp, a distance of 630 km. If he drives at a steady rate of 90 km/h, how many hours will the trip take?
7. Kareem wants to ride his bike from Golden, BC to Banff, AB. The distance is 140 km. If he rides at a steady rate of 20 km/h, how many hours will the trip take?	8. Aurelia is driving from Calgary to Edmonton at a rate of 85 km/h. The distance is 300 km. To the nearest tenth of an hour, how long will the trip take?
9. Alejandra is driving to Prince George, 450 km away. If she wants to be there in 6 hours, at what rate does she need to drive?	10. Javier is driving to Vernon, 240 km away. If he needs to be in Vernon in 3 hours, at what rate does he need to drive?
11. Philip got a ride with a friend from Calgary to Kelowna, a distance of 890 km. If the trip took 10 hours, how fast was the friend driving?	12. Aisha took the train from Spokane to Seattle. The distance is 280 miles and the trip took 3.5 hours. What was the speed of the train?

Solve a Formula for a Specific Variable

In the following exercises, use the formula $d = rt$.

13. Solve for t
a) when $d = 240 \text{ and } r = 60$
b) in general

14. Solve for t
a) when $d = 350$ and $r = 70$
b) in general

15. Solve for t
a) when $d = 175 \text{ and } r = 50$
b) in general

16. Solve for t
a) when $d = 510 \text{ and } r = 60$
b) in general

17. Solve for r
a) when $d = 420 \text{ and } t = 6$
b) in general

18. Solve for r
a) when $d = 204 \text{ and } t = 3$
b) in general

19. Solve for r
a) when $d = 180 \text{ and } t = 4.5$
b) in general

20. Solve for r
a) when $d = 160 \text{ and } t = 2.5$
b) in general

In the following exercises, use the formula $A = \frac{1}{2}bh$.

21. Solve for h
a) when $A = 176 \text{ and } b = 22$
b) in general

22. Solve for b
a) when $A = 126 \text{ and } h = 18$
b) in general

23. Solve for the principal, P for
a) $I = \$5,480$, $r = 4\%$, $t = 7$ years
b) in general

24. Solve for b
a) when $A = 65 \text{ and } h = 13$
b) in general

In the following exercises, use the formula I = Prt.

25. Solve for the time, t for
a) $I = \$2,376$, $P = \$9,000$, $r = 4.4\%$
b) in general

26. Solve for the principal, P for
a) $I = \$3,950$, $r = 6\%$, $t = 5$ years
b) in general

27. Solve the formula $2x + 3y = 12$ for y
a) when $x = 3$
b) in general

28. Solve for the time, t for
a) $I = \$624$, $P = \$6,000$, $r = 5.2\%$
b) in general

In the following exercises, solve.

29. Solve the formula $3x - y = 7$ for y
a) when $x = -2$
b) in general

30. Solve the formula $5x + 2y = 10$ for y
a) when $x = 4$
b) in general

31. Solve $a + b = 90$ for b.

32. Solve the formula $4x + y = 5$ for y
a) when $x = -3$
b) in general

33. Solve $180 = a + b + c$ for a.

34. Solve $a + b = 90$ for a.

35. Solve the formula $8x + y = 15$ for y.

36. Solve $180 = a + b + c$ for c.

37. Solve the formula $-4x + y = -6$ for y.

38. Solve the formula $9x + y = 13$ for y.

39. Solve the formula $4x + 3y = 7$ for y.

40. Solve the formula $-5x + y = -1$ for y.

41. Solve the formula $x - y = -4$ for y.

42. Solve the formula $3x + 2y = 11$ for y.

43. Solve the formula $P = 2L + 2W$ for L.

44. Solve the formula $x - y = -3$ for y.

45. Solve the formula $C = \pi d$ for d.

46. Solve the formula $P = 2L + 2W$ for W.

47. Solve the formula $V = LWH$ for L.

48. Solve the formula $C = \pi d$ for π.

49. Solve the formula $V = LWH$ for H.

Everyday Math

50. **Converting temperature.** Yon was visiting the United States and he saw that the temperature in Seattle one day was 50^{o} Fahrenheit. Solve for C in the formula $F = \frac{9}{5}C + 32$ to find the Celsius temperature.

51. **Converting temperature.** While on a tour in Greece, Tatyana saw that the temperature was 40^{o} Celsius. Solve for F in the formula $C = \frac{5}{9}(F - 32)$ to find the Fahrenheit temperature.

Writing Exercises

52. Solve the equation $5x - 2y = 10$ for x
a) when $y = 10$
b) in general
c) Which solution is easier for you, a) or b)? Why?

53. Solve the equation $2x + 3y = 6$ for y
a) when $x = -3$
b) in general
c) Which solution is easier for you, a) or b)? Why?

Answers

1. 290 miles	3. 30 miles	5. 9 hours.
7. 75 km/h	9. 3.5 hours	11. 7 hours
13. 7	15. 89 km/h	17. a) $t = 4$ b) $t = \frac{d}{r}$
19. a) $t = 3.5$ b) $t = \frac{d}{r}$	21. a) $r = 70$ b) $r = \frac{d}{t}$	23. a) $r = 40$ b) $r = \frac{d}{t}$
25. a) $h = 16$ b) $h = \frac{2A}{b}$	27. a) $b = 10$ b) $b = \frac{2A}{h}$	29. a) $P = \$13,166.67$ b) $P = \frac{I}{rt}$
31. a) $t = 2$ years b) $t = \frac{I}{\mathrm{Pr}}$	33. a) $y = -5$ b) $y = \frac{10-5x}{2}$	35. a) $y = 17$ b) $y = 5 - 4x$
37. $a = 90 - b$	39. $c = 180 - a - b$	41. $y = 13 - 9x$
43. $y = -1 + 5x$	45. $y = \frac{11-3x}{4}$	47. $y = 3 + x$
49. $W = \frac{P-2L}{2}$	51. $\pi = \frac{C}{d}$	53. $H = \frac{V}{LW}$
55. 10°C	57. Answers will vary.	

Attributions

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CHAPTER 1.7: USE A PROBLEM-SOLVING STRATEGY

Learning Objectives

By the end of this section, you will be able to:

- Approach word problems with a positive attitude
- Use a problem-solving strategy for word problems
- Solve number problems

Approach Word Problems with a Positive Attitude

"If you think you can… or think you can't… you're right."—Henry Ford

The world is full of word problems! Will my income qualify me to rent that apartment? How much punch do I need to make for the party? What size diamond can I afford to buy my girlfriend? Should I fly or drive to my family reunion?

How much money do I need to fill the car with gas? How much tip should I leave at a restaurant? How many socks should I pack for vacation? What size turkey do I need to buy for Thanksgiving dinner, and then what time do I need to put it in the oven? If my sister and I buy our mother a present, how much does each of us pay?

Now that we can solve equations, we are ready to apply our new skills to word problems. Do you know anyone who has had negative experiences in the past with word problems? Have you ever had thoughts like the student below?

Negative thoughts can be barriers to success.

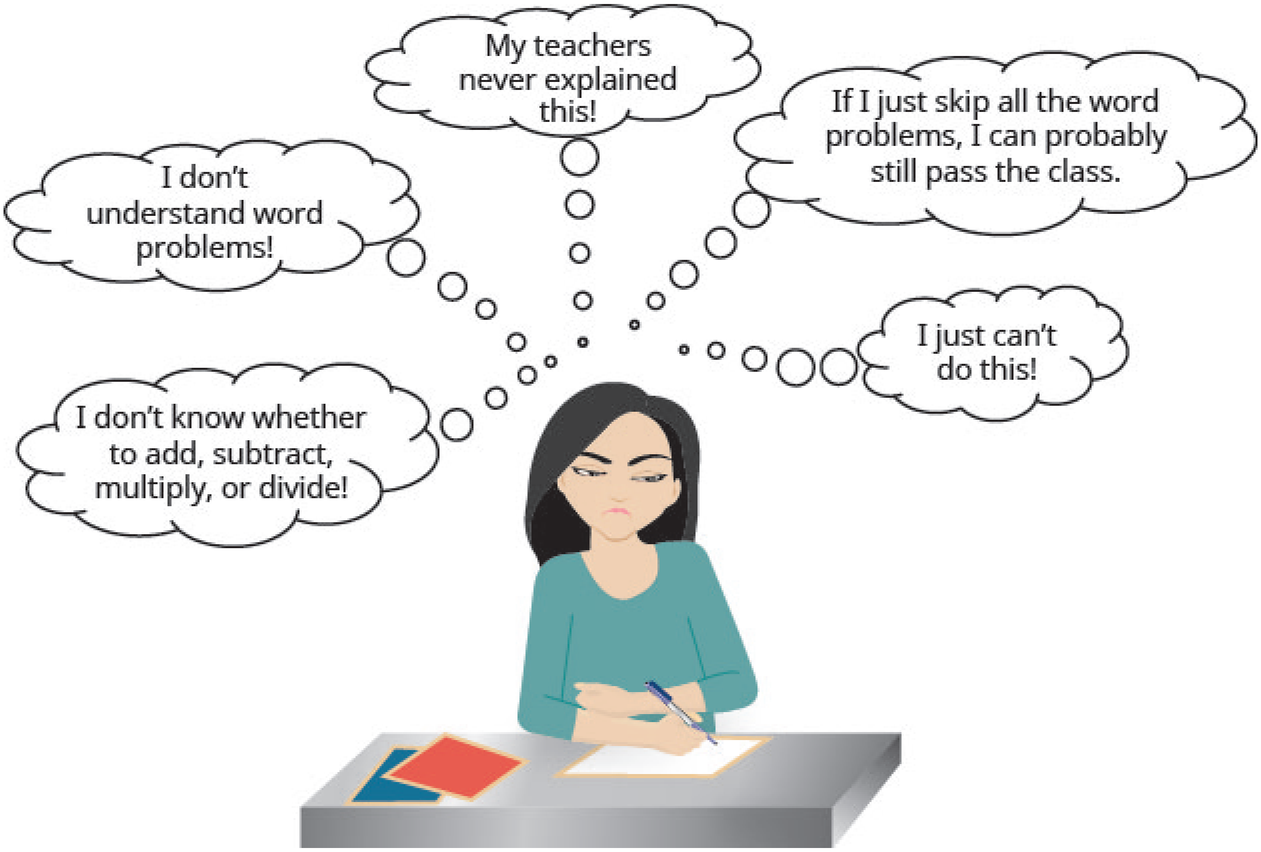

Figure .1

When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. We need to calm our fears and change our negative feelings.

Start with a fresh slate and begin to think positive thoughts. If we take control and believe we can be successful, we will be able to master word problems! Read the positive thoughts in (Figure 2) and say them out loud.

Thinking positive thoughts is a first step towards success.

Figure .2

Think of something, outside of school, that you can do now but couldn't do 3 years ago. Is it driving a car? Snowboarding? Cooking a gourmet meal? Speaking a new language? Your past experiences with word problems happened when you were younger—now you're older and ready to succeed!

Use a Problem-Solving Strategy for Word Problems

We have reviewed translating English phrases into algebraic expressions, using some basic mathematical vocabulary and symbols. We have also translated English sentences into algebraic equations and solved some word problems. The word problems applied math to everyday situations. We restated the situation in one sentence, assigned a variable, and then wrote an equation to solve the problem. This method works as long as the situation is familiar and the math is not too complicated.

Now, we'll expand our strategy so we can use it to successfully solve any word problem. We'll list the strategy here, and then we'll use it to solve some problems. We summarize below an effective strategy for problem solving.

Use a Problem-Solving Strategy to Solve Word Problems.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

EXAMPLE 1

Pilar bought a purse on sale for $18, which is one-half of the original price. What was the original price of the purse?

Solution

Step 1. Read the problem. Read the problem two or more times if necessary. Look up any unfamiliar words in a dictionary or on the internet.

- *In this problem, is it clear what is being discussed? Is every word familiar?*

Step 2. Identify what you are looking for. Did you ever go into your bedroom to get something and then forget what you were looking for? It's hard to find something if you are not sure what it is! Read the problem again and look for words that tell you what you are looking for!

- *In this problem, the words "what was the original price of the purse" tell us what we need to find.*

Step 3. Name what we are looking for. Choose a variable to represent that quantity. We can use any letter for the variable, but choose one that makes it easy to remember what it represents.

- Let $p =$ the original price of the purse.

Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Translate the English sentence into an algebraic equation.

Reread the problem carefully to see how the given information is related. Often, there is one sentence that gives this information, or it may help to write one sentence with all the important information. Look for clue words to help translate the sentence into algebra. Translate the sentence into an equation.

Restate the problem in one sentence with all the important information.	18 is one-half the original price.
Translate into an equation.	$18 = \frac{1}{2} \cdot p$

Step 5. Solve the equation using good algebraic techniques. Even if you know the solution right away, using good algebraic techniques here will better prepare you to solve problems that do not have obvious answers.

Solve the equation.	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$

Step 6. Check the answer in the problem to make sure it makes sense. We solved the equation and found that $p = 36$, which means "the original price" was $36

- *Does $36 make sense in the problem? Yes, because 18 is one-half of 36, and the purse was on sale at half the original price.*

Step 7. Answer the question with a complete sentence. The problem asked "What was the original price of the purse?"

- *The answer to the question is: "The original price of the purse was $36."*

If this were a homework exercise, our work might look like this:

Pilar bought a purse on sale for $18, which is one-half the original price. What was the original price of the purse?

	Let $p =$ the original price.
	18 is one-half the original price.
	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$
Check. Is $36 a reasonable price for a purse?	Yes.
Is 18 one half of 36?	$18 \stackrel{?}{=} \frac{1}{2} \cdot 36$
	$18 = 18\checkmark$
	The original price of the purse was $36.

TRY IT 1.1

Joaquin bought a bookcase on sale for $120, which was two-thirds of the original price. What was the original price of the bookcase?

Show answer

$180

TRY IT 1.2

Two-fifths of the songs in Mariel's playlist are country. If there are 16 country songs, what is the total number of songs in the playlist?

Show answer

40

Let's try this approach with another example.

EXAMPLE 2

Ginny and her classmates formed a study group. The number of girls in the study group was three more than twice the number of boys. There were 11 girls in the study group. How many boys were in the study group?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	How many boys were in the study group?
Step 3. Name. Choose a variable to represent the number of boys.	Let n = the number of boys.
Step 4. Translate. Restate the problem in one sentence with all the important information.	$\underbrace{\text{The number of girls (11)}}\ \underbrace{\text{was}}\ \underbrace{\text{three more than twice the number of boys}}$
Translate into an equation.	$11 \quad = \quad 2b+3$
Step 5. Solve the equation.	$11 = 2b + 3$
Subtract 3 from each side.	$11 - 3 = 2b + 3 - 3$
Simplify.	$8 = 2b$
Divide each side by 2.	$\frac{8}{2} = \frac{2b}{2}$
Simplify.	$4 = b$
Step 6. Check. First, is our answer reasonable?	Yes, having 4 boys in a study group seems OK. The problem says the number of girls was 3 more than twice the number of boys. If there are four boys, does that make eleven girls? Twice 4 boys is 8. Three more than 8 is 11.
Step 7. Answer the question.	There were 4 boys in the study group.

TRY IT 2.1

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was 3 more than twice the number of notebooks. He bought 7 textbooks. How many notebooks did he buy?

Show answer

2

TRY IT 2.2

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

Show answer
7

Solve Number Problems

Now that we have a problem solving strategy, we will use it on several different types of word problems. The first type we will work on is “number problems.” Number problems give some clues about one or more numbers. We use these clues to write an equation. Number problems don’t usually arise on an everyday basis, but they provide a good introduction to practicing the problem solving strategy outlined above.

EXAMPLE 3

The difference of a number and six is 13. Find the number.

Solution

Step 1. Read the problem. Are all the words familiar?	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let n = the number.
Step 4. Translate. Remember to look for clue words like "difference… of… and…"	
Restate the problem as one sentence.	$\underbrace{\text{The difference of the number and 6}}\ \underbrace{\text{is}}\ \underbrace{13}$
Translate into an equation.	$n - 6 \quad = \quad 13$
Step 5. Solve the equation.	$n - 6 = 13$
Simplify.	$n = 19$
Step 6. Check.	
The difference of 19 and 6 is 13. It checks!	
Step 7. Answer the question.	The number is 19.

TRY IT 3.1

The difference of a number and eight is 17. Find the number.

Show answer

25

TRY IT 3.2

The difference of a number and eleven is -7. Find the number.

Show answer

4

EXAMPLE 4

The sum of twice a number and seven is 15. Find the number.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let n = the number.
Step 4. Translate.	
Restate the problem as one sentence.	$\underbrace{\text{The sum of twice a number and 7}}\ \underbrace{\text{is}}\ \underbrace{15}$
Translate into an equation.	$2n + 7 \quad = \quad 15$
Step 5. Solve the equation.	$2n + 7 = 15$
Subtract 7 from each side and simplify.	$2n = 8$
Divide each side by 2 and simplify.	$n = 4$
Step 6. Check.	
Is the sum of twice 4 and 7 equal to 15?	$\begin{aligned} 2 \cdot 4 + 7 &\stackrel{?}{=} 15 \\ 15 &= 15\checkmark \end{aligned}$
Step 7. Answer the question.	The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

TRY IT 4.1

The sum of four times a number and two is 14. Find the number.

Show answer

3

TRY IT 4.2

The sum of three times a number and seven is 25. Find the number.

Show answer

6

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

EXAMPLE 5

One number is five more than another. The sum of the numbers is 21. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name. We have two numbers to name and need a name for each.	
Choose a variable to represent the first number.	Let $n = 1^{\text{st}}$ number.
What do we know about the second number?	One number is five more than another.
	$n + 5 = 2^{\text{nd}}$ number
Step 4. Translate. Restate the problem as one sentence with all the important information.	The sum of the 1^{st} number and the 2^{nd} number is 21.
Translate into an equation.	$\underbrace{1^{\text{st}}\text{ number}} + \underbrace{2^{\text{nd}}\text{ number}} \underbrace{=} \underbrace{21}$
Substitute the variable expressions.	$n \quad + \quad n+5 \quad = \quad 21$
Step 5. Solve the equation.	$n + n + 5 = 21$
Combine like terms.	$2n + 5 = 21$
Subtract 5 from both sides and simplify.	$2n = 16$
Divide by 2 and simplify.	$n = 8 \quad 1^{\text{st}}$ number
Find the second number, too.	$n + 5 \quad 2^{\text{nd}}$ number
	$8 + 5$
	13
Step 6. Check.	
Do these numbers check in the problem?	
Is one number 5 more than the other?	$13 \stackrel{?}{=} 8 + 5$
Is thirteen 5 more than 8? Yes.	$13 = 13\checkmark$
Is the sum of the two numbers 21?	$8 + 13 \stackrel{?}{=} 21$
	$21 = 21\checkmark$
Step 7. Answer the question.	The numbers are 8 and 13.

TRY IT 5.1

One number is six more than another. The sum of the numbers is twenty-four. Find the numbers.

Show answer

9, 15

TRY IT 5.2

The sum of two numbers is fifty-eight. One number is four more than the other. Find the numbers.

Show answer

27, 31

EXAMPLE 6

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name.	
Choose a variable.	Let $n = 1^{\text{st}}$ number.
One number is 4 less than the other.	$n - 4 = 2^{\text{nd}}$ number
Step 4. Translate.	
Write as one sentence.	The sum of the 2 numbers is negative 14.
Translate into an equation.	$\underbrace{1^{\text{st}} \text{ number}}_{n} + \underbrace{2^{\text{nd}} \text{ number}}_{n-4} \underbrace{\text{is}}_{=} \underbrace{\text{negative fourteen}}_{-14}$
Step 5. Solve the equation.	$n + n - 4 = -14$
Combine like terms.	$n + n - 4 = -14$
Add 4 to each side and simplify.	$2n - 4 = -14$
Simplify.	$2n = -10$
	$n = -5$ 1st number
	$n - 4$ 2nd number
	$-5 - 4$
	-9
Step 6. Check.	
Is −9 four less than −5?	$-5 - 4 \stackrel{?}{=} -9$
	$-9 = -9\checkmark$
Is their sum −14?	$-5 + (-9) \stackrel{?}{=} -14$
	$-14 = -14\checkmark$
Step 7. Answer the question.	The numbers are −5 and −9.

TRY IT 6.1

The sum of two numbers is negative twenty-three. One number is seven less than the other. Find the numbers.

Show answer

$-15, -8$

TRY IT 6.2

The sum of two numbers is -18. One number is 40 more than the other. Find the numbers.

Show answer

$-29, 11$

EXAMPLE 7

One number is ten more than twice another. Their sum is one. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for two numbers.
Step 3. Name.	
Choose a variable.	Let $x = 1^{\text{st}}$ number.
One number is 10 more than twice another.	$2x + 10 = 2^{\text{nd}}$ number
Step 4. Translate.	
Restate as one sentence.	Their sum is one.
	The sum of the two numbers is 1.
Translate into an equation.	$x + 2x + 10 = 1$
Step 5. Solve the equation.	
Combine like terms.	$x + 2x + 10 = 1$
Subtract 10 from each side.	$3x + 10 = 1$
Divide each side by 3.	$3x = -9$
	$x = -3$ 1^{st} number
	$2x + 10$ 2^{nd} number
	$2(-3) + 10$
	4
Step 6. Check.	
Is ten more than twice −3 equal to 4?	$2(-3) + 10 \stackrel{?}{=} 4$
	$-6 + 10 \stackrel{?}{=} 4$
	$4 = 4\checkmark$
Is their sum 1?	$-3 + 4 \stackrel{?}{=} 1$
	$1 = 1\checkmark$
Step 7. Answer the question.	The numbers are −3 and −4.

TRY IT 7.1

One number is eight more than twice another. Their sum is negative four. Find the numbers.

Show answer

$-4, 0$

TRY IT 7.2

One number is three more than three times another. Their sum is -5. Find the numbers.

Show answer

$-3, -2$

Some number problems involve consecutive integers.*Consecutive integers* are integers that immediately follow each other.

Examples of consecutive integers are:

$1, 2, 3, 4$

$-10, -9, -8, -7$

$150, 151, 152, 153$

Notice that each number is one more than the number preceding it. So if we define the first integer as *n*, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

$$\begin{array}{cl} n & 1^{\text{st}} \text{ integer} \\ n+1 & 2^{\text{nd}} \text{ consecutive integer} \\ n+2 & 3^{\text{rd}} \text{ consecutive integer } \ldots \text{ etc.} \end{array}$$

EXAMPLE 8

The sum of two consecutive integers is 47. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	two consecutive integers
Step 3. Name each number.	Let $n = 1^{\text{st}}$ integer.
	$n + 1 =$ next consecutive integer
Step 4. Translate.	
Restate as one sentence.	The sum of the integers is 47.
Translate into an equation.	$n + n + 1 = 47$
Step 5. Solve the equation.	$n + n + 1 = 47$
Combine like terms.	$2n + 1 = 47$
Subtract 1 from each side.	$2n = 46$
Divide each side by 2.	$n = 23$ 1^{st} integer
	$n + 1$ next consecutive integer
	$23 + 1$
	24
Step 6. Check.	$23 + 24 \stackrel{?}{=} 47$ $47 = 47\checkmark$
Step 7. Answer the question.	The two consecutive integers are 23 and 24.

TRY IT 8.1

The sum of two consecutive integers is 95. Find the numbers.

Show answer

47, 48

TRY IT 8.2

The sum of two consecutive integers is -31. Find the numbers.

Show answer

$-16, -15$

EXAMPLE 9

Find three consecutive integers whose sum is -42.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	three consecutive integers
Step 3. Name each of the three numbers.	Let $n = 1^{\text{st}}$ integer.
	$n+1 = 2^{\text{nd}}$ consecutive integer
	$n+2 = 3^{\text{rd}}$ consecutive integer
Step 4. Translate.	
Restate as one sentence.	The sum of the three integers is −42.
Translate into an equation.	$n+n+1+n+2=-42$
Step 5. Solve the equation.	$n+n+1+n+2=-42$
Combine like terms.	$3n+3=-42$
Subtract 3 from each side.	$3n=-45$
Divide each side by 3.	$n=-15$ 1^{st} integer
	$n+1$ 2^{nd} integer
	$-15+1$
	-14
	$n+2$ 3^{rd} integer
	$-15+2$
	-13
Step 6. Check.	$-13+(-14)+(-15) \stackrel{?}{=} -42$ $-42 = -42\checkmark$
Step 7. Answer the question.	The three consecutive integers are −13, −14, and −15.

TRY IT 9.1

Find three consecutive integers whose sum is -96.

Show answer

$-33, -32, -31$

TRY IT 9.2

Find three consecutive integers whose sum is -36.

Show answer

$-13, -12, -11$

Now that we have worked with consecutive integers, we will expand our work to include consecutive even integers and consecutive odd integers. *Consecutive even integers* are even integers that immediately follow one another. Examples of consecutive even integers are:

$$18, 20, 22$$

$$64, 66, 68$$

$$-12, -10, -8$$

Notice each integer is 2 more than the number preceding it. If we call the first one *n*, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$.

$$\begin{array}{ll} n & 1^{\text{st}} \text{ even integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive even integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive even integer} \ldots \text{ etc.} \end{array}$$

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 77, 79, and 81

$$77, 79, 81$$

$$n, n+2, n+4$$

$$\begin{array}{ll} n & 1^{\text{st}} \text{ odd integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive odd integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive odd integer} \ldots \text{ etc.} \end{array}$$

Does it seem strange to add 2 (an even number) to get from one odd integer to the next? Do you get an odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you don't have to do anything different. The pattern is still the same—to get from one odd or one even integer to the next, add 2

EXAMPLE 10

Find three consecutive even integers whose sum is 84

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	three consecutive even integers
Step 3. Name the integers.	Let $n = 1^{\text{st}}$ even integer. $n + 2 = 2^{\text{nd}}$ consecutive even integer $n + 4 = 3^{\text{rd}}$ consecutive even integer
Step 4. Translate.	
Restate as one sentence.	The sume of the three even integers is 84.
Translate into an equation.	$n + n + 2 + n + 4 \;=\; 84$
Step 5. Solve the equation.	
Combine like terms.	$n + n + 2 + n + 4 \;=\; 84$
Subtract 6 from each side.	$3n + 6 \;=\; 84$
Divide each side by 3.	$3n \;=\; 78$
	$n = 26 \quad 1^{\text{st}}$ integer
	$n + 2 \quad 2^{\text{nd}}$ integer $26 + 2$ 28
	$n + 4 \quad 3^{\text{rd}}$ integer $26 + 4$ 30
Step 6. Check.	$26 + 28 + 30 \stackrel{?}{=} 84$ $84 = 84\checkmark$
Step 7. Answer the question.	The three consecutive integers are 26, 28, and 30.

TRY IT 10.1

Find three consecutive even integers whose sum is 102

Show answer

32, 34, 36

TRY IT 10.2

Find three consecutive even integers whose sum is -24.

Show answer

$-10, -8, -6$

EXAMPLE 11

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	How much does the husband earn?
Step 3. Name.	
Choose a variable to represent the amount the husband earns.	Let $h =$ the amount the husband earns.
The wife earns $16,000 less than twice that.	$2h - 16,000$ the amount the wife earns.
Step 4. Translate.	Together the husband and wife earn $110,000.
Restate the problem in one sentence with all the important information.	$\underbrace{\text{The amount the husband earns}}$ plus $\underbrace{\text{the amount the wife earns}}$ is $110,000
Translate into an equation.	$h \quad + \quad 2h - 16,000 \quad = \quad 110,000$
Step 5. Solve the equation.	h + 2h – 16,000 = 110,000
Combine like terms.	3h – 16,000 = 110,000
Add 16,000 to both sides and simplify.	3h = 126,000
Divide each side by 3.	h = 42,000
	$42,000 amount husband earns
	2h – 16,000 amount wife earns
	2(42,000) – 16,000
	84,000 – 16,000
	68,000
Step 6. Check.	If the wife earns $68,000 and the husband earns $42,000 is the total $110,000? Yes!
Step 7. Answer the question.	The husband earns $42,000 a year.

TRY IT 11.1

According to the National Automobile Dealers Association, the average cost of a car in 2014 was 28,500. This was 1,500 less than 6 times the cost in 1975. What was the average cost of a car in 1975?

Show answer

5,000

TRY IT 11.2

The Canadian Real Estate Association (CREA) data shows that the median price of new home in the Canada in December 2018 was $470,000. This was $14,000 more than 19 times the price in December 1967. What was the median price of a new home in December 1967?

Show answer

$24,000

Key Concepts

- **Problem-Solving Strategy**

  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

- **Consecutive Integers**

  Consecutive integers are integers that immediately follow each other.

$$\begin{array}{ll} n & 1^{\text{st}} \text{ integer} \\ n+1 & 2^{\text{nd}} \text{ integer consecutive integer} \\ n+2 & 3^{\text{rd}} \text{ consecutive integer } \ldots \text{ etc.} \end{array}$$

Consecutive even integers are even integers that immediately follow one another.

n	1^{st} integer
$n+2$	2^{nd} integer consecutive integer
$n+4$	3^{rd} consecutive integer . . . etc.

Consecutive odd integers are odd integers that immediately follow one another.

n	1^{st} integer
$n+2$	2^{nd} integer consecutive integer
$n+4$	3^{rd} consecutive integer . . . etc.

Practice Makes Perfect

Use the Approach Word Problems with a Positive Attitude

In the following exercises, prepare the lists described.

1. List five positive thoughts you can say to yourself that will help you approach word problems with a positive attitude. You may want to copy them on a sheet of paper and put it in the front of your notebook, where you can read them often.

2. List five negative thoughts that you have said to yourself in the past that will hinder your progress on word problems. You may want to write each one on a small piece of paper and rip it up to symbolically destroy the negative thoughts.

Use a Problem-Solving Strategy for Word Problems

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

3. Two-thirds of the children in the fourth-grade class are girls. If there are 20 girls, what is the total number of children in the class?	4. Three-fifths of the members of the school choir are women. If there are 24 women, what is the total number of choir members?
5. Zachary has 25 country music CDs, which is one-fifth of his CD collection. How many CDs does Zachary have?	6. One-fourth of the candies in a bag of M&M's are red. If there are 23 red candies, how many candies are in the bag?
7. There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.	8. There are 18 Cub Scouts in Pack 645. The number of scouts is three more than five times the number of adult leaders. Find the number of adult leaders.
9. Huong is organizing paperback and hardback books for her club's used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong had 162 paperbacks. How many hardback books were there?	10. Jeff is lining up children's and adult bicycles at the bike shop where he works. The number of children's bicycles is nine less than three times the number of adult bicycles. There are 42 adult bicycles. How many children's bicycles are there?
11. Philip pays $1,620 in rent every month. This amount is $120 more than twice what his brother Paul pays for rent. How much does Paul pay for rent?	12. Marc just bought an SUV for $54,000. This is $7,400 less than twice what his wife paid for her car last year. How much did his wife pay for her car?
13. Laurie has $46,000 invested in stocks and bonds. The amount invested in stocks is $8,000 less than three times the amount invested in bonds. How much does Laurie have invested in bonds?	14. Erica earned a total of $50,450 last year from her two jobs. The amount she earned from her job at the store was $1,250 more than three times the amount she earned from her job at the college. How much did she earn from her job at the college?

Solve Number Problems

In the following exercises, solve each number word problem.

15. The sum of a number and eight is 12. Find the number.

16. The sum of a number and nine is 17. Find the number.

17. The difference of a number and 12 is three. Find the number.

18. The difference of a number and eight is four. Find the number.

19. The sum of three times a number and eight is 23. Find the number.

20. The sum of twice a number and six is 14. Find the number.

21.The difference of twice a number and seven is 17. Find the number.

22. The difference of four times a number and seven is 21. Find the number.

23. Three times the sum of a number and nine is 12. Find the number.

24. Six times the sum of a number and eight is 30. Find the number.

25. One number is six more than the other. Their sum is 42. Find the numbers.

26. One number is five more than the other. Their sum is 33. Find the numbers.

27. The sum of two numbers is 20. One number is four less than the other. Find the numbers.

28. The sum of two numbers is 27. One number is seven less than the other. Find the numbers.

29. The sum of two numbers is -45. One number is nine more than the other. Find the numbers.

30. The sum of two numbers is -61. One number is 35 more than the other. Find the numbers.

31. The sum of two numbers is -316. One number is 94 less than the other. Find the numbers.

32. The sum of two numbers is -284. One number is 62 less than the other. Find the numbers.

33. One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.

34. One number is 11 less than another. If their sum is increased by eight, the result is 71. Find the numbers.

35. One number is five more than another. If their sum is increased by nine, the result is 60. Find the numbers.

36. One number is eight more than another. If their sum is increased by 17, the result is 95. Find the numbers.

37. One number is one more than twice another. Their sum is -5. Find the numbers.

38. One number is six more than five times another. Their sum is six. Find the numbers.

39. The sum of two numbers is 14. One number is two less than three times the other. Find the numbers.

40. The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.

41. The sum of two consecutive integers is 77. Find the integers.

42. The sum of two consecutive integers is 89. Find the integers.

43. The sum of two consecutive integers is -23. Find the integers.

44. The sum of two consecutive integers is -37. Find the integers.

45. The sum of three consecutive integers is 78. Find the integers.

46. The sum of three consecutive integers is 60. Find the integers.

47. Find three consecutive integers whose sum is -36.

48. Find three consecutive integers whose sum is -3.

49. Find three consecutive even integers whose sum is 258.

50. Find three consecutive even integers whose sum is 222.

51. Find three consecutive odd integers whose sum is 171.

52. Find three consecutive odd integers whose sum is 291.

53. Find three consecutive even integers whose sum is -36.

54. Find three consecutive even integers whose sum is -84.

55. Find three consecutive odd integers whose sum is -213.

56. Find three consecutive odd integers whose sum is -267.

Everyday Math

57. **Sale Price.** Patty paid $35 for a purse on sale for $10 off the original price. What was the original price of the purse?	58. **Sale Price.** Travis bought a pair of boots on sale for $25 off the original price. He paid $60 for the boots. What was the original price of the boots?
59. **Buying in Bulk.** Minh spent $6.25 on five sticker books to give his nephews. Find the cost of each sticker book.	60. **Buying in Bulk.** Alicia bought a package of eight peaches for $3.20. Find the cost of each peach.
61. **Price before Sales Tax.** Tom paid $1,166.40 for a new refrigerator, including $86.40 tax. What was the price of the refrigerator?	62. **Price before Sales Tax.** Kenji paid $2,279 for a new living room set, including $129 tax. What was the price of the living room set?

Writing Exercises

63. What has been your past experience solving word problems?	64. When you start to solve a word problem, how do you decide what to let the variable represent?
65. What are consecutive odd integers? Name three consecutive odd integers between 50 and 60.	66. What are consecutive even integers? Name three consecutive even integers between -50 and -40.

Answers

1. Answers will vary	3. 30	5. 125
7. 6	9. 58	11. $750
13. $13,500	15. 4	17. 15
19. 5	21. 12	23. -5
25. 18, 24	27. 8, 12	29. $-18, -27$
31. $-111, -205$	33. 32, 46	35. 23, 28
37. $-2, -3$	39. 4, 10	41. 38, 39
43. $-11, -12$	45. 25, 26, 27	47. $-11, -12, -13$
49. 84, 86, 88	51. 55, 57, 59	53. $-10, -12, -14$
55. $-69, -71, -73$	57. $45	59. $1.25
61. $1080	63. Answers will vary	65. Consecutive odd integers are odd numbers that immediately follow each other. An example of three consecutive odd integers between 50 and 60 would be 51, 53, and 55.

Attributions

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CHAPTER 2: SYSTEMS OF EQUATIONS

CHAPTER 2.1: SLOPES AND THEIR GRAPHS

Another important property of any line or linear function is slope. Slope is a measure of steepness and indicates in some situations how fast something is changing—specifically, its rate of change. A line with a large slope, such as 10, is very steep. A line with a small slope, such as $\frac{1}{10}$, is very flat or nearly level. Lines that rise from left to right are called positive slopes and lines that sink are called negative slopes. Slope can also be used to describe the direction of a line. A line that goes up as it moves from left to right is described as having a positive slope whereas a line that goes downward has a negative slope. Slope, therefore, will define a line as rising or falling.

Slopes in real life have significance. For instance, roads with slopes that are potentially dangerous often carry warning signs. For steep slopes that are rising, extra slow moving lanes are generally provided for large trucks. For roads that have steep down slopes, runaway lanes are often provided for vehicles that lose their ability to brake.

When quantifying slope, use the measure of the rise of the line divided by its run. The symbol that represents slope is the letter m, which has unknown origins. Its first recorded usage is in an 1844 text by Matthew O'Brian, "A Treatise on Plane Co-Ordinate Geometry,"[1] which was quickly followed by George Salmon's "A Treatise on Conic Sections" (1848), in which he used m in the equation $y = mx + b$.

1. Derivation of Slope: https://services.math.duke.edu//education/webfeats/Slope/Slopederiv.html

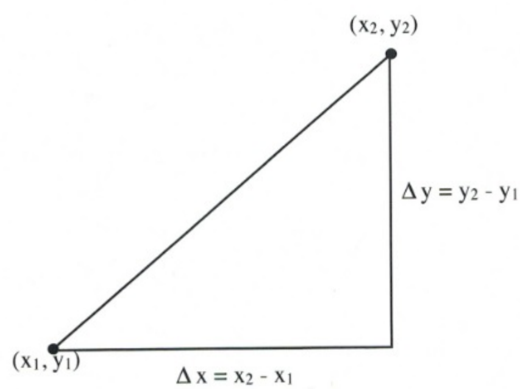
$$\text{slope} \;=\frac{\text{rise of the line}}{\text{run of the line}}$$

Since the rise of a line is shown by the change in the y-value and the run is shown by the change in the x-value, this equation is shortened to:

$$m=\frac{\Delta y}{\Delta x},\ \text{where } \Delta \text{ is the symbol for change and means final value} - \text{initial value}$$

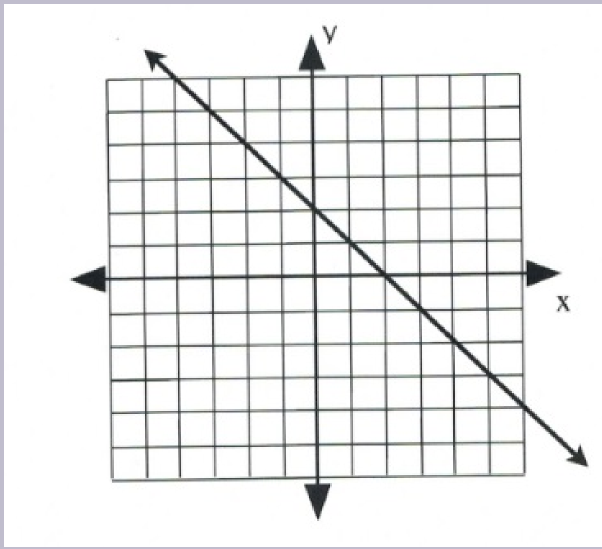
This equation is often expanded to:

$$m=\frac{y_2-y_1}{x_2-x_1}$$



Example 1

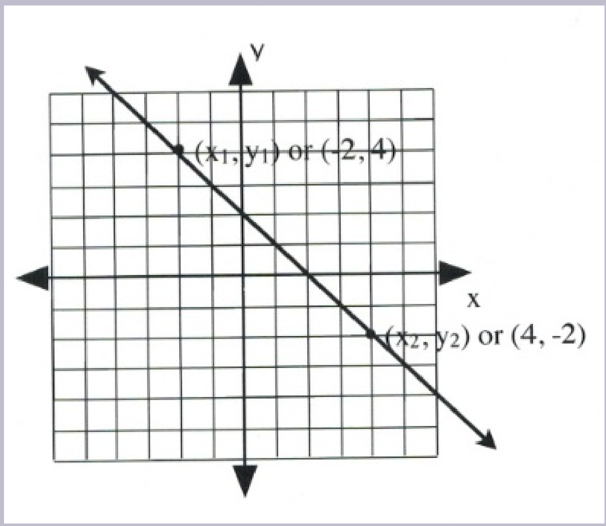
Find the slope of the following line.



First, choose two points on the line on this graph. Any points can be chosen, but they should fall on one of the corner grids. These are labelled (x_1, y_1) and (x_2, y_2).

To find the slope of this line, consider the rise, or vertical change, and the run, or horizontal change. Observe in this example that the Δy-value (the rise) goes from 4 to −2.

Therefore, $\Delta y = y_2 - y_1$, or (4 − −2), which equals (4 + 2), or 6.



The Δx-value (the run) goes from −2 to 4.

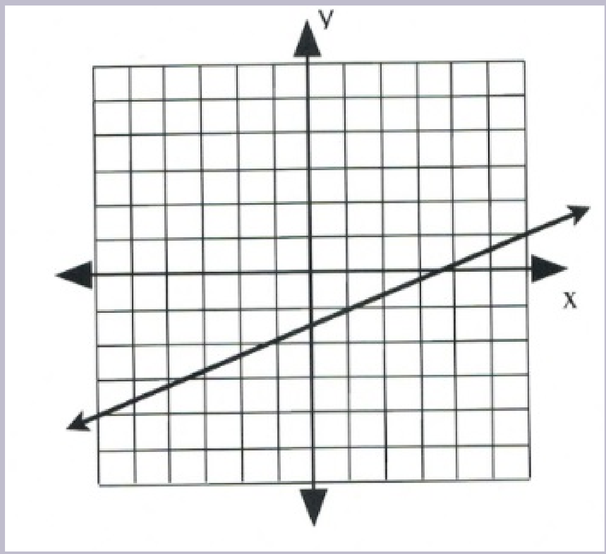
Therefore, $\Delta x = x_2 - x_1$, or (−2 − 4), which equals (−2 + −4), or −6.

This means the slope of this line is $m = \dfrac{\Delta y}{\Delta x}$, or $\dfrac{6}{-6}$, or −1.

$m = -1$

Example 2

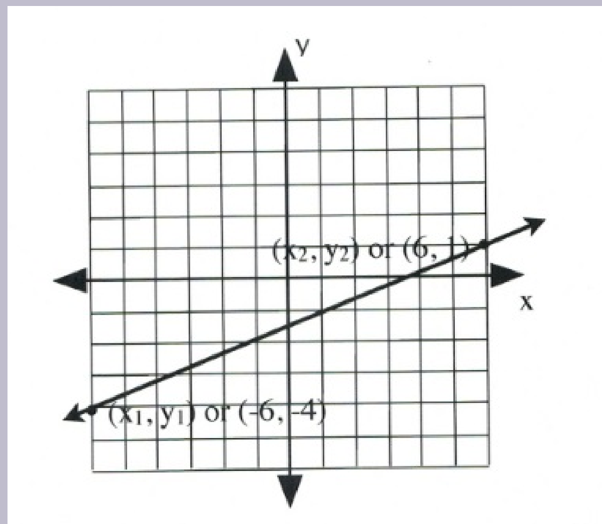
Find the slope of the following line.



First, choose two points on the line on this graph. Any points can be chosen, but to fall on a corner grid, they should be on opposite sides of the graph. These are (x_1, y_1) and (x_2, y_2).

To find the slope of this line, consider the rise, or vertical change, and the run, or horizontal change. Observe in this example that the Δy-value (the rise) goes from −4 to 1.

Therefore, $\Delta y = y_2 - y_1$, or (1 − −4), which equals 5.



The Δx-value (the run) goes from −6 to 6.

Therefore, $\Delta x = x_2 - x_1$ or (6 – −6), which equals 12.

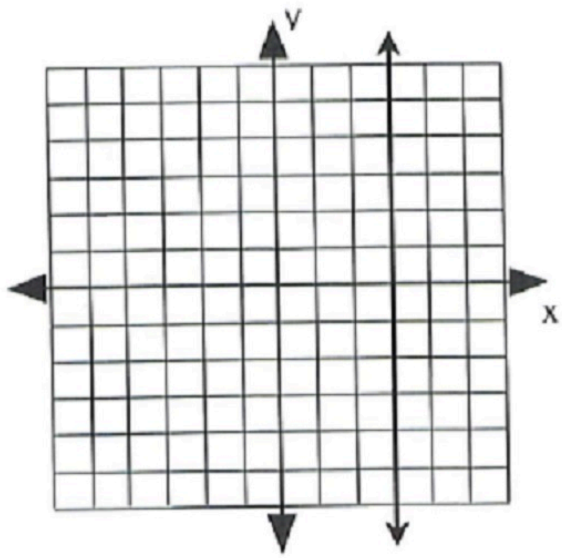
This means the slope of this line is $m = \dfrac{\Delta y}{\Delta x}$, or $\dfrac{5}{12}$, which cannot be further simplified.

$$m = \frac{5}{12}$$

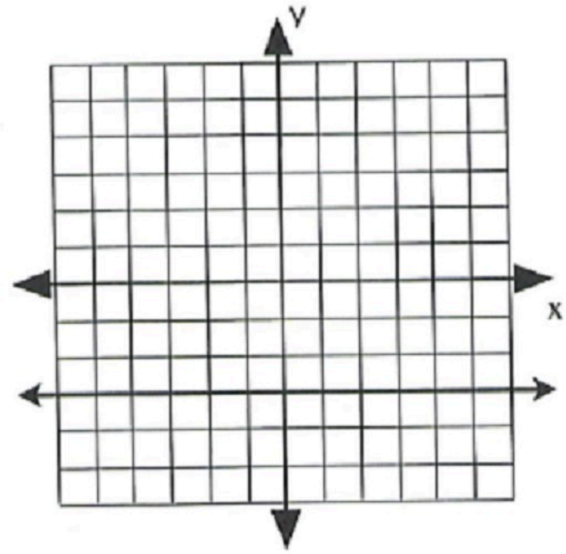
There are two special lines that have unique slopes that must be noted: lines with slopes equal to zero and slopes that are undefined.

Undefined slopes arise when the line on the graph is vertical, going straight up and down. In this case, $\Delta x = 0$, which means that zero is divided by while calculating the slope, which makes it undefined.

Zero slopes are flat, horizontal lines that do not rise or fall; therefore, $\Delta y = 0$. In this case, the slope is simply 0.



Undefined Slope since $\Delta x = 0$



Zero Slope since $\Delta y = 0$

Most often, the slope of the line must be found using data points rather than graphs. In this case, two data points are generally given, and the slope m is found by dividing Δy by Δx. This is usually done using the expanded slope equation of:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 3

Find the slope of a line that would connect the data points $(-4, 3)$ and $(2, -9)$.

Choose Point 1 to be $(-4, 3)$ and Point 2 to be $(2, -9)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-9 - 3}{2 - -4}$$

$$m = \frac{-12}{6} \text{ or } -2$$

Example 4

Find the slope of a line that would connect the data points $(-5, 3)$ and $(2, 3)$.

Choose Point 1 to be $(-5, 3)$ and Point 2 to be $(2, 3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{3 - 3}{2 - -5}$$

$$m = \frac{0}{7} \text{ or } 0$$

This is an example of a flat, horizontal line.

Example 5

Find the slope of a line that would connect the data points $(4, 3)$ and $(4, -5)$.

Choose Point 1 to be $(4, 3)$ and Point 2 to be $(4, -5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-5 - 3}{4 - 4}$$

$$m = \frac{-8}{0} \text{ or undefined}$$

This is a vertical line.

Example 6

Find the slope of a line that would connect the data points $(-4, -3)$ and $(2, 6)$.

Choose Point 1 to be $(-4, -3)$ and Point 2 to be $(2, 6)$.

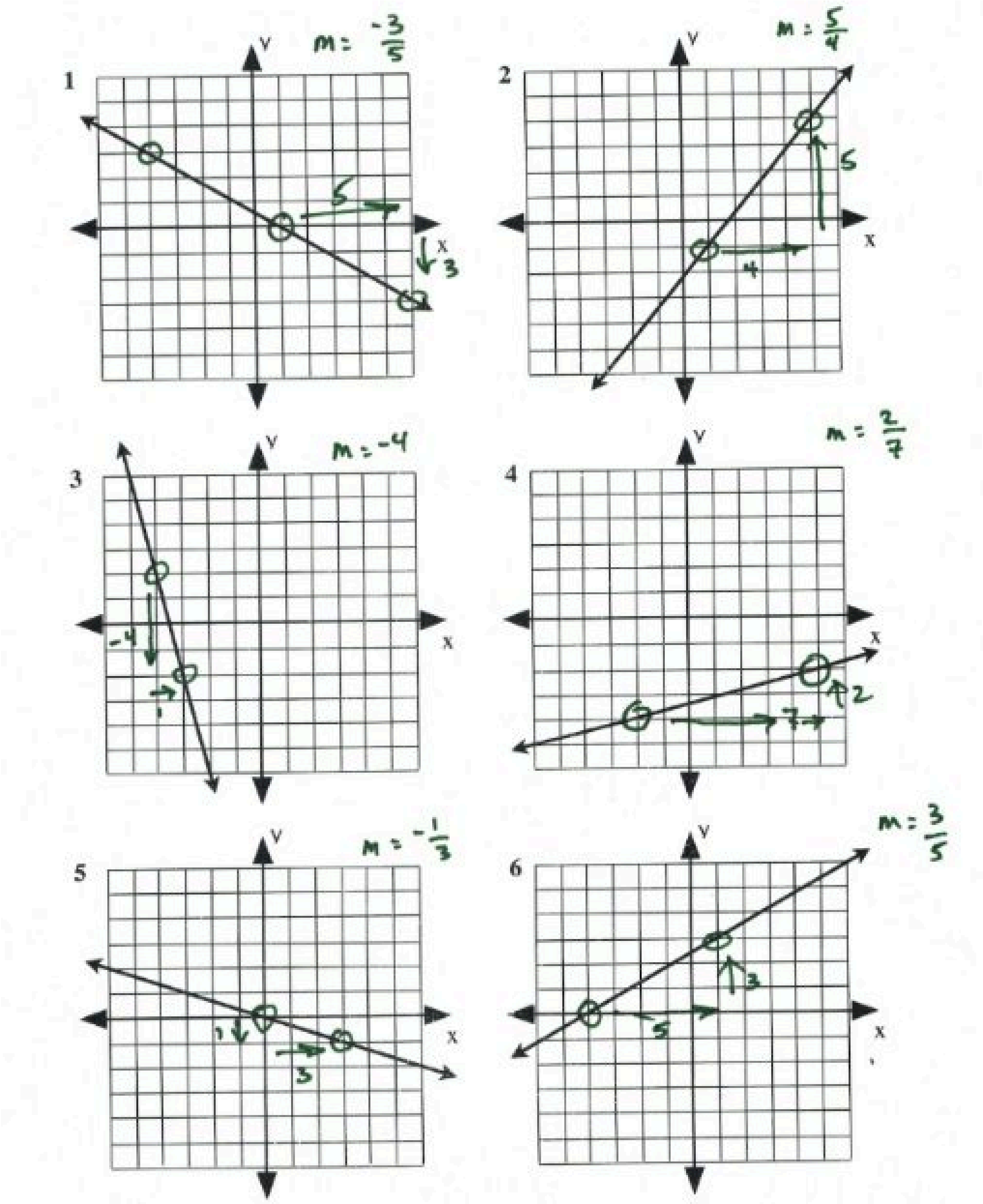
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{6 - -3}{2 - -4}$$

$$m = \frac{9}{6} \text{ or } \frac{3}{2}$$

Questions

For questions 1 to 6, find the slope of each line shown on the graph.



For questions 7 to 26, find the slope of the line that would connect each pair of points.

7. (2, 10), (−2, 15)
8. (1, 2), (−6, −12)
9. (−5, 10), (0, 0)
10. (2, −2), (7, 8)
11. (4, 6), (−8, −10)
12. (−3, 6), (9, −6)

13. $(-2\ -4), (10, -4)$
14. $(3, 5), (2, 0)$
15. $(-4, 4), (-6, 8)$
16. $(9, -6), (-7, -7)$
17. $(2, -9), (6, 4)$
18. $(-6, 2), (5, 0)$
19. $(-5, 0), (-5, 0)$
20. $(8, 11), (-3, -13)$
21. $(-7, 9), (1, -7)$
22. $(1, -2), (1, 7)$
23. $(7, -4), (-8, -9)$
24. $(-8, -5), (4, -3)$
25. $(-5, 7), (-8, 4)$
26. $(9, 5), (5, 1)$

Answers to odd questions.

1. $m = -\dfrac{3}{5}$
3. $m = -4$
5. $m = -\dfrac{1}{3}$
7. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{15 - 10}{-2 - 2} \Rightarrow \dfrac{-5}{4}$
9. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{0 - 10}{0 - -5} \Rightarrow \dfrac{-10}{5} \Rightarrow -2$
11. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{-10 - 6}{-8 - 4} \Rightarrow \dfrac{-16}{-12} \Rightarrow \dfrac{4}{3}$
13. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{-4 - -4}{10 - -2} \Rightarrow \dfrac{0}{12} \Rightarrow 0$
15. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{8 - 4}{-6 - -4} \Rightarrow \dfrac{4}{-2} \Rightarrow -2$
17. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{4 - -9}{6 - 2} \Rightarrow \dfrac{13}{4}$
19. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{0 - 0}{-5 - -5} \Rightarrow \dfrac{0}{0} \therefore \text{Undefined}$
21. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{-7 - 9}{1 - -7} \Rightarrow \dfrac{-16}{8} \Rightarrow -2$
23. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{-9 - -4}{-8 - 7} \Rightarrow \dfrac{-5}{-15} \Rightarrow \dfrac{1}{3}$
25. $m = \dfrac{\Delta y}{\Delta x} \Rightarrow \dfrac{4 - 7}{-8 - -5} \Rightarrow \dfrac{-3}{-3} \Rightarrow 1$

CHAPTER 2.2: GRAPHING LINEAR EQUATIONS

There are two common procedures that are used to draw the line represented by a linear equation. The first one is called the slope-intercept method and involves using the slope and intercept given in the equation.

If the equation is given in the form $y = mx + b$, then m gives the rise over run value and the value b gives the point where the line crosses the y-axis, also known as the y-intercept.

Example 1

Given the following equations, identify the slope and the y-intercept.

1. $y = 2x - 3 \qquad \text{Slope } (m) = 2 \qquad y\text{-intercept } (b) = -3$
2. $y = \dfrac{1}{2}x - 1 \qquad \text{Slope } (m) = \dfrac{1}{2} \qquad y\text{-intercept } (b) = -1$
3. $y = -3x + 4 \qquad \text{Slope } (m) = -3 \qquad y\text{-intercept } (b) = 4$
4. $y = \dfrac{2}{3}x \qquad \text{Slope } (m) = \dfrac{2}{3} \qquad y\text{-intercept } (b) = 0$

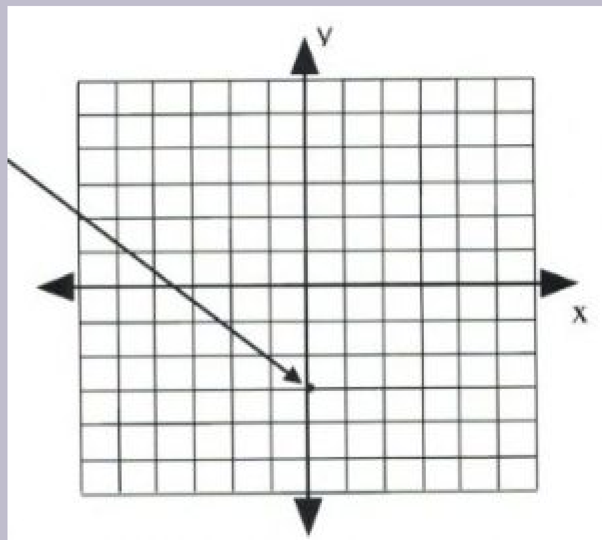
When graphing a linear equation using the slope-intercept method, start by using the value given for the y-intercept. After this point is marked, then identify other points using the slope.

This is shown in the following example.

Example 2

Graph the equation $y = 2x - 3$.

First, place a dot on the y-intercept, $y = -3$, which is placed on the coordinate $(0, -3)$.

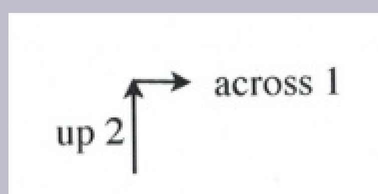


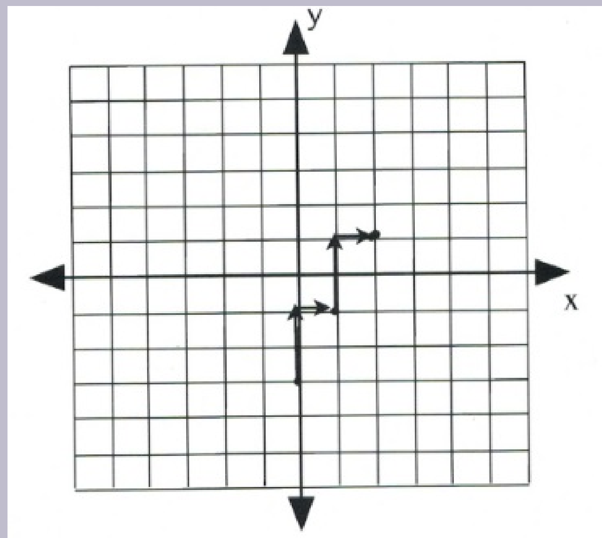
Now, place the next dot using the slope of 2.

A slope of 2 means that the line rises 2 for every 1 across.

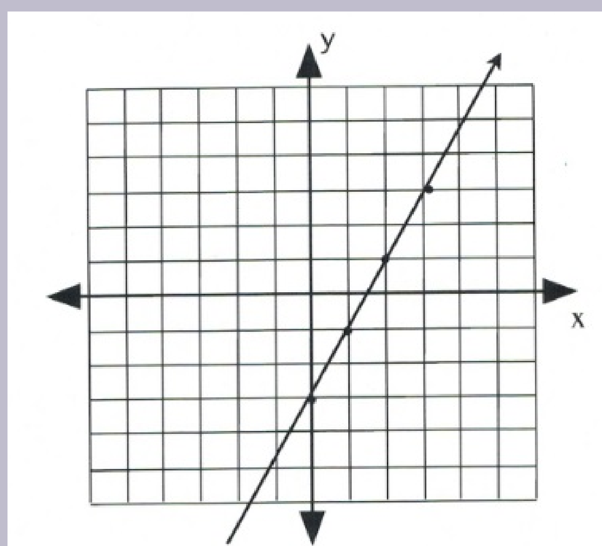
Simply, $m = 2$ is the same as $m = \dfrac{2}{1}$, where $\Delta y = 2$ and $\Delta x = 1$.

Placing these points on the graph becomes a simple counting exercise, which is done as follows:



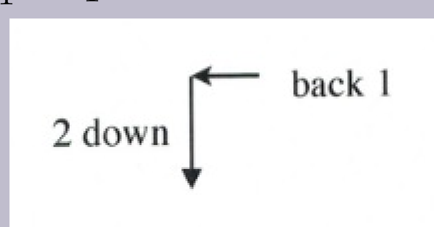


Once several dots have been drawn, draw a line through them, like so:



Note that dots can also be drawn in the reverse of what has been drawn here.

Slope is 2 when rise over run is $\frac{2}{1}$ or $\frac{-2}{-1}$, which would be drawn as follows:

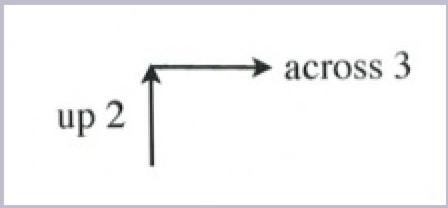


Example 3

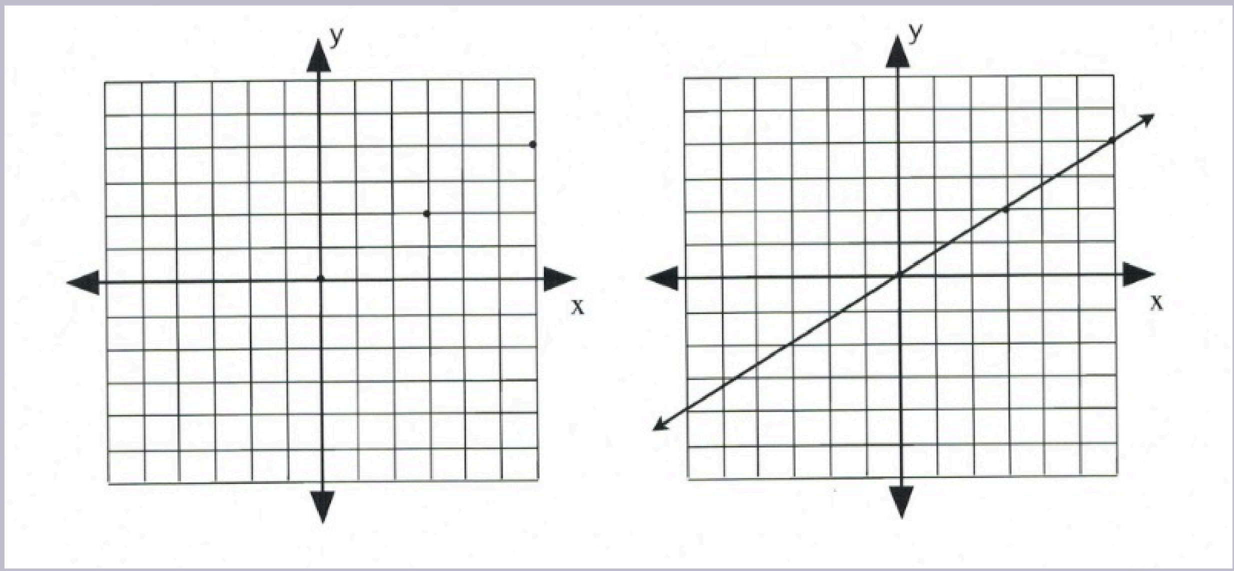
Graph the equation $y = \frac{2}{3}x$.

First, place a dot on the y-intercept, $(0, 0)$.

Now, place the dots according to the slope, $\frac{2}{3}$.



This will generate the following set of dots on the graph. All that remains is to draw a line through the dots.



The second method of drawing lines represented by linear equations and functions is to identify the two intercepts of the linear equation. Specifically, find x when $y = 0$ and find y when $x = 0$.

Example 4

Graph the equation $2x + y = 6$.

To find the first coordinate, choose $x = 0$.

This yields:

$$\begin{array}{rcrcl} 2(0) & + & y & = & 6 \\ & & y & = & 6 \end{array}$$

Coordinate is $(0, 6)$.

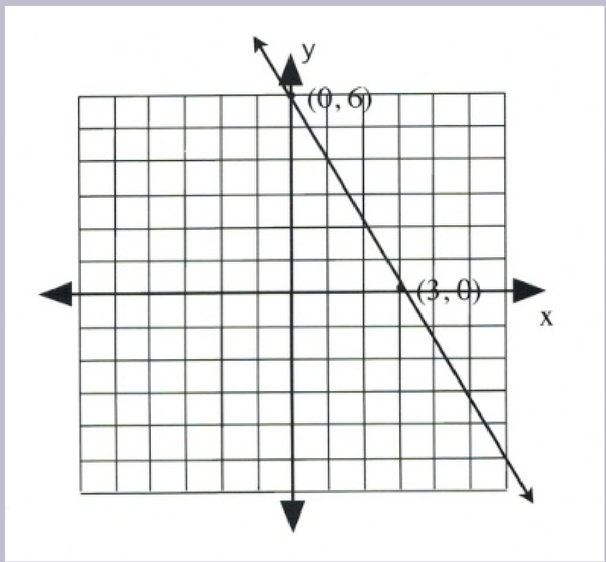
Now choose $y = 0$.

This yields:

$$\begin{array}{rcrcl} 2x & + & 0 & = & 6 \\ & & 2x & = & 6 \\ & & x & = & \frac{6}{2} \text{ or } 3 \end{array}$$

Coordinate is $(3, 0)$.

Draw these coordinates on the graph and draw a line through them.



Example 5

Graph the equation $x + 2y = 4$.

To find the first coordinate, choose $x = 0$.

This yields:

$$\begin{array}{rcrcl} (0) & + & 2y & = & 4 \\ & & y & = & \frac{4}{2} \text{ or } 2 \end{array}$$

Coordinate is $(0, 2)$.

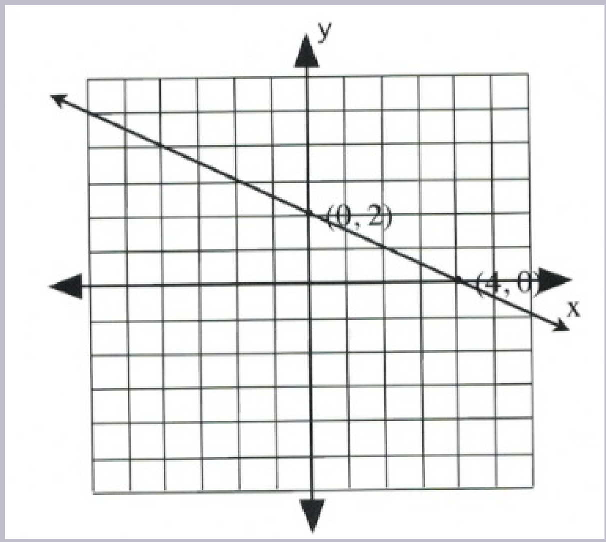
Now choose $y = 0$.

This yields:

$$\begin{array}{rcrcl} x & + & 2(0) & = & 4 \\ & & x & = & 4 \end{array}$$

Coordinate is $(4, 0)$.

Draw these coordinates on the graph and draw a line through them.



Example 6

Graph the equation $2x + y = 0$.

To find the first coordinate, choose $x = 0$.

This yields:

$$\begin{array}{rcrcr} 2(0) & + & y & = & 0 \\ & & y & = & 0 \end{array}$$

Coordinate is $(0, 0)$.

Since the intercept is $(0, 0)$, finding the other intercept yields the same coordinate. In this case, choose any value of convenience.

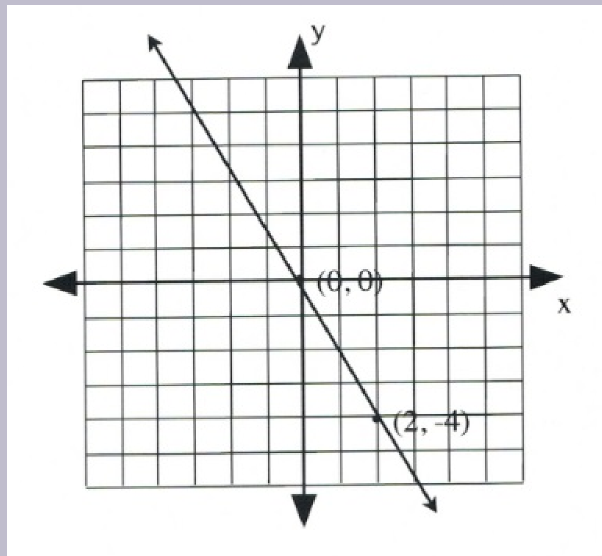
Choose $x = 2$.

This yields:

$$\begin{array}{rcrcr} 2(2) & + & y & = & 0 \\ 4 & + & y & = & 0 \\ -4 & & & & -4 \\ & & y & = & -4 \end{array}$$

Coordinate is $(2, -4)$.

Draw these coordinates on the graph and draw a line through them.



Questions

For questions 1 to 10, sketch each linear equation using the slope-intercept method.

1. $y = -\frac{1}{4}x - 3$
2. $y = \frac{3}{2}x - 1$
3. $y = -\frac{5}{4}x - 4$
4. $y = -\frac{3}{5}x + 1$
5. $y = -\frac{4}{3}x + 2$
6. $y = \frac{5}{3}x + 4$
7. $y = \frac{3}{2}x - 5$
8. $y = -\frac{2}{3}x - 2$
9. $y = -\frac{4}{5}x - 3$
10. $y = \frac{1}{2}x$

For questions 11 to 20, sketch each linear equation using the x- and y-intercepts.

11. $x + 4y = -4$
12. $2x - y = 2$
13. $2x + y = 4$
14. $3x + 4y = 12$
15. $2x - y = 2$
16. $4x + 3y = -12$
17. $x + y = -5$
18. $3x + 2y = 6$
19. $x - y = -2$
20. $4x - y = -4$

For questions 21 to 28, sketch each linear equation using any method.

21. $y = -\frac{1}{2}x + 3$
22. $y = 2x - 1$
23. $y = -\frac{5}{4}x$
24. $y = -3x + 2$
25. $y = -\frac{3}{2}x + 1$
26. $y = \frac{1}{3}x - 3$
27. $y = \frac{3}{2}x + 2$
28. $y = 2x - 2$

For questions 29 to 40, reduce and sketch each linear equation using any method.

29. $y + 3 = -\frac{4}{5}x + 3$
30. $y - 4 = \frac{1}{2}x$
31. $x + 5y = -3 + 2y$
32. $3x - y = 4 + x - 2y$
33. $4x + 3y = 5(x + y)$
34. $3x + 4y = 12 - 2y$
35. $2x - y = 2 - y$ (tricky)
36. $7x + 3y = 2(2x + 2y) + 6$
37. $x + y = -2x + 3$
38. $3x + 4y = 3y + 6$

39. $2(x+y)=-3(x+y)+5$
40. $9x-y=4x+5$

Answers to odd questions.

1. $m=2$
3. $m=4$
5. $$\begin{array}{rrrrrrrl} & x & - & y & = & 4 & & \\ -x & & & & & -x & & \\ & & & (-y & = & -x & + & 4)\ (-1) \\ & & & y & = & x & - & 4 \\ & & & m & = & 1 & & \end{array}$$
7. $$\begin{array}{rcl} y & = & \frac{1}{3}x \\ \therefore m & = & \frac{1}{3} \\ m_{\perp} & = & -1 \div \frac{1}{3} \text{ or} \\ m_{\perp} & = & -3 \end{array}$$
9. $$\begin{array}{rcl} m & = & -\frac{1}{3} \\ m_{\perp} & = & -1 \div -\frac{1}{3} \\ m_{\perp} & = & -1 \cdot -\frac{3}{1} = 3 \end{array}$$
11. $$\begin{array}{rrrrrrrr} & x & - & 3y & = & -6 & & \\ -x & & & & & -x & & \\ & & & \frac{-3y}{-3} & = & \frac{-x}{-3} & - & \frac{6}{-3} \\ & & & y & = & \frac{1}{3}x & + & 2 \\ & & & m_{\perp} & = & -1 & \div & \frac{1}{3} \\ & & & m_{\perp} & = & -3 & & \end{array}$$

13. $$m = \frac{2}{5}$$

$$\begin{array}{rcrcrcr} y & - & y_1 & = & m(x & - & x_1) \\ y & - & 4 & = & \frac{2}{5}(x & - & 1) \\ y & - & 4 & = & \frac{2}{5}x & - & \frac{2}{5} \\ & + & 4 & & & + & 4 \\ & & y & = & \frac{2}{5}x & + & \frac{18}{5} \end{array}$$

15. $$m = \frac{1}{2}$$

$$\begin{array}{rcrcrcr} y & - & y_1 & = & m(x & - & x_1) \\ y & - & 4 & = & \frac{1}{2}(x & - & 3) \\ y & - & 4 & = & \frac{1}{2}x & - & \frac{3}{2} \\ & + & 4 & & & + & 4 \\ & & y & = & \frac{1}{2}x & + & \frac{5}{2} \end{array}$$

17. $$m = -\frac{3}{5}$$

$$\begin{array}{rcrcrcr} y & - & y_1 & = & m(x & - & x_1) \\ y & - & 3 & = & -\frac{3}{5}(x & - & 2) \\ y & - & 3 & = & -\frac{3}{5}x & + & \frac{6}{5} \\ & + & 3 & & & + & 3 \\ & & y & = & -\frac{3}{5}x & + & \frac{21}{5} \end{array}$$

19.
$$\begin{array}{rrrrrrrrr} -x & + & y & = & 1 & & & & \\ +x & & & & +x & & & & \\ & & y & = & x & + & 1 & & \\ & & \therefore m & = & 1 & & & & \end{array}$$

$$\begin{array}{rrrrrrrrr} y & - & y_1 & = & m(x & - & x_1) & & \\ y & - & -5 & = & 1(x & - & 1) & & \\ y & + & 5 & = & x & - & 1 & & \\ -y & - & 5 & & -y & - & 5 & & \\ & & 0 & = & x & - & y & - & 6 \end{array}$$

21.
$$\begin{array}{rrrrrrr} 5x & + & y & = & -3 & & \\ -5x & & & & -5x & & \\ & & y & = & -5x & - & 3 \\ & & \therefore m & = & -5 & & \end{array}$$

$$\begin{array}{rrrrrrrrrr} y & - & y_1 & = & m(x & - & x_1) & & & \\ y & - & 2 & = & -5(x & - & 5) & & & \\ y & - & 2 & = & -5x & + & 25 & & & \\ -y & + & 2 & & -y & + & 2 & & & \\ & & (0 & = & -5x & - & y & + & 27) & (-1) \\ & & 0 & = & 5x & + & y & - & 27 & \end{array}$$

23.
$$\begin{array}{rrrrr} -4x & + & y & = & 0 \\ +4x & & & & +4x \\ & & y & = & 4x \\ & & \therefore m & = & 4 \end{array}$$

$$\begin{array}{rrrrrrrrr} y & - & y_1 & = & m(x & - & x_1) & & \\ y & - & 2 & = & 4(x & - & 4) & & \\ y & - & 2 & = & 4x & - & 16 & & \\ -y & + & 2 & & -y & + & 2 & & \\ & & 0 & = & 4x & - & y & - & 14 \end{array}$$

25. $y = -3$

27. $x = -3$

29. $y = -1$

31. $x = -2$

33. $y = 3$

35. $x = 5$

CHAPTER 2.3: PERPENDICULAR AND PARALLEL LINES

Perpendicular, parallel, horizontal, and vertical lines are special lines that have properties unique to each type. Parallel lines, for instance, have the same slope, whereas perpendicular lines are the opposite and have negative reciprocal slopes. Vertical lines have a constant x-value, and horizontal lines have a constant y-value.

Two equations govern perpendicular and parallel lines:

For parallel lines, the slope of the first line is the same as the slope for the second line. If the slopes of these two lines are called m_1 and m_2, then $m_1 = m_2$.

$$\text{The rule for parallel lines is } m_1 = m_2$$

Perpendicular lines are slightly more difficult to understand. If one line is rising, then the other must be falling, so both lines have slopes going in opposite directions. Thus, the slopes will always be negative to one another. The other feature is that the slope at which one is rising or falling will be exactly flipped for the other one. This means that the slopes will always be negative reciprocals to each other. If the slopes of these two lines are called m_1 and m_2, then $m_1 = \dfrac{-1}{m_2}$.

$$\text{The rule for perpendicular lines is } m_1 = \frac{-1}{m_2}$$

Example 1

Find the slopes of the lines that are parallel and perpendicular to $y = 3x + 5$.

The parallel line has the identical slope, so its slope is also 3.

The perpendicular line has the negative reciprocal to the other slope, so it is $-\dfrac{1}{3}$.

Example 2

Find the slopes of the lines that are parallel and perpendicular to $y = -\frac{2}{3}x - 4$.

The parallel line has the identical slope, so its slope is also $-\frac{2}{3}$.

The perpendicular line has the negative reciprocal to the other slope, so it is $\frac{3}{2}$.

Typically, questions that are asked of students in this topic are written in the form of "Find the equation of a line passing through point (x, y) that is perpendicular/parallel to $y = mx + b$." The first step is to identify the slope that is to be used to solve this equation, and the second is to use the described methods to arrive at the solution like previously done. For instance:

Example 3

Find the equation of the line passing through the point $(2, 4)$ that is parallel to the line $y = 2x - 3$.

The first step is to identify the slope, which here is the same as in the given equation, $m = 2$.

Now, simply use the methods from before:

$$\begin{aligned} m &= \frac{y - y_1}{x - x_1} \\ 2 &= \frac{y - 4}{x - 2} \end{aligned}$$

Clearing the fraction by multiplying both sides by $(x - 2)$ leaves:

$$2(x - 2) = y - 4 \text{ or } 2x - 4 = y - 4$$

Now put this equation in one of the three forms. For this example, use the standard form:

$$\begin{array}{rcrcrcr} 2x & - & 4 & = & y & - & 4 \\ -y & + & 4 & & -y & + & 4 \\ 2x & - & y & = & 0 & & \end{array}$$

Example 4

Find the equation of the line passing through the point $(1, 3)$ that is perpendicular to the line $y = \frac{3}{2}x + 4$.

The first step is to identify the slope, which here is the negative reciprocal to the one in the given equation, so $m = -\frac{2}{3}$.

Now, simply use the methods from before:

$$\begin{array}{rcl} m & = & \dfrac{y - y_1}{x - x_1} \\ -\dfrac{2}{3} & = & \dfrac{y - 3}{x - 1} \end{array}$$

First, clear the fraction by multiplying both sides by $3(x - 1)$. This leaves:

$$-2(x - 1) = 3(y - 3)$$

which reduces to:

$$-2x + 2 = 3y - 9$$

Now put this equation in one of the three forms. For this example, choose the general form:

$$\begin{array}{rcrcrcrcr} -2x & & & + & 2 & = & 3y & - & 9 \\ & & -3y & + & 9 & & -3y & + & 9 \\ -2x & - & 3y & + & 11 & = & 0 & & \end{array}$$

For the general form, the coefficient in front of the x must be positive. So for this equation, multiply the entire equation by -1 to make $-2x$ positive.

$$(-2x - 3y + 11 = 0)(-1)$$
$$2x + 3y - 11 = 0$$

Questions that are looking for the vertical or horizontal line through a given point are the easiest to do and the most commonly confused.

Vertical lines always have a single x-value, yielding an equation like $x = \text{constant}$.

Horizontal lines always have a single y-value, yielding an equation like $y = \text{constant}$.

Example 5

Find the equation of the vertical and horizontal lines through the point $(-2, 4)$.

The vertical line has the same x-value, so the equation is $x = -2$.

The horizontal line has the same y-value, so the equation is $y = 4$.

Questions

For questions 1 to 6, find the slope of any line that would be parallel to each given line.

1. $y = 2x + 4$
2. $y = -\dfrac{2}{3}x + 5$
3. $y = 4x - 5$
4. $y = -10x - 5$
5. $x - y = 4$
6. $6x - 5y = 20$

For questions 7 to 12, find the slope of any line that would be perpendicular to each given line.

7. $y = \frac{1}{3}x$
8. $y = -\frac{1}{2}x - 1$
9. $y = -\frac{1}{3}x$
10. $y = \frac{4}{5}x$
11. $x - 3y = -6$
12. $3x - y = -3$

For questions 13 to 18, write the slope-intercept form of the equation of each line using the given point and line.

13. (1, 4) and parallel to $y = \frac{2}{5}x + 2$
14. (5, 2) and perpendicular to $y = \frac{1}{3}x + 4$
15. (3, 4) and parallel to $y = \frac{1}{2}x - 5$
16. (1, −1) and perpendicular to $y = -\frac{3}{4}x + 3$
17. (2, 3) and parallel to $y = -\frac{3}{5}x + 4$
18. (−1, 3) and perpendicular to $y = -3x - 1$

For questions 19 to 24, write the general form of the equation of each line using the given point and line.

19. (1, −5) and parallel to $-x + y = 1$
20. (1, −2) and perpendicular to $-x + 2y = 2$
21. (5, 2) and parallel to $5x + y = -3$
22. (1, 3) and perpendicular to $-x + y = 1$
23. (4, 2) and parallel to $-4x + y = 0$
24. (3, −5) and perpendicular to $3x + 7y = 0$

For questions 25 to 36, write the equation of either the horizontal or the vertical line that runs through each point.

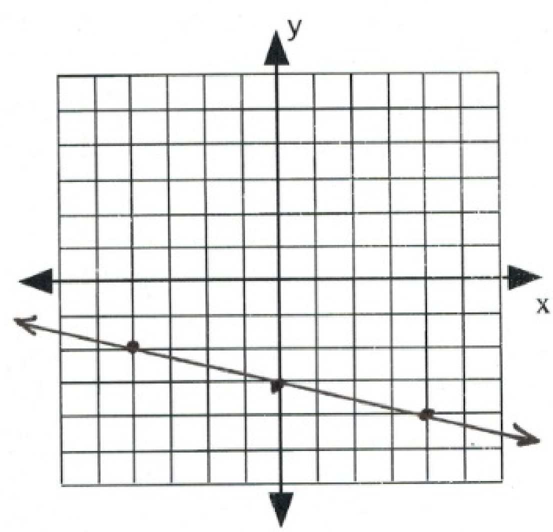
25. Horizontal line through (4, −3)
26. Vertical line through (−5, 2)
27. Vertical line through (−3,1)
28. Horizontal line through (−4, 0)

29. Horizontal line through $(-4, -1)$
30. Vertical line through $(2, 3)$
31. Vertical line through $(-2, -1)$
32. Horizontal line through $(-5, -4)$
33. Horizontal line through $(4, 3)$
34. Vertical line through $(-3, -5)$
35. Vertical line through $(5, 2)$
36. Horizontal line through $(5, -1)$

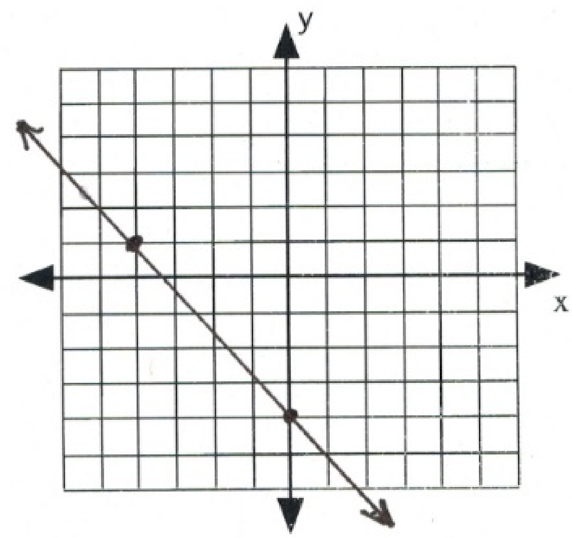
Answers to odd questions

For questions 1 to 9, sketch the linear equation using the slope intercept method.

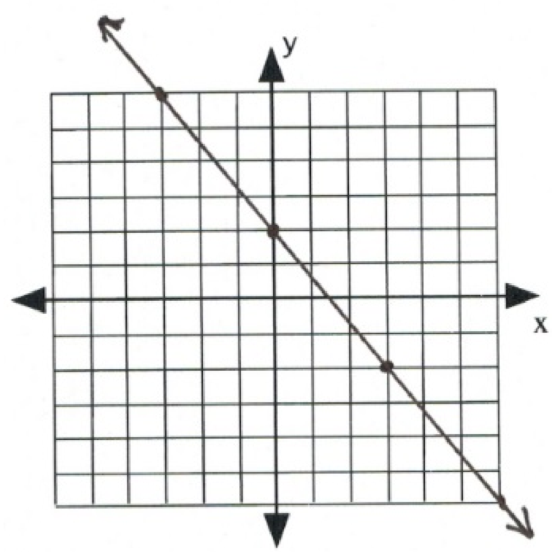
1. $y = -\dfrac{1}{4}x - 3$



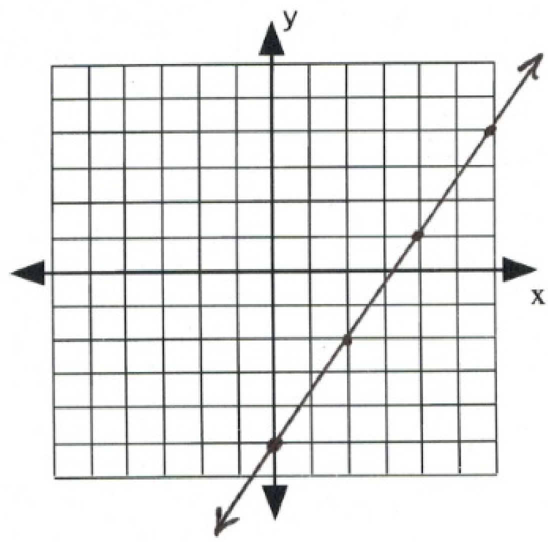
3. $y = -\dfrac{5}{4}x - 4$



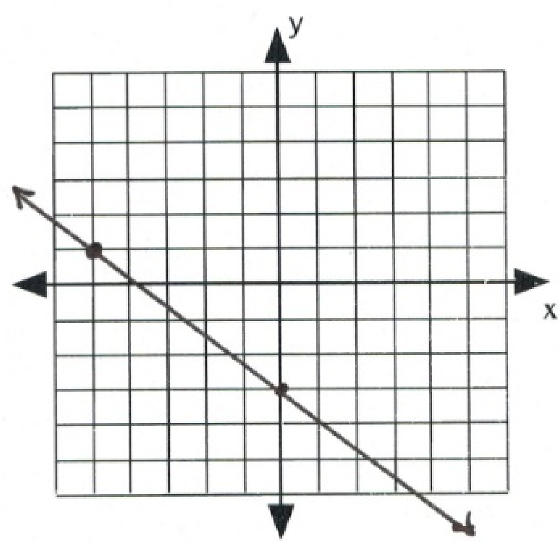
5. $y = -\frac{4}{3}x + 2$



7. $y = \frac{3}{2}x - 5$

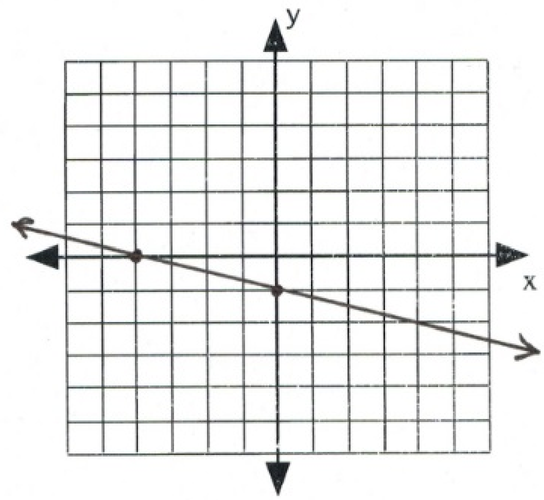


9. $y = -\frac{4}{5}x - 3$

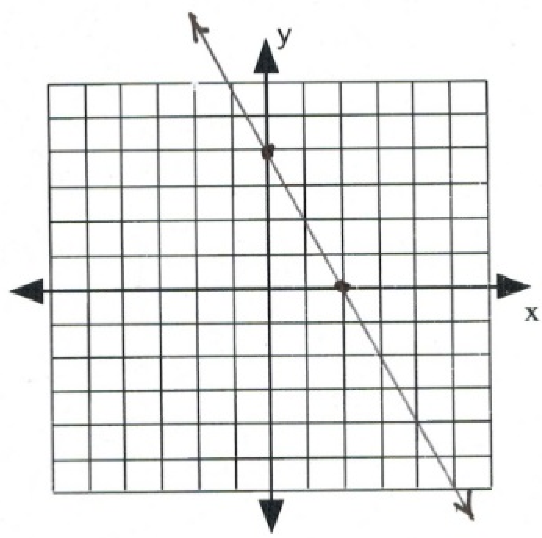


For questions 11 to 19, sketch the linear equation using the x and y intercepts.

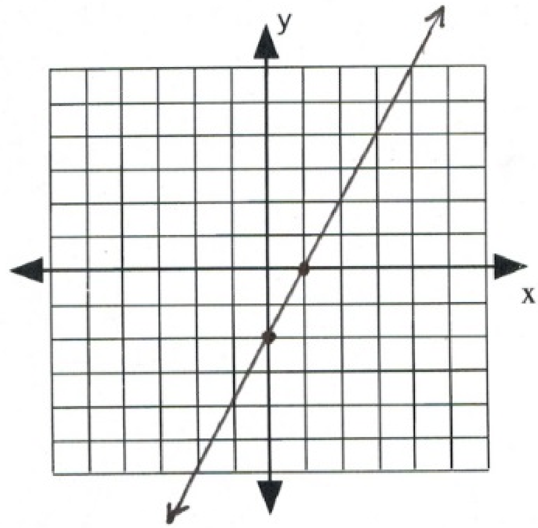
11. $x + 4y = -4$



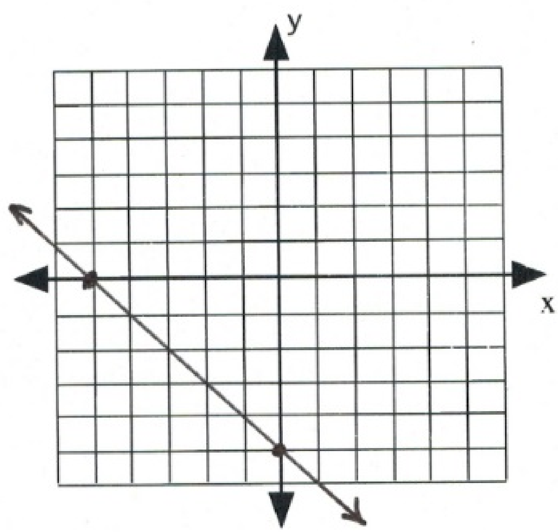
13. $2x + y = 4$



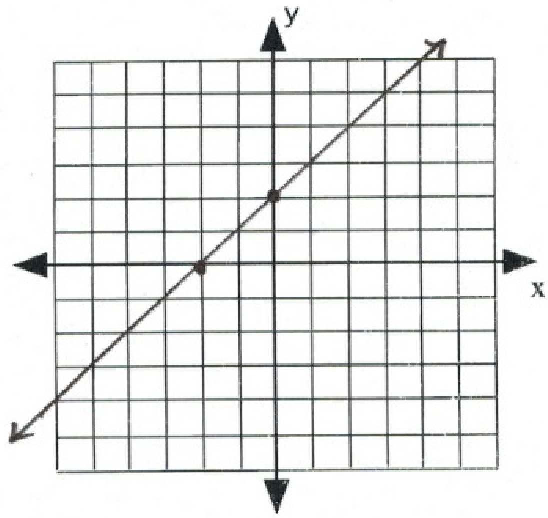
15. $2x - y = 2$



17. $x + y = -5$

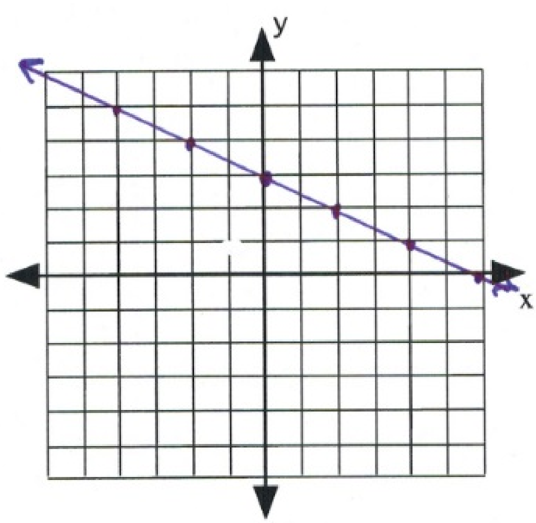


19. $x - y = -2$

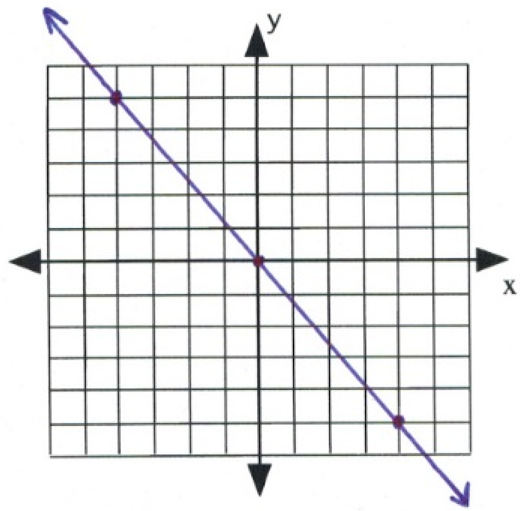


For questions 21 to 27, sketch the linear equation using any method.

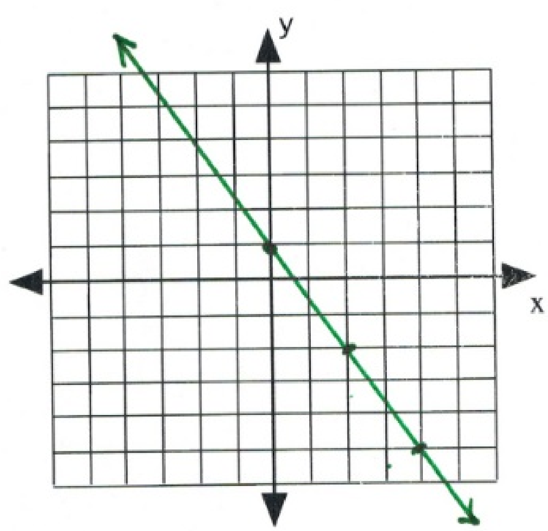
21. $y = -\dfrac{1}{2}x + 3$



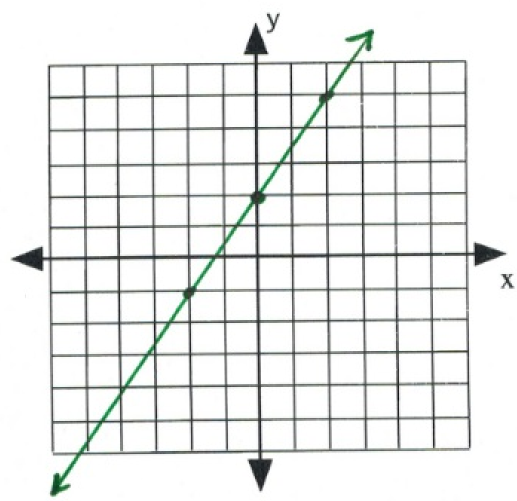
23. $y = -\dfrac{5}{4}x$



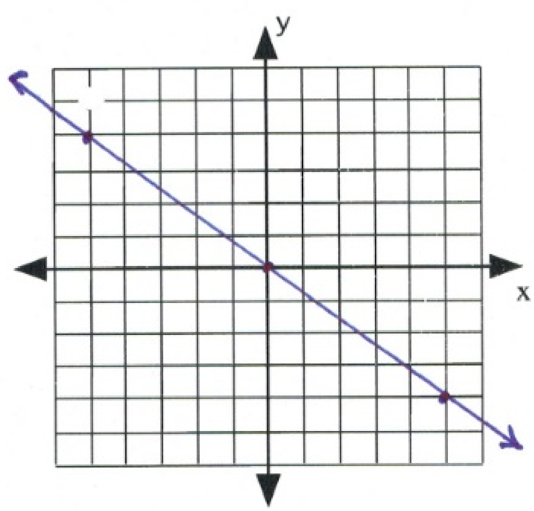
25. $y = -\frac{3}{2}x + 1$



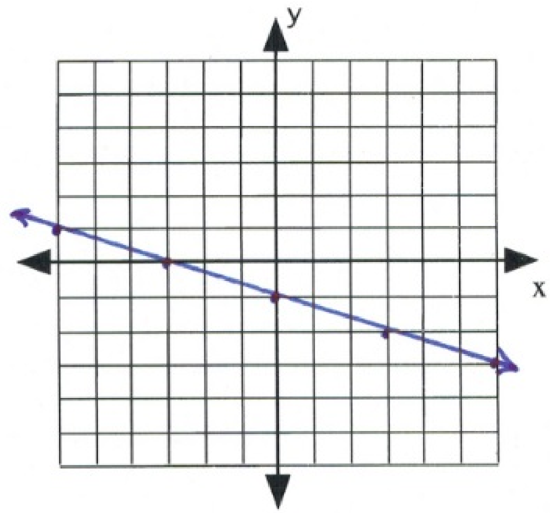
27. $y = \frac{3}{2}x + 2$



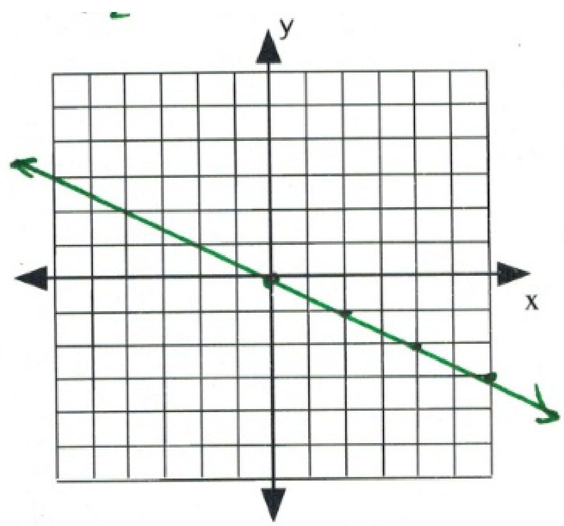
29.
$$\begin{array}{rrrrrrr} y & + & 3 & = & -\frac{4}{5}x & + & 3 \\ & - & 3 & & & - & 3 \\ & & y & = & -\frac{4}{5}x & & \end{array}$$



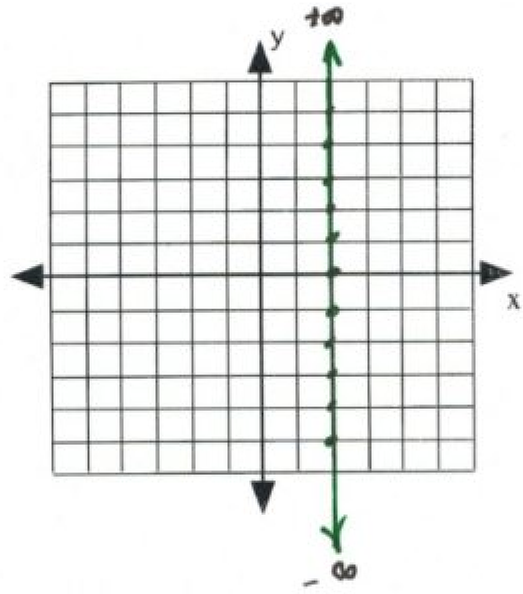
31.
$$\begin{array}{rrrrrrr} x & + & 5y & = & -3 & + & 2y \\ -x & - & 2y & & -x & - & 2y \\ & & 3y & = & -x & - & 3 \\ & & y & = & -\frac{1}{3}x & - & 1 \end{array}$$



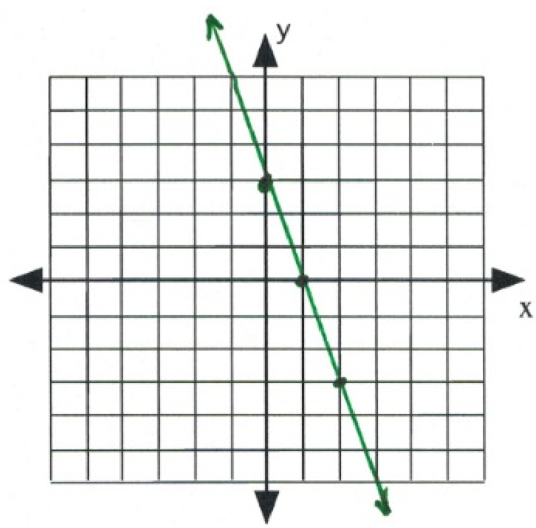
33.
$$\begin{array}{rcrcrcrcr} 4x & + & 3y & = & 5(x & + & y) \\ 4x & + & 3y & = & 5x & + & 5y \\ -4x & - & 5y & & -4x & - & 5y \\ & & \dfrac{-2y}{-2} & = & \dfrac{x}{-2} & & \\ & & y & = & -\dfrac{1}{2}x & & \end{array}$$



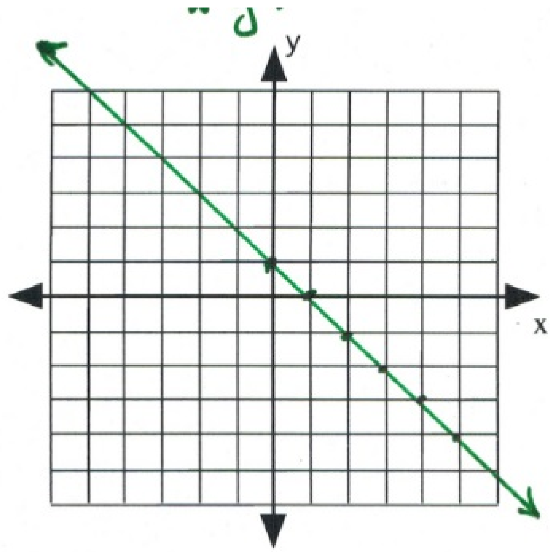
35.
$$\begin{array}{rcrcrcr} 2x & - & y & = & 2 & - & y \\ & + & y & & & + & y \\ & & 2x & = & 2 & & \\ & & x & = & 1 & & \end{array}$$



$$
\begin{array}{rrrrrrrr}
 & x & + & y & = & -2x & + & 3 \\
37. & -x & & & & -\ x & & \\
 & & & y & = & -3x & + & 3
\end{array}
$$



$$
\begin{array}{rrrrrrrrrrr}
 & 2(x & + & y) & = & -3(x & + & y) & + & 5 \\
 & 2x & + & 2y & = & -3x & - & 3y & + & 5 \\
39. & -2x & + & 3y & & -2x & + & 3y & & \\
 & & & 5y & = & -5x & + & 5 & & \\
 & & & y & = & -x & + & 1 & &
\end{array}
$$


y
x

CHAPTER 2.4: INTRODUCTION TO SYSTEMS OF EQUATIONS

Figure 1. Enigma machines like this one, once owned by Italian dictator Benito Mussolini, were used by government and military officials for enciphering and deciphering top-secret communications during World War II. (credit: Dave Addey, Flickr)

By 1943, it was obvious to the Nazi regime that defeat was imminent unless it could build a weapon with unlimited destructive power, one that had never been seen before in the history of the world. In September, Adolf Hitler ordered German scientists to begin building an atomic bomb. Rumors and whispers began to spread from across the ocean. Refugees and diplomats told of the experiments happening in Norway. However, Franklin D. Roosevelt wasn't sold, and even doubted British Prime Minister Winston Churchill's warning. Roosevelt wanted undeniable proof. Fortunately, he soon received the proof he wanted when a group of mathematicians cracked the "Enigma" code, proving beyond a doubt that Hitler was building an atomic bomb. The next day, Roosevelt gave the order that the United States begin work on the same.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

CHAPTER 2.5: SYSTEMS OF LINEAR EQUATIONS: TWO VARIABLES

Learning Objectives

In this section, you will:

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

Figure 1. (credit: Thomas Sørenes)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. A system of linear equations consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x + y &= \ 15 \\ 3x - y &= \ 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair (4, 7) is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$\begin{aligned} 2\left(4\right) + \left(7\right) &= 15 \text{True} \\ 3\left(4\right) - \left(7\right) &= 5 \ \ \text{True} \end{aligned}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same *y*-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different *y*-intercepts. There are no points common to both lines; hence, there is no solution to the system.

Types of Linear Systems

There are three types of systems of linear equations in two variables, and three types of solutions.

- An independent system has exactly one solution pair (x, y) . The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

(Figure) compares graphical representations of each type of system.

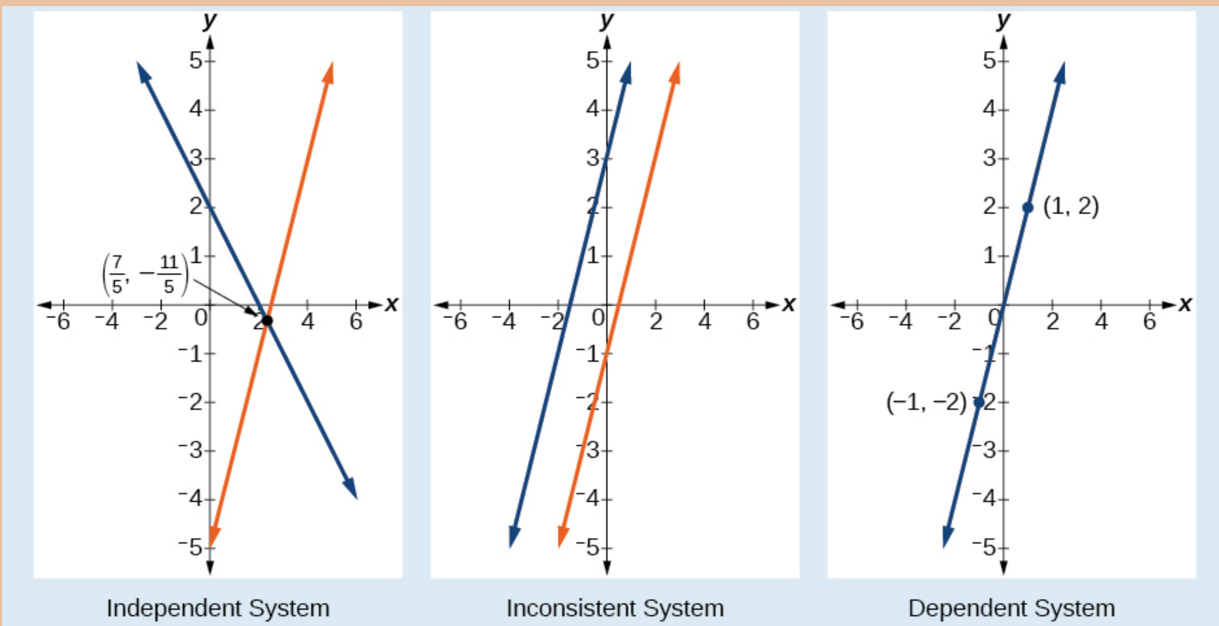


Figure 2.

How To

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution.

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$x + 3y = 8$$
$$2x - 9 = y$$

Show Solution

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned}(5) + 3(1) &= 8 \\ 8 &= 8 \qquad \text{True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \qquad \text{True}\end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See (Figure).

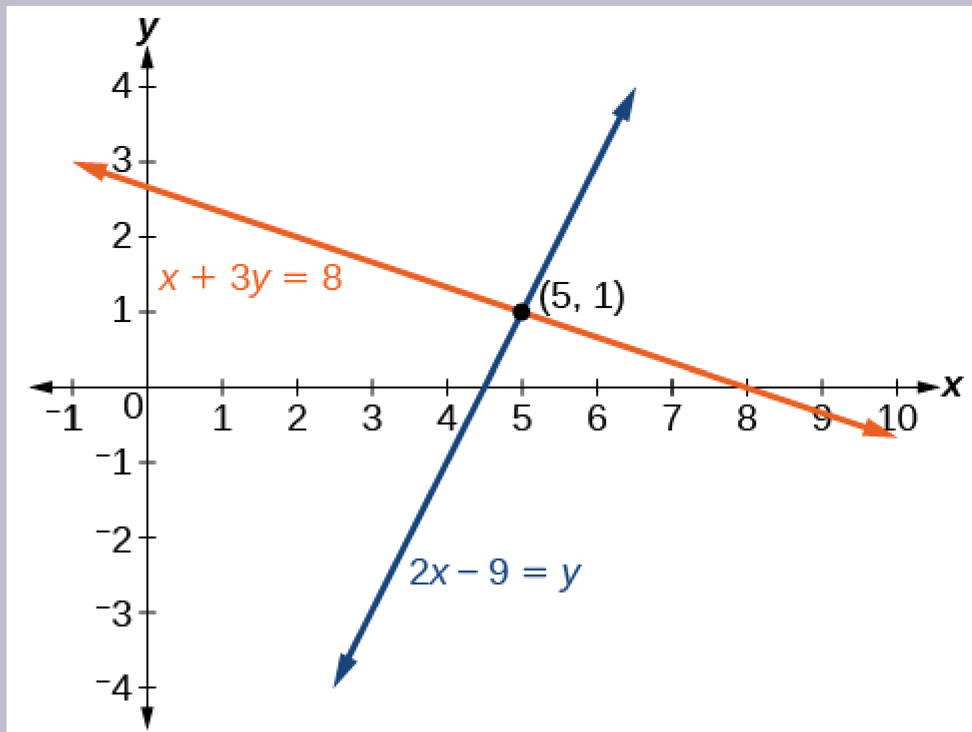


Figure 3.

Try It

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$\begin{aligned} 5x - 4y &= 20 \\ 2x + 1 &= 3y \end{aligned}$$

Show Solution

Not a solution.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Show Solution

Solve the first equation for y.

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for y.

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in (Figure).

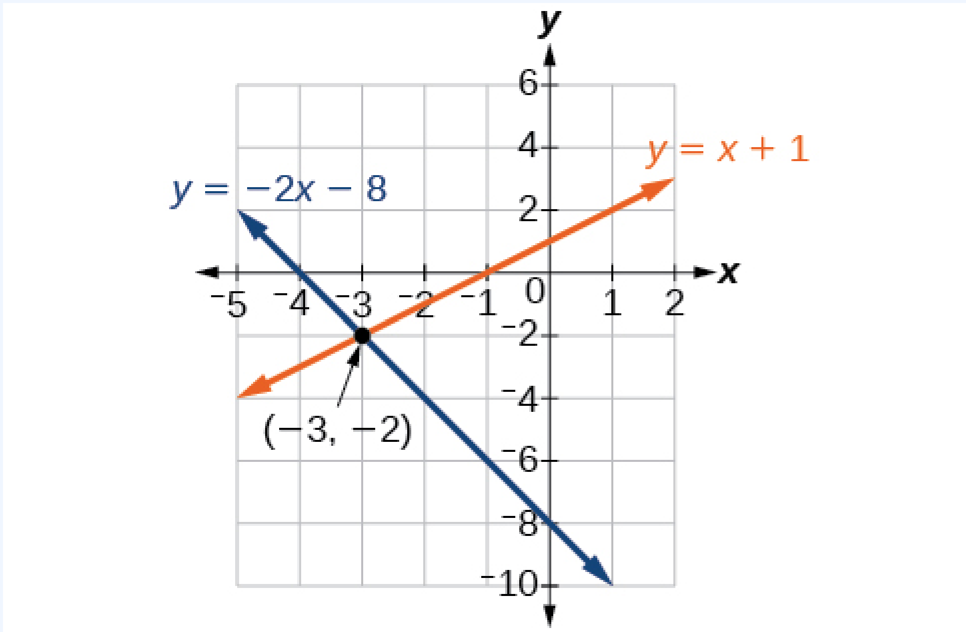


Figure 4.

The lines appear to intersect at the point $(-3, -2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{array}{ll} 2\left(-3\right)+\left(-2\right)=-8 & \\ \quad -8=-8 & \text{True} \\ \left(-3\right)-\left(-2\right)=-1 & \\ \quad -1=-1 & \text{True} \end{array}$$

The solution to the system is the ordered pair $(-3, -2)$, so the system is independent.

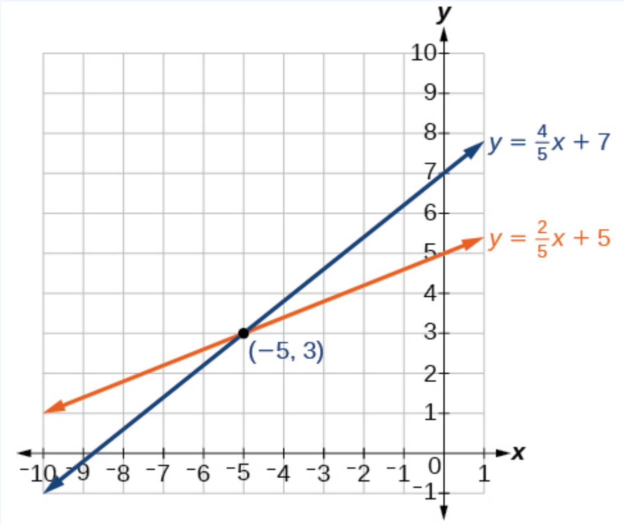
Try It

Solve the following system of equations by graphing.

Show Solution

$$\begin{aligned} 2x - 5y &= -25 \\ -4x + 5y &= 35 \end{aligned}$$

The solution to the system is the ordered pair $(-5, 3)$.



Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the substitution method, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How To

Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$-x+y=-5$$
$$2x-5y=1$$

Show Solution

First, we will solve the first equation for y.

$$-x+y=-5$$
$$y=x-5$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned} 2x - 5y &= 1 \\ 2x - 5(x-5) &= 1 \\ 2x - 5x + 25 &= 1 \\ -3x &= -24 \\ x &= 8 \end{aligned}$$

Now, we substitute $x = 8$ into the first equation and solve for y.

$$\begin{aligned} -(8) + y &= -5 \\ y &= 3 \end{aligned}$$

Our solution is $(8, 3)$.

Check the solution by substituting $(8, 3)$ into both equations.

$$\begin{aligned} -x + y &= -5 & \\ -(8) + (3) &= -5 & \text{True} \\ 2x - 5y &= 1 & \\ 2(8) - 5(3) &= 1 & \text{True} \end{aligned}$$

Try It

Solve the following system of equations by substitution.

$$x = y + 3$$
$$4 = 3x - 2y$$

Show Solution

$(-2, -5)$

Can the substitution method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of 1 or –1 so that we do not have to deal with fractions.

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How To

Given a system of equations, solve using the addition method.

1. Write both equations with x– and y-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Solving a System by the Addition Method

Solve the given system of equations by addition.

$x+2y=-1$
$-x+y=3$

Show Solution

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, –1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\frac{\begin{array}{r} x+2y=-1 \\ -x+y=3 \end{array}}{3y=2}$$

Now that we have eliminated $x,$ we can solve the resulting equation for $y.$

$$\begin{aligned} 3y&=2 \\ y&=\tfrac{2}{3} \end{aligned}$$

Then, we substitute this value for y into one of the original equations and solve for $x.$

$$\begin{aligned} -x+y&=3 \\ -x+\tfrac{2}{3}&=3 \\ -x&=3-\tfrac{2}{3} \\ -x&=\tfrac{7}{3} \\ x&=-\tfrac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3},\frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y&=-1 \\ \left(-\tfrac{7}{3}\right)+2\left(\tfrac{2}{3}\right)&= \\ -\tfrac{7}{3}+\tfrac{4}{3}&= \\ -\tfrac{3}{3}&= \\ -1&=-1 \qquad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See (Figure) to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

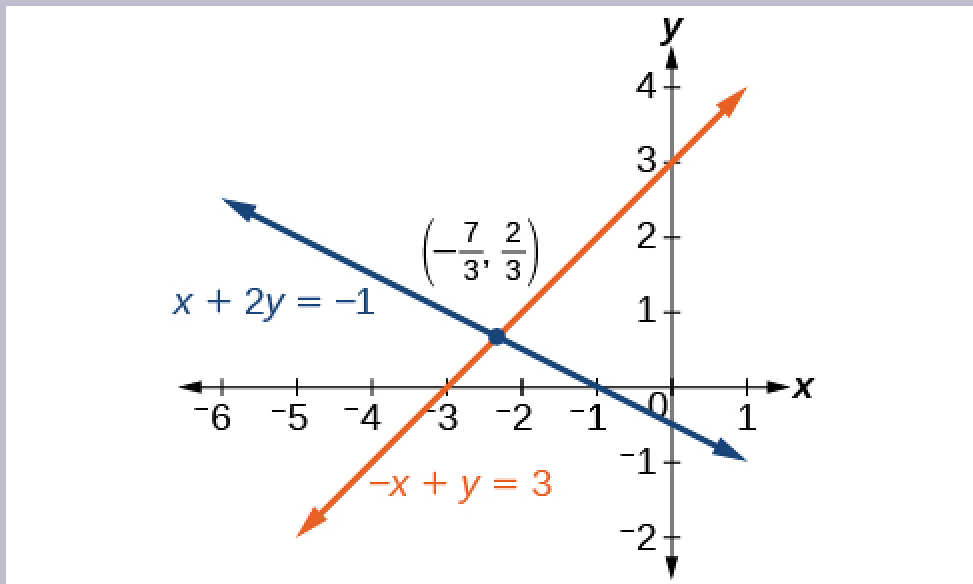


Figure 5.

Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$3x + 5y = -11$$

$$x - 2y = 11$$

Show Solution

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by $-3,$ the *x*-terms will add to zero.

$$\begin{array}{ll} x-2y=11 & \\ -3\left(x-2y\right)=-3\left(11\right) & \text{Multiply both sides by }-3. \\ -3x+6y=-33 & \text{Use the distributive property.} \end{array}$$

Now, let's add them.

$$\begin{array}{r} 3x+5y=-11 \\ -3x+6y=-33 \\ \hline \end{array}$$

$$\begin{array}{l} 11y=-44 \\ y=-4 \end{array}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.

$$\begin{array}{r} 3x+5y=-11 \\ 3x+5\left(-4\right)=-11 \\ 3x-20=-11 \\ 3x=9 \\ x=3 \end{array}$$

Our solution is the ordered pair $\left(3,-4\right)$. See (Figure). Check the solution in the original second equation.

$$\begin{array}{ll} x-2y=11 & \\ \left(3\right)-2\left(-4\right)=3+8 & \\ 11=11 & \text{True} \end{array}$$

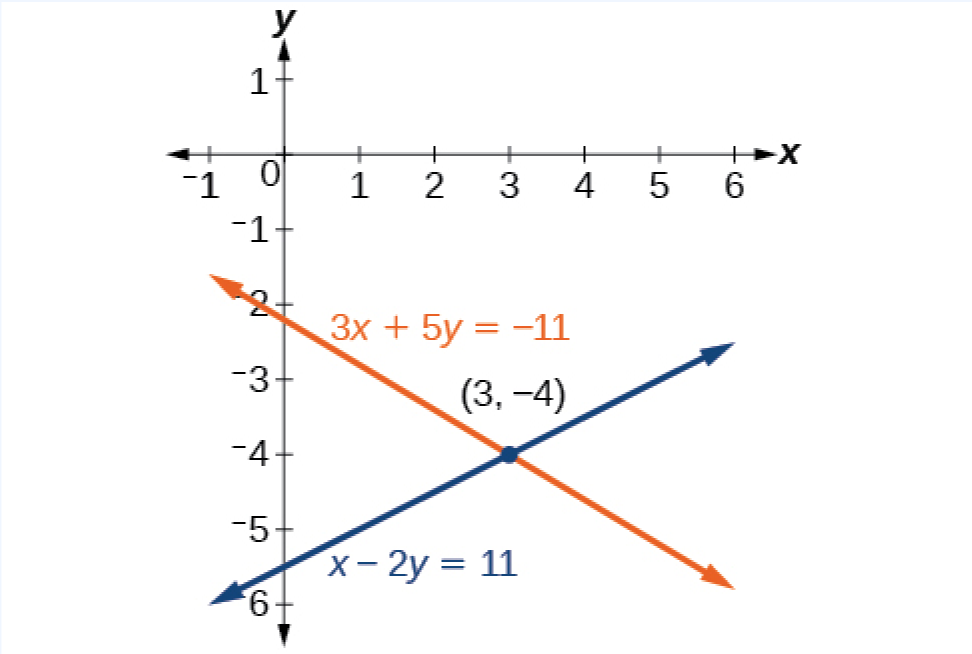


Figure 6.

Try It

Solve the system of equations by addition.

$$\begin{aligned} 2x - 7y &= 2 \\ 3x + y &= -20 \end{aligned}$$

Show Solution

$(-6, -2)$

Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$2x + 3y = -16$$
$$5x - 10y = 30$$

Show Solution

One equation has $2x$ and the other has $5x.$ The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by $2.$

$$-5\left(2x + 3y\right) = -5\left(-16\right)$$
$$-10x - 15y = 80$$
$$2\left(5x - 10y\right) = 2\left(30\right)$$
$$10x - 20y = 60$$

Then, we add the two equations together.

$$-10x - 15y = 80$$
$$10x - 20y = 60$$

$$-35y = 140$$
$$y = -4$$

Substitute $y = -4$ into the original first equation.

$$2x + 3\left(-4\right) = -16$$
$$2x - 12 = -16$$
$$2x = -4$$
$$x = -2$$

The solution is $\left(-2, -4\right).$ Check it in the other equation.

$$5x - 10y = 30$$
$$5\left(-2\right) - 10\left(-4\right) = 30$$
$$-10 + 40 = 30$$
$$30 = 30$$

See (Figure).

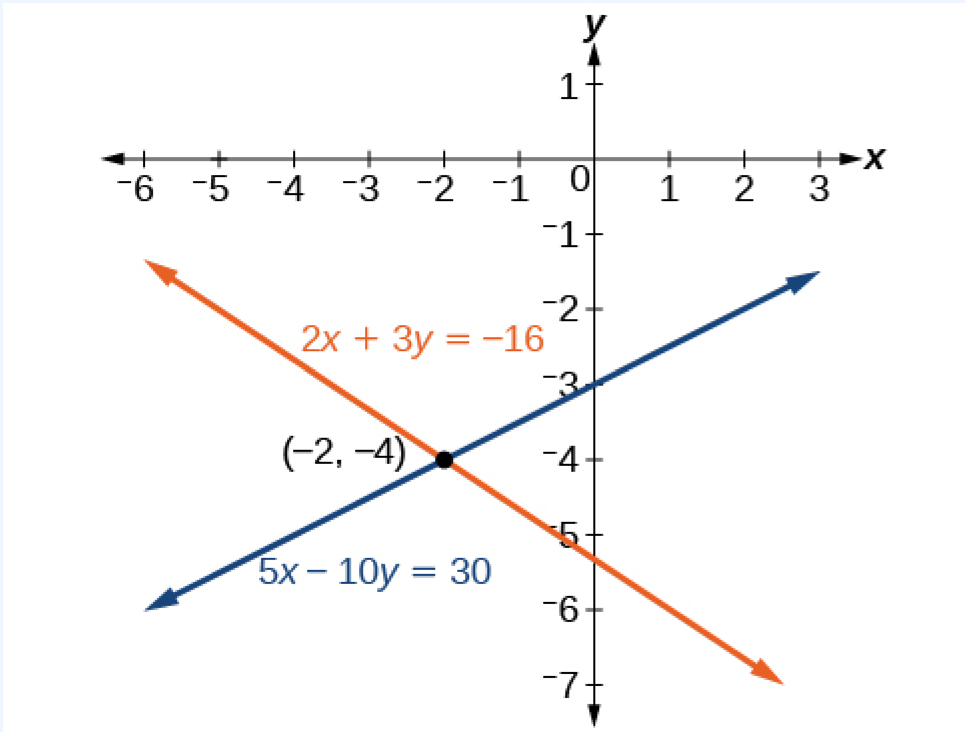


Figure 7.

Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$\frac{x}{3}+\frac{y}{6}=3$$
$$\frac{x}{2}-\frac{y}{4}=\ 1$$

Show Solution

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$6\left(\frac{x}{3}+\frac{y}{6}\right)=6\left(3\right)$$
$$2x+y=18$$
$$4\left(\frac{x}{2}-\frac{y}{4}\right)=4\left(1\right)$$
$$2x-y=4$$

Now multiply the second equation by -1 so that we can eliminate the *x*-variable.

$$-1\left(2x-y\right)=-1\left(4\right)$$
$$-2x+y=-4$$

Add the two equations to eliminate the *x*-variable and solve the resulting equation.

$$2x+y=18$$
$$-2x+y=-4$$

$$2y=14$$
$$y=7$$

Substitute $y=7$ into the first equation.

$$2x+\left(7\right)=18$$
$$2x=11$$
$$x=\frac{11}{2}$$
$$=5.5$$

The solution is $\left(\frac{11}{2},7\right)$. Check it in the other equation.

$$\frac{x}{2}-\frac{y}{4}=1$$
$$\frac{\frac{11}{2}}{2}-\frac{7}{4}=1$$
$$\frac{11}{4}-\frac{7}{4}=1$$
$$\frac{4}{4}=1$$

Try It

Solve the system of equations by addition.

$$2x + 3y = 8$$
$$3x + 5y = 10$$

Show Solution

$(10, -4)$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y -intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

Solving an Inconsistent System of Equations

Solve the following system of equations.

$$x = 9 - 2y$$
$$x + 2y = 13$$

Show Solution

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x + 2y &= 13 \\ (9 - 2y) + 2y &= 13 \\ 9 + 0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13.$ Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$x = 9 - 2y$$
$$2y = -x + 9$$
$$y = -\frac{1}{2}x + \frac{9}{2}$$

We then convert the second equation expressed to slope-intercept form.

$$x + 2y = 13$$
$$2y = -x + 13$$
$$y = -\frac{1}{2}x + \frac{13}{2}$$

Comparing the equations, we see that they have the same slope but different *y*-intercepts. Therefore, the lines are parallel and do not intersect.

$$y = -\frac{1}{2}x + \frac{9}{2}$$
$$y = -\frac{1}{2}x + \frac{13}{2}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in (Figure).

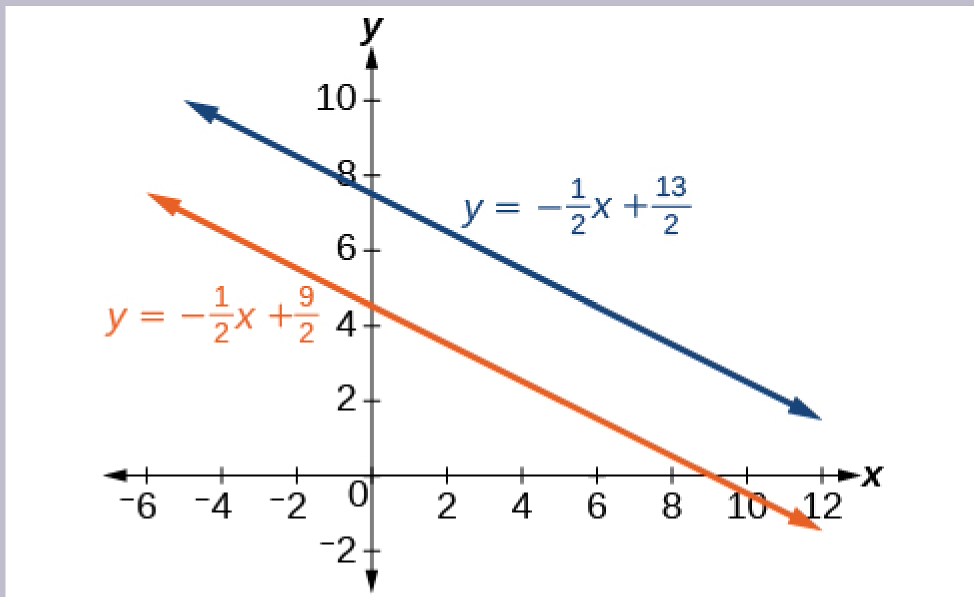


Figure 8.

Try It

Solve the following system of equations in two variables.

$$2y - 2x = 2$$
$$2y - 2x = 6$$

Show Solution

No solution. It is an inconsistent system.

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0 = 0$.

Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the addition method.

$$\begin{aligned} x + 3y &= 2 \\ 3x + 9y &= 6 \end{aligned}$$

Show Solution

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x -variable.

$$\begin{aligned} x + 3y &= 2 \\ (-3)(x + 3y) &= (-3)(2) \\ -3x - 9y &= -6 \end{aligned}$$

Now add the equations.

$$\begin{array}{rrl} & -3x - 9y & = -6 \\ + & 3x + 9y & = 6 \\ \hline & 0 & = 0 \end{array}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$x + 3y = 2$$
$$3y = -x + 2$$
$$y = -\frac{1}{3}x + \frac{2}{3}$$
$$3x + 9y = 6$$
$$9y = -3x + 6$$
$$y = -\frac{3}{9}x + \frac{6}{9}$$
$$y = -\frac{1}{3}x + \frac{2}{3}$$

See (Figure). Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x + \frac{2}{3}\right)$.

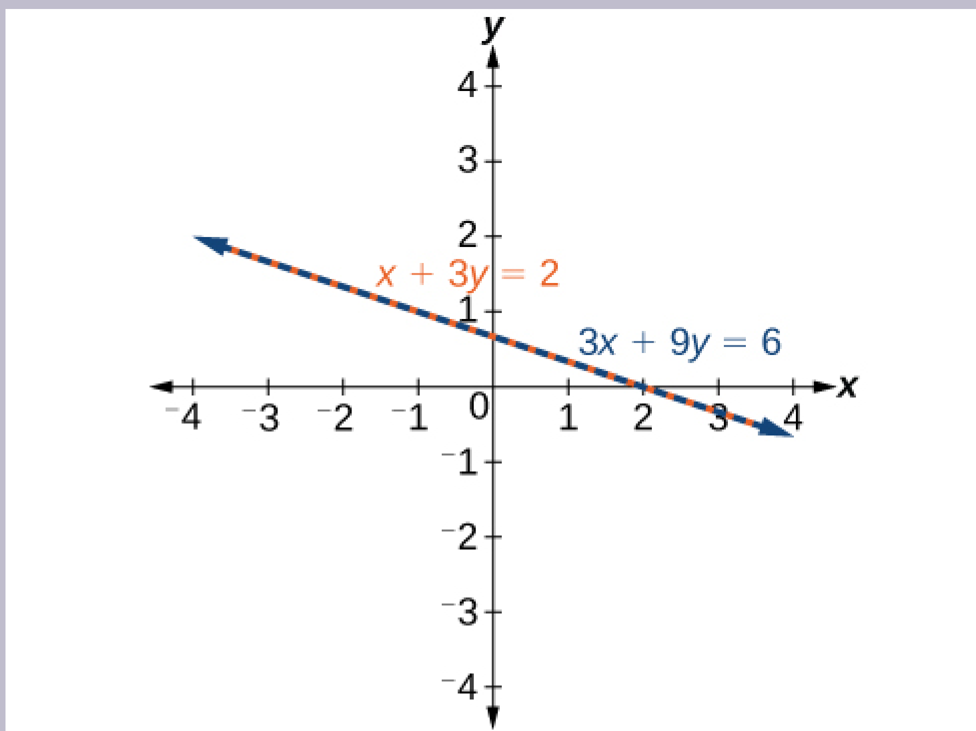


Figure 9.

Try It

Solve the following system of equations in two variables.

$$\begin{aligned} y - 2x &= 5 \\ -3y + 6x &= -15 \end{aligned}$$

Show Solution

The system is dependent so there are infinite solutions of the form $(x, 2x+5)$.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where $x =$ quantity and $p =$ price. The revenue function is shown in orange in (Figure).

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in (Figure). The x -axis represents quantity in hundreds of units. The *y*-axis represents either cost or revenue in hundreds of dollars.

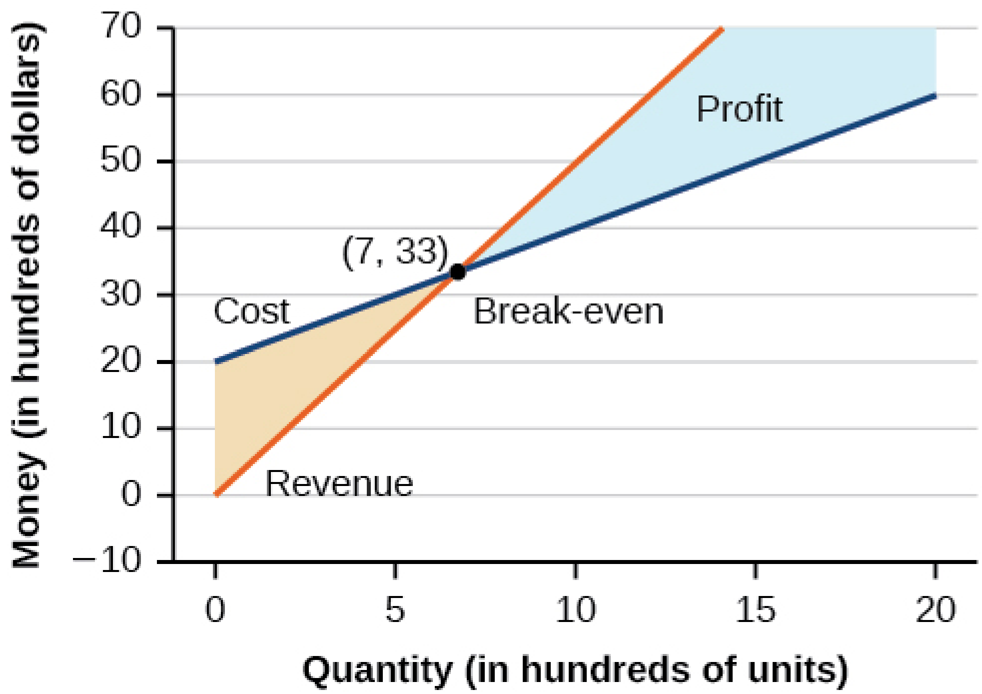


Figure 10.

The point at which the two lines intersect is called the break-even point. We can see from the graph that if 700 units are produced, the cost is $3,300 and the revenue is also $3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P\left(x\right)=R\left(x\right)-C\left(x\right)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C\left(x\right)=0.85x+35,000$ and the revenue function $R\left(x\right)=1.55x$, find the break-even point and the profit function.

Show Solution

Write the system of equations using y to replace function notation.

$$\begin{array}{l} y=0.85x+35,000 \\ y=1.55x \end{array}$$

Substitute the expression $0.85x+35,000$ from the first equation into the second equation and solve for x.

$$\begin{array}{r} 0.85x+35,000=1.55x \\ 35,000=0.7x \\ 50,000=x \end{array}$$

Then, we substitute $x=50,000$ into either the cost function or the revenue function.

$$1.55\left(50,000\right)=77,500$$

The break-even point is $\left(50,000,77,500\right)$.

The profit function is found using the formula $P\left(x\right)=R\left(x\right)-C\left(x\right)$.

$$\begin{array}{l} P\left(x\right)=1.55x-\left(0.85x+35,000\right) \\ \quad =0.7x-35,000 \end{array}$$

The profit function is $P\left(x\right)=0.7x-35,000.$

Analysis

The cost to produce 50,000 units is $77,500, and the revenue from the sales of 50,000 units is also $77,500. To make a profit, the business must produce and sell more than 50,000 units. See (Figure).

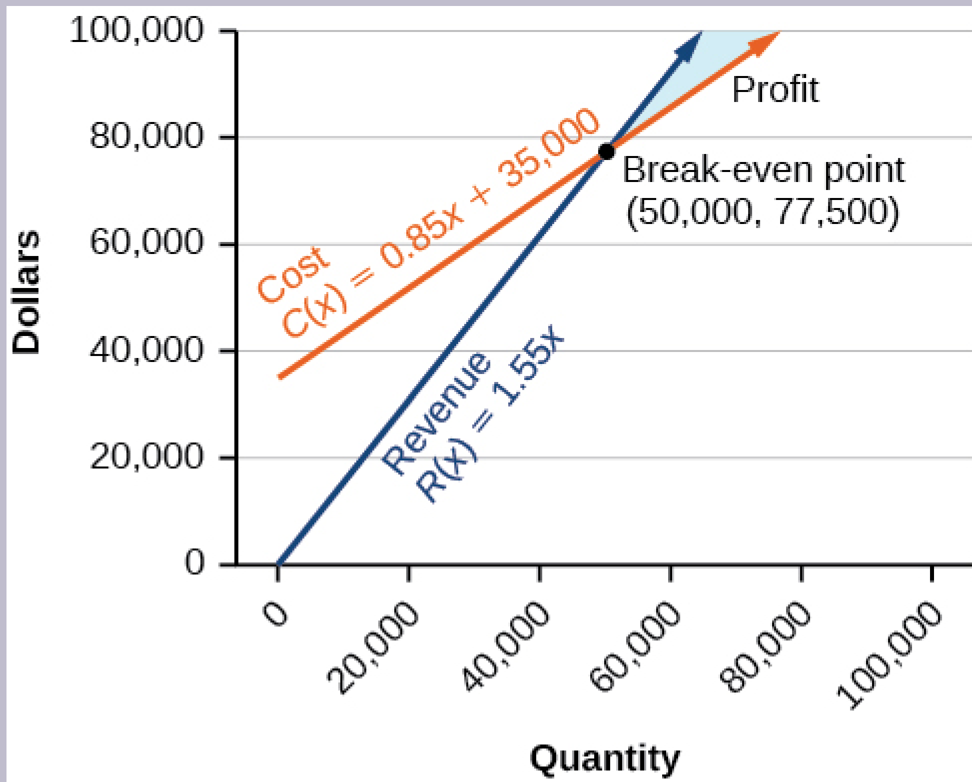


Figure 12.

We see from the graph in (Figure) that the profit function has a negative value until $x = 50,000,$ when the graph crosses the *x*-axis. Then, the graph emerges into positive *y*-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

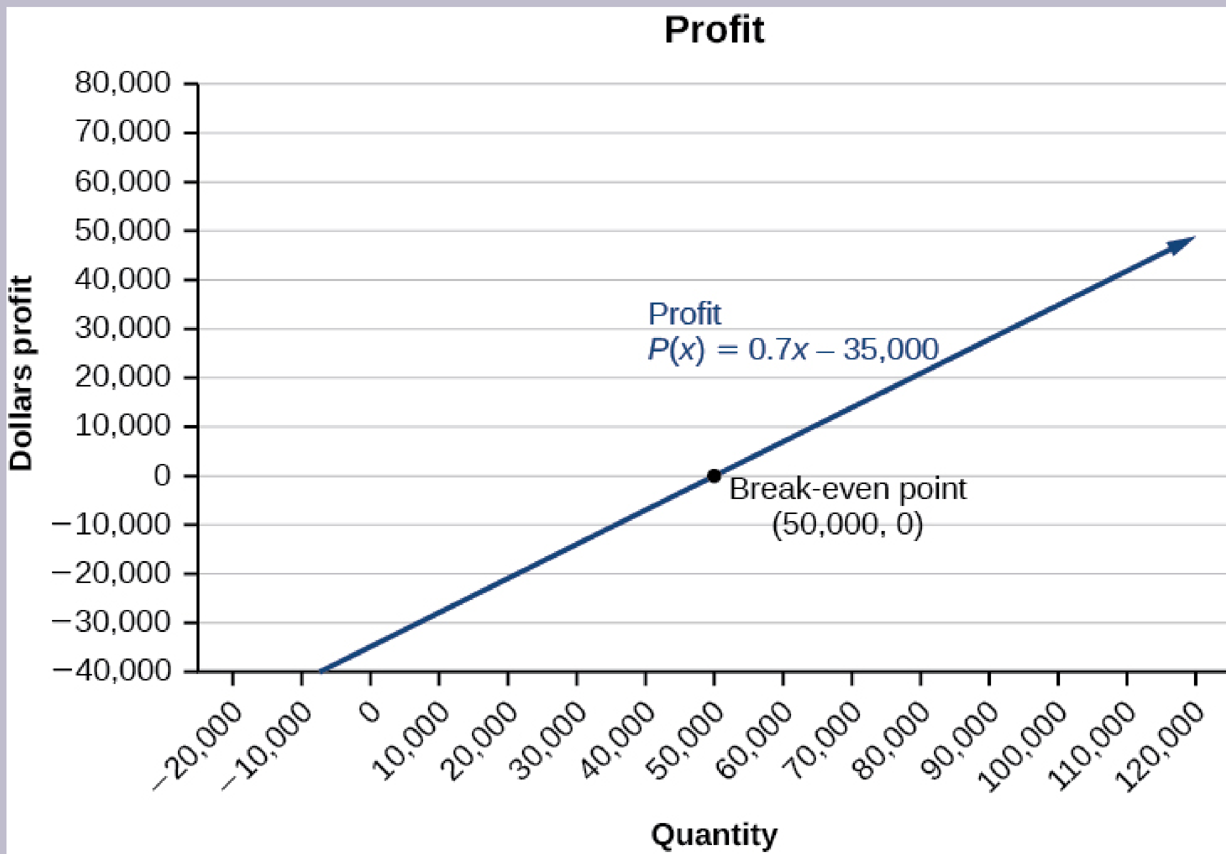


Figure 13.

Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000.$ How many children and how many adults bought tickets?

Show Solution

Let *c* = the number of children and *a* = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c + a = 2,000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70,000$. We can use this to write an equation for the revenue.

$$25c + 50a = 70,000$$

We now have a system of linear equations in two variables.

$$\begin{aligned} c + a &= 2,000 \\ 25c + 50a &= 70,000 \end{aligned}$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$\begin{aligned} c + a &= 2,000 \\ a &= 2,000 - c \end{aligned}$$

Substitute the expression $2,000 - c$ in the second equation for a and solve for c.

$$\begin{aligned} 25c + 50\,(2,000 - c) &= 70,000 \\ 25c + 100,000 - 50c &= 70,000 \\ -25c &= -30,000 \\ c &= 1,200 \end{aligned}$$

Substitute $c = 1,200$ into the first equation to solve for a.

$$\begin{aligned} 1,200 + a &= 2,000 \\ a &= 800 \end{aligned}$$

We find that $1,200$ children and 800 adults bought tickets to the circus that day.

Try It

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1,650$ meal tickets were bought for a total of $\$14,200$, how many children and how many adults bought meal tickets?

Show Solution
700 children, 950 adults

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations

Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See (Figure).
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See (Figure).
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See (Figure).
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See (Figure).
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See (Figure), (Figure), and (Figure).
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See (Figure).

- The solution to a system of dependent equations will always be true because both equations describe the same line. See (Figure).
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See (Figure) and (Figure).

Section Exercises

Verbal

1. Can a system of linear equations have exactly two solutions? Explain why or why not.

Show Solution
No, you can either have zero, one, or infinitely many. Examine graphs.

2. If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.

3. If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

Show Solution
This means there is no realistic break-even point. By the time the company produces one unit they are already making profit.

4. If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

5. Given a system of equations, explain at least two different methods of solving that system.

Show Solution
You can solve by substitution (isolating x or y), graphically, or by addition.

Algebraic

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

6. $\begin{array}{r} 5x - y = 4 \\ x + 6y = 2 \end{array}$ and $(4, 0)$

7. $\begin{array}{r} -3x - 5y = 13 \\ -x + 4y = 10 \end{array}$ and $(-6, 1)$

Show Solution
Yes

8. $\begin{array}{r} 3x + 7y = 1 \\ 2x + 4y = 0 \end{array}$ and $(2, 3)$

9. $\begin{array}{r} -2x + 5y = 7 \\ 2x + 9y = 7 \end{array}$ and $(-1, 1)$

Show Solution
Yes

10. $\begin{array}{r} x + 8y = 43 \\ 3x - 2y = -1 \end{array}$ and $(3, 5)$

For the following exercises, solve each system by substitution.

11. $\begin{array}{r} x + 3y = 5 \\ 2x + 3y = 4 \end{array}$

Show Solution

$(-1, 2)$

12. $$\begin{aligned} 3x - 2y &= 18 \\ 5x + 10y &= -10 \end{aligned}$$

13. $$\begin{aligned} 4x + 2y &= -10 \\ 3x + 9y &= 0 \end{aligned}$$

Show Solution

$(-3, 1)$

14. $$\begin{aligned} 2x + 4y &= -3.8 \\ 9x - 5y &= 1.3 \end{aligned}$$

15. $$\begin{aligned} -2x + 3y &= 1.2 \\ -3x - 6y &= 1.8 \end{aligned}$$

Show Solution

$\left(-\frac{3}{5}, 0\right)$

16. $$\begin{aligned} x - 0.2y &= 1 \\ -10x + 2y &= 5 \end{aligned}$$

17. $$\begin{aligned} 3x + 5y &= 9 \\ 30x + 50y &= -90 \end{aligned}$$

Show Solution

No solutions exist.

18. $-3x + y = 2$
$12x - 4y = -8$

19. $\frac{1}{2}x + \frac{1}{3}y = 16$
$\frac{1}{6}x + \frac{1}{4}y = 9$

Show Solution
$\left(\frac{72}{5}, \frac{132}{5}\right)$

20. $-\frac{1}{4}x + \frac{3}{2}y = 11$
$-\frac{1}{8}x + \frac{1}{3}y = 3$

For the following exercises, solve each system by addition.

21. $-2x + 5y = -42$
$7x + 2y = 30$

Show Solution
$(6, -6)$

22. $6x - 5y = -34$
$2x + 6y = 4$

23. $5x - y = -2.6$
$-4x - 6y = 1.4$

Show Solution
$\left(-\frac{1}{2}, \frac{1}{10}\right)$

24. $7x - 2y = 3$
$4x + 5y = 3.25$

25. $-\mathrm{x}+2y=-1$
$5x-10y=6$

Show Solution
No solutions exist.

26. $7x+6y=2$
$-28x-24y=-8$

27. $\frac{5}{6}x+\frac{1}{4}y=0$
$\frac{1}{8}x-\frac{1}{2}y=-\frac{43}{120}$

Show Solution
$\left(-\frac{1}{5},\frac{2}{3}\right)$

28. $\frac{1}{3}x+\frac{1}{9}y=\frac{2}{9}$
$-\frac{1}{2}x+\frac{4}{5}y=-\frac{1}{3}$

29. $-0.2x+0.4y=0.6$
$x-2y=-3$

Show Solution
$\left(x,\frac{x+3}{2}\right)$
Infinite Solutions

30. $-0.1x+0.2y=0.6$
$5x-10y=1$

For the following exercises, solve each system by any method.

31. $5x+9y=16$
$x+2y=4$

Show Solution

$(-4, 4)$

32. $\begin{array}{l} 6x - 8y = -0.6 \\ 3x + 2y = 0.9 \end{array}$

33. $\begin{array}{l} 5x - 2y = 2.25 \\ 7x - 4y = 3 \end{array}$

Show Solution

$\left(\frac{1}{2}, \frac{1}{8}\right)$

34. $\begin{array}{r} x - \frac{5}{12}y = -\frac{55}{12} \\ -6x + \frac{5}{2}y = \frac{55}{2} \end{array}$

35. $\begin{array}{l} 7x - 4y = \frac{7}{6} \\ 2x + 4y = \frac{1}{3} \end{array}$

Show Solution

$\left(\frac{1}{6}, 0\right)$

36. $\begin{array}{l} 3x + 6y = 11 \\ 2x + 4y = 9 \end{array}$

37. $\begin{array}{l} \frac{7}{3}x - \frac{1}{6}y = 2 \\ -\frac{21}{6}x + \frac{3}{12}y = -3 \end{array}$

Show Solution

$(x, 2(7x - 6))$
Infinite Solutions

38. $\frac{1}{2}x + \frac{1}{3}y = \frac{1}{3}$
$\frac{3}{2}x + \frac{1}{4}y = -\frac{1}{8}$

39. $2.2x + 1.3y = -0.1$
$4.2x + 4.2y = 2.1$

Show Solution
$\left(-\frac{5}{6}, \frac{4}{3}\right)$

40. $0.1x + 0.2y = 2$
$0.35x - 0.3y = 0$

Graphical

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

41. $3x - y = 0.6$
$x - 2y = 1.3$

Show Solution
Consistent with one solution

42. $-x + 2y = 4$
$2x - 4y = 1$

43. $x + 2y = 7$
$2x + 6y = 12$

Show Solution

Consistent with one solution

44. $$\begin{array}{r} 3x - 5y = 7 \\ x - 2y = 3 \end{array}$$

45. $$\begin{array}{r} 3x - 2y = 5 \\ -9x + 6y = -15 \end{array}$$

Show Solution

Dependent with infinitely many solutions

Technology

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

46. $$\begin{array}{r} 0.1x + 0.2y = 0.3 \\ -0.3x + 0.5y = 1 \end{array}$$

47. $$\begin{array}{r} -0.01x + 0.12y = 0.62 \\ 0.15x + 0.20y = 0.52 \end{array}$$

Show Solution

$(-3.08, 4.91)$

48. $$\begin{array}{r} 0.5x + 0.3y = 4 \\ 0.25x - 0.9y = 0.46 \end{array}$$

49. $$\begin{array}{r} 0.15x + 0.27y = 0.39 \\ -0.34x + 0.56y = 1.8 \end{array}$$

Show Solution

$(-1.52, 2.29)$

50. $-0.71x + 0.92y = 0.13$
$0.83x + 0.05y = 2.1$

Extensions

For the following exercises, solve each system in terms of $A, B, C, D, E,$ and F where $A - F$ are nonzero numbers. Note that $A \neq B$ and $AE \neq BD$.

51. $x + y = A$
$x - y = B$

Show Solution

$\left(\frac{A+B}{2}, \frac{A-B}{2}\right)$

52. $x + Ay = 1$
$x + By = 1$

53. $Ax + y = 0$
$Bx + y = 1$

Show Solution

$\left(\frac{-1}{A-B}, \frac{A}{A-B}\right)$

54. $Ax + By = C$
$x + y = 1$

55. $Ax + By = C$
$Dx + Ey = F$

Show Solution
$\left(\frac{CE-BF}{BD-AE}, \frac{AF-CD}{BD-AE}\right)$

Real-World Applications

For the following exercises, solve for the desired quantity.

56. A stuffed animal business has a total cost of production $C = 12x + 30$ and a revenue function $R = 20x.$ Find the break-even point.

57. A fast-food restaurant has a cost of production $C\left(x\right) = 11x + 120$ and a revenue function $R\left(x\right) = 5x.$ When does the company start to turn a profit?

Show Solution
They never turn a profit.

58. A cell phone factory has a cost of production $C\left(x\right) = 150x + 10,000$ and a revenue function $R\left(x\right) = 200x.$ What is the break-even point?

59. A musician charges $C\left(x\right) = 64x + 20,000,$ where x is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

Show Solution
$(1,250, 100,000)$

60. A guitar factory has a cost of production $C\left(x\right) = 75x + 50,000.$ If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

61. Find two numbers whose sum is 28 and difference is 13.

Show Solution
The numbers are 7.5 and 20.5.

62. A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.

63. The startup cost for a restaurant is $120,000, and each meal costs $10 for the restaurant to make. If each meal is then sold for $15, after how many meals does the restaurant break even?

Show Solution
24,000

64. A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

65. A total of 1,595 first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?

Show Solution
790 sophomores, 805 freshman

66. 276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

67. There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?

Show Solution
56 men, 74 women

68. A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.

69. If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many
gallons of 10% and 60% solutions were mixed?

Show Solution
10 gallons of 10% solution, 15 gallons of 60% solution

70. An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?

71. An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

Show Solution
Swan Peak: $750,000, Riverside: $350,000

72. If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and

the other that pays $2\frac{7}{8}\%$ interest, and the investor earns $737.50 annual interest, how much was invested in each account?

73. If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?

Show Solution
$12,500 in the first account, $10,500 in the second account.

74. CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?

75. A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?

Show Solution
High-tops: 45, Low-tops: 15

76. A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?

77. Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child's ticket and the price of the adult ticket?

Show Solution

Infinitely many solutions. We need more information.

Glossary

addition method
: an algebraic technique used to solve systems of linear equations in which the equations are added in a way that eliminates one variable, allowing the resulting equation to be solved for the remaining variable; substitution is then used to solve for the first variable

break-even point
: the point at which a cost function intersects a revenue function; where profit is zero

consistent system
: a system for which there is a single solution to all equations in the system and it is an independent system, or if there are an infinite number of solutions and it is a dependent system

cost function
: the function used to calculate the costs of doing business; it usually has two parts, fixed costs and variable costs

dependent system
: a system of linear equations in which the two equations represent the same line; there are an infinite number of solutions to a dependent system

inconsistent system
: a system of linear equations with no common solution because they represent parallel lines, which have no point or line in common

independent system
: a system of linear equations with exactly one solution pair (x, y)

profit function
: the profit function is written as $P(x) = R(x) - C(x)$, revenue minus cost

revenue function
: the function that is used to calculate revenue, simply written as $R = xp$, where $x =$

quantity and $p =$ price

substitution method

an algebraic technique used to solve systems of linear equations in which one of the two equations is solved for one variable and then substituted into the second equation to solve for the second variable

system of linear equations

a set of two or more equations in two or more variables that must be considered simultaneously.

CHAPTER 2.6: SYSTEMS OF LINEAR EQUATIONS: THREE VARIABLES

Learning Objectives

In this section, you will:

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

Figure 1. (credit: “Elembis,” Wikimedia Commons)

John received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z) , which we call an ordered triple. A system in upper triangular form looks like the following:

$$\begin{aligned}&Ax + By + Cz = D\\&Ey + Fz = G\\&Hz = K\end{aligned}$$

The third equation can be solved for z, and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The **solution set** to a three-by-three system is an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

Number of Possible Solutions

(Figure) and (Figure) illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

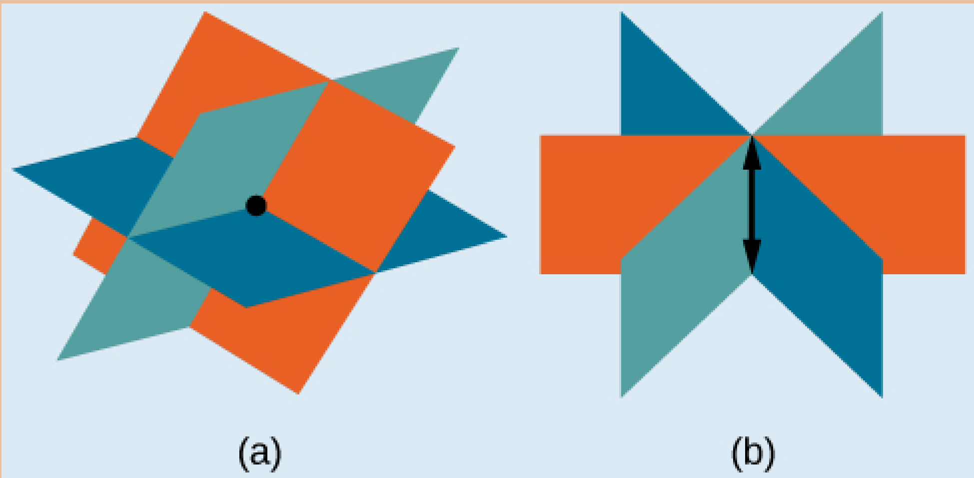


Figure 2. (a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.

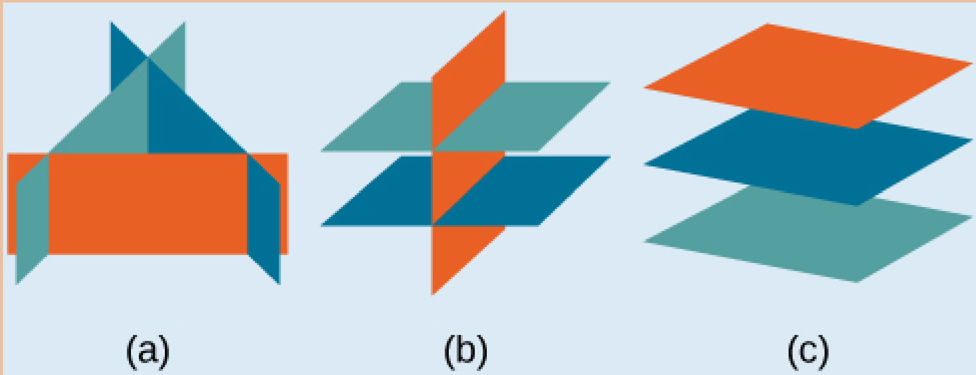


Figure 3. All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

Show Solution

We will check each equation by substituting in the values of the ordered triple for $x, y,$ and z.

$$\begin{array}{r} x + y + z = 2 \\ (3) + (-2) + (1) = 2 \\ \text{True} \end{array} \qquad \begin{array}{r} 6x - 4y + 5z = 31 \\ 6(3) - 4(-2) + 5(1) = 31 \\ 18 + 8 + 5 = 31 \\ \text{True} \end{array} \qquad \begin{array}{r} 5x + 2y + 2z = 13 \\ 5(3) + 2(-2) + 2(1) = 13 \\ 15 - 4 + 2 = 13 \\ \text{True} \end{array}$$

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

How To

Given a linear system of three equations, solve for three unknowns.

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x-2y+3z&=9 \quad &(1)\\ -x+3y-z&=-6 \quad &(2)\\ 2x-5y+5z&=17 \quad &(3)\end{aligned}$$

Show Solution

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$\begin{array}{rl} x-2y+3z=9 & (1)\\ -x+3y-z=-6 & (2)\\ \hline y+2z=3 & (3)\end{array}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

$$\begin{array}{rl} -2x+4y-6z=-18 & (1)\ \text{multiplied by } -2\\ 2x-5y+5z=17 & (3)\\ \hline -y-z=-1 & (5)\end{array}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

$$\begin{array}{rl} y + 2z = 3 & (4) \\ -y - z = -1 & (5) \\ \hline z = 2 & (6) \end{array}$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$\begin{array}{ll} x - 2y + 3z = 9 & (1) \\ y + 2z = 3 & (4) \\ z = 2 & (6) \end{array}$$

Next, we back-substitute $z = 2$ into equation (4) and solve for y.

$$\begin{array}{l} y + 2(2) = 3 \\ y + 4 = 3 \\ y = -1 \end{array}$$

Finally, we can back-substitute $z = 2$ and $y = -1$ into equation (1). This will yield the solution for x.

$$\begin{array}{r} x - 2(-1) + 3(2) = 9 \\ x + 2 + 6 = 9 \\ x = 1 \end{array}$$

The solution is the ordered triple $(1, -1, 2)$. See (Figure).

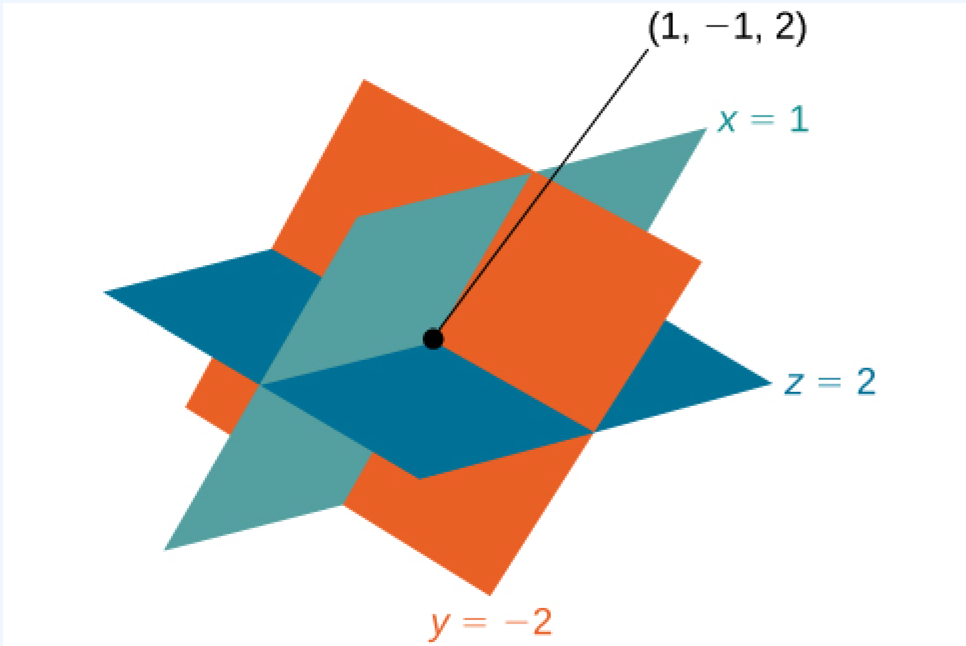


Figure 4.

Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of \$12,000 in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested \$4,000 more in mutual funds than he invested in municipal bonds. The total interest earned in one year was \$670. How much did he invest in each type of fund?

Show Solution

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$\begin{aligned}x &= \text{amount invested in money-market fund}\\ y &= \text{amount invested in municipal bonds}\\ z &= \text{amount invested in mutual funds}\end{aligned}$$

The first equation indicates that the sum of the three principal amounts is $12,000.

$$x+y+z=12,000$$

We form the second equation according to the information that John invested $4,000 more in mutual funds than he invested in municipal bonds.

$$z=y+4,000$$

The third equation shows that the total amount of interest earned from each fund equals $670.

$$0.03x+0.04y+0.07z=670$$

Then, we write the three equations as a system.

$$\begin{aligned}x+y+z&=12,000\\ -y+z&=4,000\\ 0.03x+0.04y+0.07z&=670\end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$\begin{aligned}x+\ y+z\ &=12,000 \quad (1)\\ -y+z\ &=4,000 \quad (2)\\ 3x+4y+7z&=67,000 \quad (3)\end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$\begin{aligned}x+\ y+\ z&=12,000\\ 3x+4y+7z&=67,000\\ -y\ +\ z&=4,000\end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.

$$\begin{aligned}x+y+z\ &=12,000\\ y+4z&=31,000\\ -y+z\ &=4,000\end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$\begin{aligned}x+y+\ z&=12,000\\ y+4z&=31,000\\ 5z\ &=35,000\end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

$$5z = 35,000$$
$$z = 7,000$$

$$y + 4\left(7,000\right) = 31,000$$
$$y = 3,000$$

$$x + 3,000 + 7,000 = 12,000$$
$$x = 2,000$$

John invested \$2,000 in a money-market fund, \$3,000 in municipal bonds, and \$7,000 in mutual funds.

Try It

Solve the system of equations in three variables.

$$2x + y - 2z = -1$$
$$3x - 3y - z = 5$$
$$x - 2y + 3z = 6$$

Show Solution

$$\left(1, -1, 1\right)$$

Identifying Inconsistent Systems of Equations Containing Three

Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3=7$ or some other contradiction.

Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$\begin{aligned} x-3y+z&=4 && (1)\\ -x+2y-5z&=3 && (2)\\ 5x-13y+13z&=8 && (3) \end{aligned}$$

Show Solution

Looking at the coefficients of x, we can see that we can eliminate x by adding equation (1) to equation (2).

$$\begin{array}{l} x-3y+z=4\ \ (1)\\ \underline{-x+2y-5z=3\ \ (2)}\\ -y-4z=7\ \ (4) \end{array}$$

Next, we multiply equation (1) by -5 and add it to equation (3).

$$\begin{array}{ll} -5x+15y-5z=-20 & (1)\ \text{multiplied by}\ -5\\ 5x-13y+13z=8 & (3)\\ \hline 2y+8z=-12 & (5) \end{array}$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{array}{l} -2y-8z=14\ \ (4)\ \text{multiplied by}\ 2\\ 2y+8z=-12\ \ (5)\\ \hline 0=2 \end{array}$$

The final equation $0=2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

Try It

Solve the system of three equations in three variables.

$$\begin{aligned} x+y+z&=2 \\ y-3z&=1 \\ 2x+y+5z&=0 \end{aligned}$$

Show Solution

No solution.

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{aligned} 2x + y - 3z &= 0 \quad (1) \\ 4x + 2y - 6z &= 0 \quad (2) \\ x - y + z &= 0 \quad (3) \end{aligned}$$

Show Solution

First, we can multiply equation (1) by -2 and add it to equation (2).

$$\begin{aligned} -4x - 2y + 6z &= 0 \text{ equation } (1) \text{ multiplied by } -2 \\ 4x + 2y - 6z &= 0 \quad (2) \end{aligned}$$

$$0 = 0$$

We do not need to proceed any further. The result we get is an identity, $0 = 0,$ which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by $-2,$ and adding it to equation (1). We then perform the same steps as above and find the same result, $0 = 0.$

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$\begin{aligned} 2x + y - 3z &= 0 \\ x - y + z &= 0 \end{aligned}$$

$$3x - 2z = 0$$

We then solve the resulting equation for $z.$

$$\begin{aligned} 3x - 2z &= 0 \\ z &= \tfrac{3}{2}x \end{aligned}$$

We back-substitute the expression for z into one of the equations and solve for $y.$

$$\begin{aligned} 2x + y - 3\left(\tfrac{3}{2}x\right) &= 0 \\ 2x + y - \tfrac{9}{2}x &= 0 \\ y &= \tfrac{9}{2}x - 2x \\ y &= \tfrac{5}{2}x \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for $x.$

Analysis

As shown in (Figure), two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

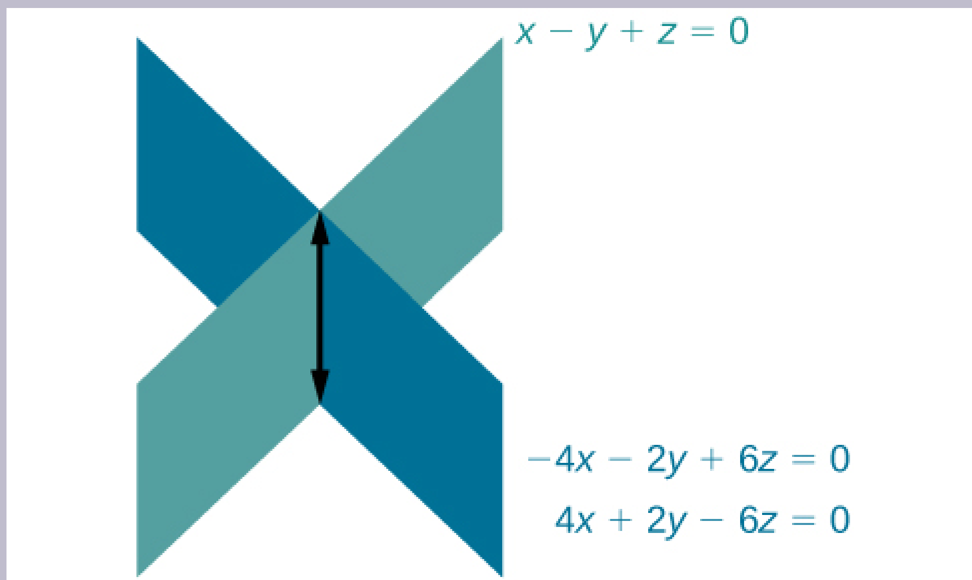


Figure 5.

Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Try It

Solve the following system.

$$\begin{aligned} x+y+z&=7\\ 3x-2y-z&=4\\ x+6y+5z&=24 \end{aligned}$$

Show Solution

Infinite number of solutions of the form $(x, 4x-11, -5x+18)$.

Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

Key Concepts

- A solution set is an ordered triple $\{(x, y, z)\}$ that represents the intersection of three planes in space. See (Figure).
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See (Figure).
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See (Figure).
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See (Figure).

- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See (Figure).
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

Section Exercises

Verbal

1. Can a linear system of three equations have exactly two solutions? Explain why or why not

Show Solution
No, there can be only one, zero, or infinitely many solutions.

2. If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.

3. If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.

Show Solution
Not necessarily. There could be zero, one, or infinitely many solutions. For example, $(0,0,0)$ is not a solution to the system below, but that does not mean that it has no solution.

$$\begin{aligned} 2x + 3y - 6z &= 1 \\ -4x - 6y + 12z &= -2 \\ x + 2y + 5z &= 10 \end{aligned}$$

4. Using the method of addition, is there only one way to solve the system?

5. Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

Show Solution
Every system of equations can be solved graphically, by substitution, and by addition. However, systems of three equations become very complex to solve graphically so other methods are usually preferable.

Algebraic

For the following exercises, determine whether the ordered triple given is the solution to the system of equations.

6. $\begin{array}{l} 2x - 6y + 6z = -12 \\ x + 4y + 5z = -1 \\ -\mathrm{x} + 2y + 3z = -1 \end{array}$ and $(0, 1, -1)$

7. $\begin{array}{l} 6x - y + 3z = 6 \\ 3x + 5y + 2z = 0 \\ x + y = 0 \end{array}$ and $(3, -3, -5)$

Show Solution

No

8. $\begin{array}{c} 6x-7y+z=2 \\ -\mathrm{x}-y+3z=4 \\ 2x+y-z=1 \end{array}$ and $(4,2,-6)$

9. $\begin{array}{c} x-y=0 \\ x-z=5 \\ x-y+z=-1 \end{array}$ and $(4,4,-1)$

Show Solution

Yes

10. $\begin{array}{c} -\mathrm{x}-y+2z=3 \\ 5x+8y-3z=4 \\ -\mathrm{x}+3y-5z=-5 \end{array}$ and $(4,1,-7)$

For the following exercises, solve each system by substitution.

11. $\begin{array}{c} 3x-4y+2z=-15 \\ 2x+4y+z=16 \\ 2x+3y+5z=20 \end{array}$

Show Solution

$(-1, 4, 2)$

12. $$\begin{aligned} 5x - 2y + 3z &= 20 \\ 2x - 4y - 3z &= -9 \\ x + 6y - 8z &= 21 \end{aligned}$$

13. $$\begin{aligned} 5x + 2y + 4z &= 9 \\ -3x + 2y + z &= 10 \\ 4x - 3y + 5z &= -3 \end{aligned}$$

Show Solution

$\left(-\frac{85}{107}, \frac{312}{107}, \frac{191}{107}\right)$

14. $$\begin{aligned} 4x - 3y + 5z &= 31 \\ -x + 2y + 4z &= 20 \\ x + 5y - 2z &= -29 \end{aligned}$$

15. $$\begin{aligned} 5x - 2y + 3z &= 4 \\ -4x + 6y - 7z &= -1 \\ 3x + 2y - z &= 4 \end{aligned}$$

Show Solution

$\left(1, \frac{1}{2}, 0\right)$

16. $$\begin{array}{l} 4x + 6y + 9z = 0 \\ -5x + 2y - 6z = 3 \\ 7x - 4y + 3z = -3 \end{array}$$

For the following exercises, solve each system by Gaussian elimination.

17. $$\begin{array}{l} 2x - y + 3z = 17 \\ -5x + 4y - 2z = -46 \\ 2y + 5z = -7 \end{array}$$

Show Solution

$(4, -6, 1)$

18. $$\begin{array}{l} 5x - 6y + 3z = 50 \\ -x + 4y = 10 \\ 2x - z = 10 \end{array}$$

19. $$\begin{array}{l} 2x + 3y - 6z = 1 \\ -4x - 6y + 12z = -2 \\ x + 2y + 5z = 10 \end{array}$$

Show Solution

$\left(x, \frac{1}{27}\left(65-16x\right), \frac{x+28}{27}\right)$

20. $$\begin{array}{r} 4x+6y-2z=8 \\ 6x+9y-3z=12 \\ -2x-3y+z=-4 \end{array}$$

21. $$\begin{array}{r} 2x+3y-4z=5 \\ -3x+2y+z=11 \\ -x+5y+3z=4 \end{array}$$

Show Solution

$\left(-\frac{45}{13}, \frac{17}{13}, -2\right)$

22. $$\begin{array}{r} 10x+2y-14z=8 \\ -\text{x}-2y-4z=-1 \\ -12x-6y+6z=-12 \end{array}$$

23. $$\begin{array}{r} x+y+z=14 \\ 2y+3z=-14 \\ -16y-24z=-112 \end{array}$$

Show Solution

No solutions exist

24. $$\begin{aligned} 5x - 3y + 4z &= -1 \\ -4x + 2y - 3z &= 0 \\ -\text{x} + 5y + 7z &= -11 \end{aligned}$$

25. $$\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 0 \\ x - z &= 0 \end{aligned}$$

Show Solution

$(0, 0, 0)$

26. $$\begin{aligned} 3x + 2y - 5z &= 6 \\ 5x - 4y + 3z &= -12 \\ 4x + 5y - 2z &= 15 \end{aligned}$$

---

27. $$\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 0 \\ x - z &= 1 \end{aligned}$$

Show Solution

$\left(\frac{4}{7}, -\frac{1}{7}, -\frac{3}{7}\right)$

28. $$\begin{aligned} 3x - \tfrac{1}{2}y - z &= -\tfrac{1}{2} \\ 4x + z &= 3 \\ -x + \tfrac{3}{2}y &= \tfrac{5}{2} \end{aligned}$$

29.
$$\begin{aligned} 6x - 5y + 6z &= 38 \\ \tfrac{1}{5}x - \tfrac{1}{2}y + \tfrac{3}{5}z &= 1 \\ -4x - \tfrac{3}{2}y - z &= -74 \end{aligned}$$

Show Solution

$(7, 20, 16)$

30.
$$\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{5}y + \tfrac{2}{5}z &= -\tfrac{13}{10} \\ \tfrac{1}{4}x - \tfrac{2}{5}y - \tfrac{1}{5}z &= -\tfrac{7}{20} \\ -\tfrac{1}{2}x - \tfrac{3}{4}y - \tfrac{1}{2}z &= -\tfrac{5}{4} \end{aligned}$$

31.
$$\begin{aligned} -\tfrac{1}{3}x - \tfrac{1}{2}y - \tfrac{1}{4}z &= \tfrac{3}{4} \\ -\tfrac{1}{2}x - \tfrac{1}{4}y - \tfrac{1}{2}z &= 2 \\ -\tfrac{1}{4}x - \tfrac{3}{4}y - \tfrac{1}{2}z &= -\tfrac{1}{2} \end{aligned}$$

Show Solution

$(-6, 2, 1)$

32.
$$\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{4}y + \tfrac{3}{4}z &= 0 \\ \tfrac{1}{4}x - \tfrac{1}{10}y + \tfrac{2}{5}z &= -2 \\ \tfrac{1}{8}x + \tfrac{1}{5}y - \tfrac{1}{8}z &= 2 \end{aligned}$$

33.
$$\begin{aligned} \tfrac{4}{5}x - \tfrac{7}{8}y + \tfrac{1}{2}z &= 1 \\ -\tfrac{4}{5}x - \tfrac{3}{4}y + \tfrac{1}{3}z &= -8 \\ -\tfrac{2}{5}x - \tfrac{7}{8}y + \tfrac{1}{2}z &= -5 \end{aligned}$$

Show Solution

$(5, 12, 15)$

34. $$\begin{array}{l} -\frac{1}{3}x - \frac{1}{8}y + \frac{1}{6}z = -\frac{4}{3} \\ -\frac{2}{3}x - \frac{7}{8}y + \frac{1}{3}z = -\frac{23}{3} \\ -\frac{1}{3}x - \frac{5}{8}y + \frac{5}{6}z = 0 \end{array}$$

35. $$\begin{array}{l} -\frac{1}{4}x - \frac{5}{4}y + \frac{5}{2}z = -5 \\ -\frac{1}{2}x - \frac{5}{3}y + \frac{5}{4}z = \frac{55}{12} \\ -\frac{1}{3}x - \frac{1}{3}y + \frac{1}{3}z = \frac{5}{3} \end{array}$$

Show Solution

$(-5, -5, -5)$

36. $$\begin{array}{l} \frac{1}{40}x + \frac{1}{60}y + \frac{1}{80}z = \frac{1}{100} \\ -\frac{1}{2}x - \frac{1}{3}y - \frac{1}{4}z = -\frac{1}{5} \\ \frac{3}{8}x + \frac{3}{12}y + \frac{3}{16}z = \frac{3}{20} \end{array}$$

37. $$\begin{array}{l} 0.1x - 0.2y + 0.3z = 2 \\ 0.5x - 0.1y + 0.4z = 8 \\ 0.7x - 0.2y + 0.3z = 8 \end{array}$$

Show Solution

$(10, 10, 10)$

38. $\begin{array}{l} 0.2x + 0.1y - 0.3z = 0.2 \\ 0.8x + 0.4y - 1.2z = 0.1 \\ 1.6x + 0.8y - 2.4z = 0.2 \end{array}$

39. $\begin{array}{l} 1.1x + 0.7y - 3.1z = -1.79 \\ 2.1x + 0.5y - 1.6z = -0.13 \\ 0.5x + 0.4y - 0.5z = -0.07 \end{array}$

Show Solution

$\left(\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\right)$

40. $\begin{array}{l} 0.5x - 0.5y + 0.5z = 10 \\ 0.2x - 0.2y + 0.2z = 4 \\ 0.1x - 0.1y + 0.1z = 2 \end{array}$

41. $\begin{array}{l} 0.1x + 0.2y + 0.3z = 0.37 \\ 0.1x - 0.2y - 0.3z = -0.27 \\ 0.5x - 0.1y - 0.3z = -0.03 \end{array}$

Show Solution

$\left(\frac{1}{2}, \frac{2}{5}, \frac{4}{5}\right)$

42. $$\begin{aligned} 0.5x - 0.5y - 0.3z &= 0.13 \\ 0.4x - 0.1y - 0.3z &= 0.11 \\ 0.2x - 0.8y - 0.9z &= -0.32 \end{aligned}$$

43. $$\begin{aligned} 0.5x + 0.2y - 0.3z &= 1 \\ 0.4x - 0.6y + 0.7z &= 0.8 \\ 0.3x - 0.1y - 0.9z &= 0.6 \end{aligned}$$

Show Solution

$(2, 0, 0)$

44. $$\begin{aligned} 0.3x + 0.3y + 0.5z &= 0.6 \\ 0.4x + 0.4y + 0.4z &= 1.8 \\ 0.4x + 0.2y + 0.1z &= 1.6 \end{aligned}$$

45. $$\begin{aligned} 0.8x + 0.8y + 0.8z &= 2.4 \\ 0.3x - 0.5y + 0.2z &= 0 \\ 0.1x + 0.2y + 0.3z &= 0.6 \end{aligned}$$

Show Solution

$(1, 1, 1)$

Extensions

For the following exercises, solve the system for x, y, and z.

46. $$\begin{aligned} x+y+z&=3 \\ \frac{x-1}{2}+\frac{y-3}{2}+\frac{z+1}{2}&=0 \\ \frac{x-2}{3}+\frac{y+4}{3}+\frac{z-3}{3}&=\frac{2}{3} \end{aligned}$$

47. $$\begin{aligned} 5x-3y-\frac{z+1}{2}&=\frac{1}{2} \\ 6x+\frac{y-9}{2}+2z&=-3 \\ \frac{x+8}{2}-4y+z&=4 \end{aligned}$$

Show Solution

$\left(\frac{128}{557}, \frac{23}{557}, \frac{28}{557}\right)$

48. $$\begin{aligned} \frac{x+4}{7}-\frac{y-1}{6}+\frac{z+2}{3}&=1 \\ \frac{x-2}{4}+\frac{y+1}{8}-\frac{z+8}{12}&=0 \\ \frac{x+6}{3}-\frac{y+2}{3}+\frac{z+4}{2}&=3 \end{aligned}$$

49. $$\begin{aligned} \frac{x-3}{6}+\frac{y+2}{2}-\frac{z-3}{3}&=2 \\ \frac{x+2}{4}+\frac{y-5}{2}+\frac{z+4}{2}&=1 \\ \frac{x+6}{2}-\frac{y-3}{2}+z+1&=9 \end{aligned}$$

Show Solution

$(6, -1, 0)$

50. $\frac{x-1}{3} + \frac{y+3}{4} + \frac{z+2}{6} = 1$
$4x + 3y - 2z = 11$
$0.02x + 0.015y - 0.01z = 0.065$

Real-World Applications

51. Three even numbers sum up to 108. The smaller is half the larger and the middle number is $\frac{3}{4}$ the larger. What are the three numbers?

Show Solution
24, 36, 48

52. Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?

53. At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

Show Solution
70 grandparents, 140 parents, 190 children

54. An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?

55. Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was \$82. She forgot to save the individual receipts but remembered that your groceries were \$0.05 cheaper than half of her groceries, and that your other roommate's groceries were \$2.10 more than your groceries. How much was each of your share of the groceries?

Show Solution

Your share was \$19.95, Sarah's share was \$40, and your other roommate's share was \$22.05.

56. Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was \$100.75. Your supplies were bought with 5% tax, John's with 8% tax, and your third roommate's with 9% sales tax. The total amount of money spent without taxes is \$93.50. If your supplies before tax were \$1 more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.

57. Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is \$82,000. The office manager makes \$4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total \$78,000. What is the annual salary of each of the co-workers?

Show Solution

There are infinitely many solutions; we need more information

58. At a carnival, \$2,914.25 in receipts were taken at the end of the day. The cost of a child's ticket was \$20.50, an adult ticket was \$29.75, and a senior citizen ticket was \$15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?

59. A local band sells out for their concert. They sell all 1,175 tickets for a total purse of \$28,112.50. The tickets were priced at \$20 for student tickets, \$22.50 for children, and \$29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?

Show Solution
500 students, 225 children, and 450 adults

60. In a bag, a child has 325 coins worth \$19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?

61. Last year, at Haven's Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of \$140,000. This year, due to inflation, the same cars would cost \$151,830. The cost of the BMW increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year's Jeep was \$7,000 less than the price of last year's BMW, what was the price of each of the three cars last year?

Show Solution
The BMW was \$49,636, the Jeep was \$42,636, and the Toyota was \$47,727.

62. A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested \$80,500 into three accounts, one that paid

4% simple interest, one that paid $3\frac{1}{8}\%$ simple interest, and one that paid $2\frac{1}{2}\%$ simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

63. You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?

Show Solution
$400,000 in the account that pays 3% interest, $500,000 in the account that pays 4% interest, and $100,000 in the account that pays 2% interest.

64. You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money in the account that pays 4% compared to 3%, how much did you invest in each account?

65. The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?[1]

1. "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Show Solution
The United States consumed 26.3%, Japan 7.1%, and China 6.4% of the world's oil.

66. The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?[2]

67. The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?[3]

Show Solution
Saudi Arabia imported 16.8%, Canada imported 15.1%, and Mexico 15.0%

68. The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?[4]

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2. "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.
3. "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.
4. "USA: The coming global oil crisis," accessed April 6, 2014, http://www.oilcrisis.com/us/.

69. At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?

Show Solution

Birds were 19.3%, fish were 18.6%, and mammals were 17.1% of endangered species

70. Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?[5]

Glossary

solution set

the set of all ordered pairs or triples that satisfy all equations in a system of equations

5. “The United States Meat Industry at a Glance,” accessed April 6, 2014, http://www.meatami.com/ht/d/sp/i/47465/pid/47465.

CHAPTER 2.7: SOLVING SYSTEMS WITH CRAMER'S RULE

Learning Objectives

In this section, you will:

- Evaluate 2×2 determinants.
- Use Cramer's Rule to solve a system of equations in two variables.
- Evaluate 3×3 determinants.
- Use Cramer's Rule to solve a system of three equations in three variables.
- Know the properties of determinants.

We have learned how to solve systems of equations in two variables and three variables, and by multiple methods: substitution, addition, Gaussian elimination, using the inverse of a matrix, and graphing. Some of these methods are easier to apply than others and are more appropriate in certain situations. In this section, we will study two more strategies for solving systems of equations.

Evaluating the Determinant of a 2×2 Matrix

A determinant is a real number that can be very useful in mathematics because it has multiple applications, such as calculating area, volume, and other quantities. Here, we will use determinants to reveal whether a matrix is invertible by using the entries of a square matrix to determine whether there is a solution to the system of equations. Perhaps one of the more interesting applications, however, is their use in cryptography. Secure signals or messages are sometimes sent encoded in a matrix. The data can only be decrypted with an invertible matrix and the determinant. For our purposes, we focus on the determinant as an indication of the invertibility of the matrix. Calculating the determinant of a matrix involves following the specific patterns that are outlined in this section.

Find the Determinant of a 2 × 2 Matrix

The determinant of a $2 \quad 2$ matrix, given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Notice the change in notation. There are several ways to indicate the determinant, including $\det(A)$ and replacing the brackets in a matrix with straight lines, $|A|$.

Finding the Determinant of a 2 × 2 Matrix

Find the determinant of the given matrix.

$$A = \begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$$

Show Solution

$$\begin{aligned} \det(A) &= \begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix} \\ &= 5(3) - (-6)(2) \\ &= 27 \end{aligned}$$

Using Cramer's Rule to Solve a System of Two Equations in Two Variables

We will now introduce a final method for solving systems of equations that uses determinants. Known as Cramer's Rule, this technique dates back to the middle of the 18th century and is named for its innovator, the Swiss mathematician Gabriel Cramer (1704-1752), who introduced it in 1750 in Introduction à l'Analyse des lignes Courbes algébriques. Cramer's Rule is a viable and efficient method for finding solutions to systems with an arbitrary number of unknowns, provided that we have the same number of equations as unknowns.

Cramer's Rule will give us the unique solution to a system of equations, if it exists. However, if the system has no solution or an infinite number of solutions, this will be indicated by a determinant of zero. To find out if the system is inconsistent or dependent, another method, such as elimination, will have to be used.

To understand Cramer's Rule, let's look closely at how we solve systems of linear equations using basic row operations. Consider a system of two equations in two variables.

$$a_1x + b_1y = c_1 \quad (1)$$
$$a_2x + b_2y = c_2 \quad (2)$$

We eliminate one variable using row operations and solve for the other. Say that we wish to solve for x. If equation (2) is multiplied by the opposite of the coefficient of y in equation (1), equation (1) is multiplied by the coefficient of y in equation (2), and we add the two equations, the variable y will be eliminated.

$$\begin{array}{ll} b_2a_1x + b_2b_1y = b_2c_1 & \text{Multiply } R_1 \text{ by } b_2 \\ -b_1a_2x - b_1b_2y = -b_1c_2 & \text{Multiply } R_2 \text{ by } -b_1 \\ \hline b_2a_1x - b_1a_2x = b_2c_1 - b_1c_2 & \end{array}$$

Now, solve for x.

$$b_2a_1x - b_1a_2x = b_2c_1 - b_1c_2$$
$$x\left(b_2a_1 - b_1a_2\right) = b_2c_1 - b_1c_2$$
$$x = \frac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Similarly, to solve for y, we will eliminate x.

$$\begin{array}{ll} a_2a_1x + a_2b_1y = a_2c_1 & \text{Multiply } R_1 \text{ by } a_2 \\ -a_1a_2x - a_1b_2y = -a_1c_2 & \text{Multiply } R_2 \text{ by } -a_1 \\ \hline a_2b_1y - a_1b_2y = a_2c_1 - a_1c_2 & \end{array}$$

Solving for y gives

$$a_2b_1y - a_1b_2y = a_2c_1 - a_1c_2$$
$$y\left(a_2b_1 - a_1b_2\right) = a_2c_1 - a_1c_2$$
$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Notice that the denominator for both x and y is the determinant of the coefficient matrix.

We can use these formulas to solve for x and y, but Cramer's Rule also introduces new notation:

- D : determinant of the coefficient matrix

- D_x : determinant of the numerator in the solution of x
  $x = \frac{D_x}{D}$
- D_y : determinant of the numerator in the solution of y
  $y = \frac{D_y}{D}$

The key to Cramer's Rule is replacing the variable column of interest with the constant column and calculating the determinants. We can then express x and y as a quotient of two determinants.

Cramer's Rule for 2×2 Systems

Cramer's Rule is a method that uses determinants to solve systems of equations that have the same number of equations as variables.

Consider a system of two linear equations in two variables.

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

The solution using Cramer's Rule is given as

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ D \neq 0;\ \ y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ D \neq 0.$$

If we are solving for x, the x column is replaced with the constant column. If we are solving for y, the y column is replaced with the constant column.

Using Cramer's Rule to Solve a 2 × 2 System

Solve the following 2 2 system using Cramer's Rule.

$$12x + 3y = 15$$
$$2x - 3y = 13$$

Show Solution

Solve for x.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{-45-39}{-36-6} = \frac{-84}{-42} = 2$$

Solve for y.

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{156-30}{-36-6} = -\frac{126}{42} = -3$$

The solution is $(2, -3)$.

Try It

Use Cramer's Rule to solve the 2 × 2 system of equations.

$$\begin{aligned} x + 2y &= -11 \\ -2x + y &= -13 \end{aligned}$$

Show Solution

$(3, -7)$

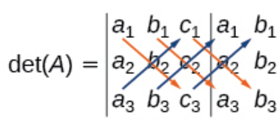
Evaluating the Determinant of a 3 × 3 Matrix

Finding the determinant of a 2×2 matrix is straightforward, but finding the determinant of a 3×3 matrix is more complicated. One method is to augment the 3×3 matrix with a repetition of the first two columns, giving a 3×5 matrix. Then we calculate the sum of the products of entries *down* each of the three diagonals (upper left to lower right), and subtract the products of entries *up* each of the three diagonals (lower left to upper right). This is more easily understood with a visual and an example.

Find the determinant of the 3×3 matrix.

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

1. Augment A with the first two columns.
$$\det(A) = \left| \begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array} \right|$$
2. From upper left to lower right: Multiply the entries down the first diagonal. Add the result to the product of entries down the second diagonal. Add this result to the product of the entries down the third diagonal.
3. From lower left to upper right: Subtract the product of entries up the first diagonal. From this result subtract the product of entries up the second diagonal. From this result, subtract the product of entries up the third diagonal.



The algebra is as follows:

$$|A| = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

Finding the Determinant of a 3 × 3 Matrix

Find the determinant of the 3 × 3 matrix given

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

Show Solution

Augment the matrix with the first two columns and then follow the formula. Thus,

$$\begin{aligned}|A| &= \left|\begin{array}{ccc|cc} 0 & 2 & 1 & 0 & 2 \\ 3 & -1 & 1 & 3 & -1 \\ 4 & 0 & 1 & 4 & 0 \end{array}\right| \\ &= 0(-1)(1) + 2(1)(4) + 1(3)(0) - 4(-1)(1) - 0(1)(0) - 1(3)(2) \\ &= 0 + 8 + 0 + 4 - 0 - 6 \\ &= 6\end{aligned}$$

Try It

Find the determinant of the 3 × 3 matrix.

$$\det(A) = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

Show Solution

-10

Can we use the same method to find the determinant of a larger matrix?

No, this method only works for $2\quad 2$ *and* $3\quad 3$ *matrices. For larger matrices it is best to use a graphing utility or computer software.*

Using Cramer's Rule to Solve a System of Three Equations in Three Variables

Now that we can find the determinant of a 3 × 3 matrix, we can apply Cramer's Rule to solve a system of three equations in three variables. Cramer's Rule is straightforward, following a pattern consistent with Cramer's Rule for 2 × 2 matrices. As the order of the matrix increases to 3 × 3, however, there are many more calculations required.

When we calculate the determinant to be zero, Cramer's Rule gives no indication as to whether the system has no solution or an infinite number of solutions. To find out, we have to perform elimination on the system.

Consider a 3 × 3 system of equations.

$$\begin{aligned} a_1x+b_1y+c_1z&=d_1\\ a_2x+b_2y+c_2z&=d_2\\ a_3x+b_3y+c_3z&=d_3 \end{aligned}$$

$$x=\frac{D_x}{D},\, y=\frac{D_y}{D},\, z=\frac{D_z}{D},\, D\neq 0$$

where

$$D=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x=\begin{vmatrix} d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3 \end{vmatrix},\ D_y=\begin{vmatrix} a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3 \end{vmatrix},\ D_z=\begin{vmatrix} a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3 \end{vmatrix}$$

If we are writing the determinant D_x, we replace the x column with the constant column. If we are writing the determinant D_y, we replace the y column with the constant column. If we are writing the determinant D_z, we replace the z column with the constant column. Always check the answer.

Solving a 3 × 3 System Using Cramer's Rule

Find the solution to the given 3 × 3 system using Cramer's Rule.

$$\begin{aligned} x+y-z&=6\\ 3x-2y+z&=-5\\ x+3y-2z&=14 \end{aligned}$$

Show Solution

Use Cramer's Rule.

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix}, D_x = \begin{vmatrix} 6 & 1 & -1 \\ -5 & -2 & 1 \\ 14 & 3 & -2 \end{vmatrix}, D_y = \begin{vmatrix} 1 & 6 & -1 \\ 3 & -5 & 1 \\ 1 & 14 & -2 \end{vmatrix}, D_z = \begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & -5 \\ 1 & 3 & 14 \end{vmatrix}$$

Then,

$x = \frac{D_x}{D} = \frac{-3}{-3} = 1$
$y = \frac{D_y}{D} = \frac{-9}{-3} = 3$
$z = \frac{D_z}{D} = \frac{6}{-3} = -2$

The solution is $(1, 3, -2)$.

Try It

Use Cramer's Rule to solve the 3 × 3 matrix.

$$\begin{aligned} x - 3y + 7z &= 13 \\ x + y + z &= 1 \\ x - 2y + 3z &= 4 \end{aligned}$$

Show Solution

$\left(-2, \frac{3}{5}, \frac{12}{5}\right)$

Using Cramer's Rule to Solve an Inconsistent System

Solve the system of equations using Cramer's Rule.

$3x - 2y = 4\,(1)$
$6x - 4y = 0\,(2)$

Show Solution

We begin by finding the determinants $D, D_x, \text{and } D_y.$

$$D = \begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = 3\,(-4) - 6\,(-2) = 0$$

We know that a determinant of zero means that either the system has no solution or it has an infinite number of solutions. To see which one, we use the process of elimination. Our goal is to eliminate one of the variables.

1. Multiply equation (1) by $-2.$
2. Add the result to equation (2) .

$$\begin{array}{rl} -6x + 4y & = -8 \\ 6x - 4y & = \ \ 0 \\ \hline 0 & = -8 \end{array}$$

We obtain the equation $0 = -8,$ which is false. Therefore, the system has no solution. Graphing the system reveals two parallel lines. See (Figure).

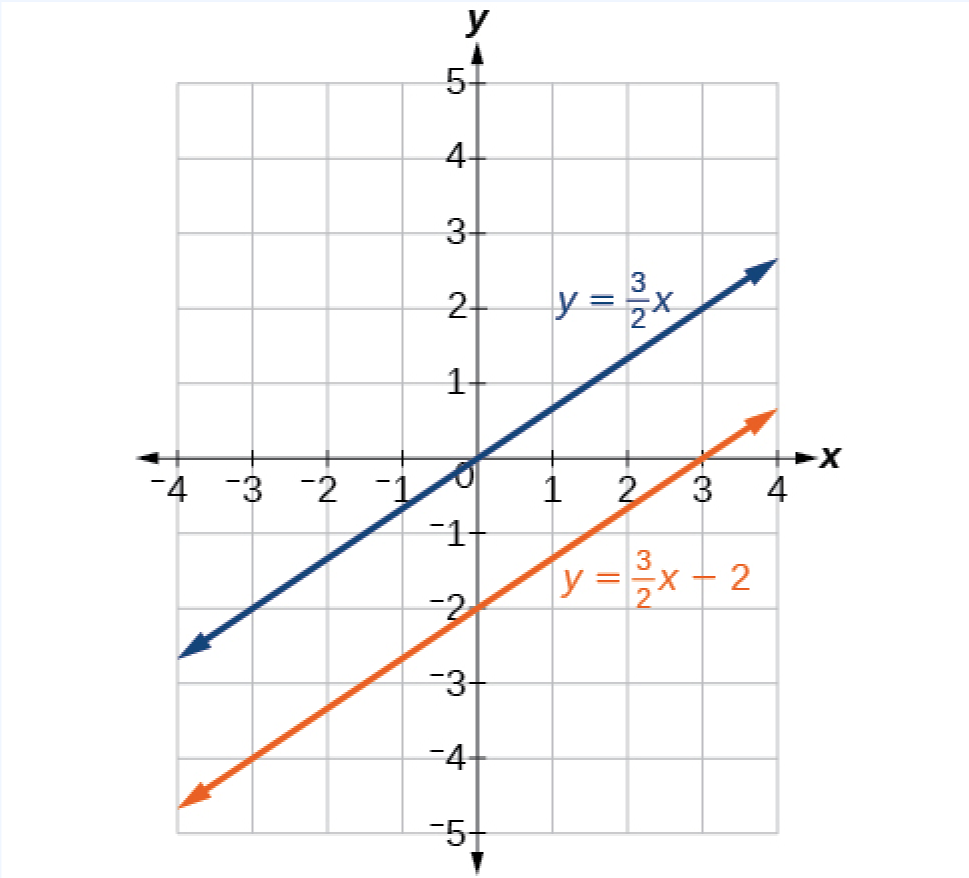


Figure 1.

Use Cramer's Rule to Solve a Dependent System

Solve the system with an infinite number of solutions.

$$\begin{aligned} x - 2y + 3z &= 0 \quad (1) \\ 3x + y - 2z &= 0 \quad (2) \\ 2x - 4y + 6z &= 0 \quad (3) \end{aligned}$$

Show Solution

Let's find the determinant first. Set up a matrix augmented by the first two columns.

$$\left|\begin{array}{ccc|cc} 1 & -2 & 3 & 1 & -2 \\ 3 & 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & 2 & -4 \end{array}\right|$$

Then,

$$1(1)(6)+(-2)(-2)(2)+3(3)(-4)-2(1)(3)-(-4)(-2)(1)-6(3)(-2)=0$$

As the determinant equals zero, there is either no solution or an infinite number of solutions. We have to perform elimination to find out.

1. Multiply equation (1) by -2 and add the result to equation (3):
$$\frac{\begin{array}{r} -2x+4y-6x=0 \\ 2x-4y+6z=0 \end{array}}{0=0}$$
2. Obtaining an answer of $0=0$, a statement that is always true, means that the system has an infinite number of solutions. Graphing the system, we can see that two of the planes are the same and they both intersect the third plane on a line. See (Figure).

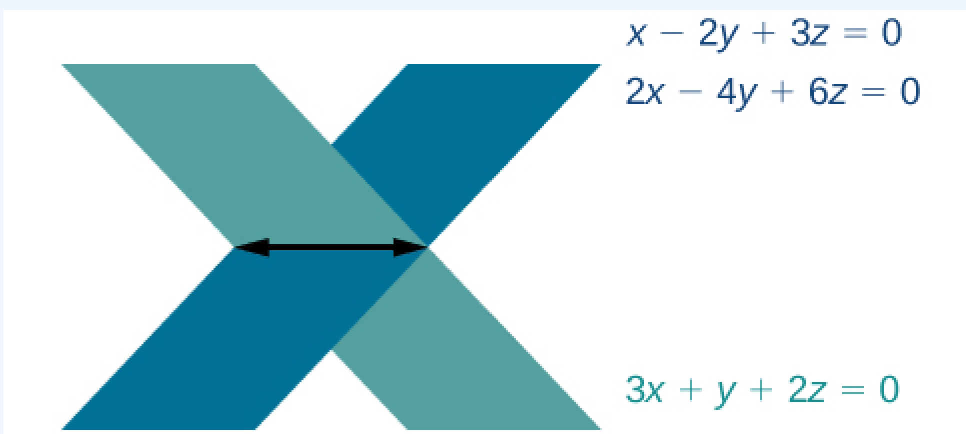


Figure 2.

Understanding Properties of Determinants

There are many properties of determinants. Listed here are some properties that may be helpful in calculating the determinant of a matrix.

Properties of Determinants

1. If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
2. When two rows are interchanged, the determinant changes sign.
3. If either two rows or two columns are identical, the determinant equals zero.
4. If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.
5. The determinant of an inverse matrix A^{-1} is the reciprocal of the determinant of the matrix A.
6. If any row or column is multiplied by a constant, the determinant is multiplied by the same factor.

Illustrating Properties of Determinants

Illustrate each of the properties of determinants.

Show Solution

Property 1 states that if the matrix is in upper triangular form, the determinant is the product of the entries down the main diagonal.

$$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 1\\ 0 & 0 & -1\end{bmatrix}$$

Augment A with the first two columns.

$$A=\left[\begin{array}{ccc|cc}1 & 2 & 3 & 1 & 2\\ 0 & 2 & 1 & 0 & 2\\ 0 & 0 & -1 & 0 & 0\end{array}\right.$$

Then

$$\begin{aligned}\det(A) &= 1(2)(-1)+2(1)(0)+3(0)(0)-0(2)(3)-0(1)(1)+1(0)(2)\\ &= -2\end{aligned}$$

Property 2 states that interchanging rows changes the sign. Given

$$A=\begin{bmatrix}-1 & 5\\ 4 & -3\end{bmatrix},\ \det(A)=(-1)(-3)-(4)(5)=3-20=-17$$

$$B=\begin{bmatrix}4 & -3\\ -1 & 5\end{bmatrix},\ \det(B)=(4)(5)-(-1)(-3)=20-3=17$$

Property 3 states that if two rows or two columns are identical, the determinant equals zero.

$$A = \left[\begin{array}{ccc|cc} 1 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ -1 & 2 & 2 & -1 & 2 \end{array}\right]$$

$$\begin{aligned} \det(A) &= 1(2)(2) + 2(2)(-1) + 2(2)(2) + 1(2)(2) - 2(2)(1) - 2(2)(2) \\ &= 4 - 4 + 8 + 4 - 4 - 8 = 0 \end{aligned}$$

Property 4 states that if a row or column equals zero, the determinant equals zero. Thus,

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad \det(A) = 1(0) - 2(0) = 0$$

Property 5 states that the determinant of an inverse matrix A^{-1} is the reciprocal of the determinant A. Thus,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \det(A) = 1(4) - 3(2) = -2$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}, \det\left(A^{-1}\right) = -2\left(-\frac{1}{2}\right) - \left(\frac{3}{2}\right)(1) = -\frac{1}{2}$$

Property 6 states that if any row or column of a matrix is multiplied by a constant, the determinant is multiplied by the same factor. Thus,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \det(A) = 1(4) - 2(3) = -2$$

$$B = \begin{bmatrix} 2(1) & 2(2) \\ 3 & 4 \end{bmatrix}, \det(B) = 2(4) - 3(4) = -4$$

Using Cramer's Rule and Determinant Properties to Solve a System

Find the solution to the given 3 × 3 system.

$$\begin{aligned} 2x + 4y + 4z &= 2 & (1) \\ 3x + 7y + 7z &= -5 & (2) \\ x + 2y + 2z &= 4 & (3) \end{aligned}$$

Show Solution

Using Cramer's Rule, we have

$$D = \begin{vmatrix} 2 & 4 & 4 \\ 3 & 7 & 7 \\ 1 & 2 & 2 \end{vmatrix}$$

Notice that the second and third columns are identical. According to Property 3, the determinant will be zero, so there is either no solution or an infinite number of solutions. We have to perform elimination to find out.

1. Multiply equation (3) by –2 and add the result to equation (1).

$$\begin{array}{c} -2x - 4y - 4x = -8 \\ 2x + 4y + 4z = 2 \\ \hline 0 = -6 \end{array}$$

Obtaining a statement that is a contradiction means that the system has no solution.

Access these online resources for additional instruction and practice with Cramer's Rule.

- Solve a System of Two Equations Using Cramer's Rule
- Solve a Systems of Three Equations using Cramer's Rule

Key Concepts

- The determinant for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. See (Figure).
- Cramer's Rule replaces a variable column with the constant column. Solutions are $x = \frac{D_x}{D}, y = \frac{D_y}{D}$. See (Figure).
- To find the determinant of a 3×3 matrix, augment with the first two columns. Add the three diagonal entries (upper left to lower right) and subtract the three diagonal entries (lower left to upper right). See (Figure).
- To solve a system of three equations in three variables using Cramer's Rule, replace a variable column with the constant column for each desired solution: $x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$.

See (Figure).

- Cramer's Rule is also useful for finding the solution of a system of equations with no solution or infinite solutions. See (Figure) and (Figure).
- Certain properties of determinants are useful for solving problems. For example:
    - If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
    - When two rows are interchanged, the determinant changes sign.
    - If either two rows or two columns are identical, the determinant equals zero.
    - If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.
    - The determinant of an inverse matrix A^{-1} is the reciprocal of the determinant of the matrix $A.$
    - If any row or column is multiplied by a constant, the determinant is multiplied by the same factor. See (Figure) and (Figure).

Section Exercises

Verbal

1. Explain why we can always evaluate the determinant of a square matrix.

Show Solution
A determinant is the sum and products of the entries in the matrix, so you can always evaluate that product—even if it does end up being 0.

2. Examining Cramer's Rule, explain why there is no unique solution to the system when the determinant of your matrix is 0. For simplicity, use a $2\ 2$ matrix.

3. Explain what it means in terms of an inverse for a matrix to have a 0 determinant.

Show Solution
The inverse does not exist.

4. The determinant of $2\ 2$ matrix A is 3. If you switch the rows and multiply the first row by 6 and the second row by 2, explain how to find the determinant and provide the answer.

Algebraic

For the following exercises, find the determinant.

5. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

Show Solution
-2

6. $\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$

7. $\begin{vmatrix} 2 & -5 \\ -1 & 6 \end{vmatrix}$

Show Solution
7

8. $\begin{vmatrix} -8 & 4 \\ -1 & 5 \end{vmatrix}$

9. $\begin{vmatrix} 1 & 0 \\ 3 & -4 \end{vmatrix}$

Show Solution
-4

10. $\begin{vmatrix} 10 & 20 \\ 0 & -10 \end{vmatrix}$

11. $\begin{vmatrix} 10 & 0.2 \\ 5 & 0.1 \end{vmatrix}$

Show Solution
0

12. $\begin{vmatrix} 6 & -3 \\ 8 & 4 \end{vmatrix}$

13. $\begin{vmatrix} -2 & -3 \\ 3.1 & 4,000 \end{vmatrix}$

Show Solution
$-7,990.7$

14. $\begin{vmatrix} -1.1 & 0.6 \\ 7.2 & -0.5 \end{vmatrix}$

15. $\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix}$

Show Solution
3

16. $\begin{vmatrix} -1 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

17. $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

Show Solution
-1

18. $\begin{vmatrix} 2 & -3 & 1 \\ 3 & -4 & 1 \\ -5 & 6 & 1 \end{vmatrix}$

19. $\begin{vmatrix} -2 & 1 & 4 \\ -4 & 2 & -8 \\ 2 & -8 & -3 \end{vmatrix}$

Show Solution
224

20. $\begin{vmatrix} 6 & -1 & 2 \\ -4 & -3 & 5 \\ 1 & 9 & -1 \end{vmatrix}$

21. $\begin{vmatrix} 5 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -6 & -3 \end{vmatrix}$

Show Solution
15

22. $\begin{vmatrix} 1.1 & 2 & -1 \\ -4 & 0 & 0 \\ 4.1 & -0.4 & 2.5 \end{vmatrix}$

23. $\begin{vmatrix} 2 & -1.6 & 3.1 \\ 1.1 & 3 & -8 \\ -9.3 & 0 & 2 \end{vmatrix}$

Show Solution
-17.03

24. $\begin{vmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & -\frac{1}{6} & \frac{1}{7} \\ 0 & 0 & \frac{1}{8} \end{vmatrix}$

For the following exercises, solve the system of linear equations using Cramer's Rule.

25. $\begin{aligned} 2x - 3y &= -1 \\ 4x + 5y &= 9 \end{aligned}$

Show Solution

$(1, 1)$

26. $$\begin{aligned} 5x - 4y &= 2 \\ -4x + 7y &= 6 \end{aligned}$$

27. $$\begin{aligned} 6x - 3y &= 2 \\ -8x + 9y &= -1 \end{aligned}$$

Show Solution

$\left(\frac{1}{2}, \frac{1}{3}\right)$

28. $$\begin{aligned} 2x + 6y &= 12 \\ 5x - 2y &= 13 \end{aligned}$$

29. $$\begin{aligned} 4x + 3y &= 23 \\ 2x - y &= -1 \end{aligned}$$

Show Solution

$(2, 5)$

30. $$\begin{aligned} 10x - 6y &= 2 \\ -5x + 8y &= -1 \end{aligned}$$

31. $$\begin{aligned} 4x - 3y &= -3 \\ 2x + 6y &= -4 \end{aligned}$$

Show Solution

$\left(-1, -\frac{1}{3}\right)$

32. $$\begin{aligned} 4x - 5y &= 7 \\ -3x + 9y &= 0 \end{aligned}$$

33. $$\begin{aligned} 4x + 10y &= 180 \\ -3x - 5y &= -105 \end{aligned}$$

Show Solution

$(15, 12)$

34. $$\begin{aligned} 8x - 2y &= -3 \\ -4x + 6y &= 4 \end{aligned}$$

For the following exercises, solve the system of linear equations using Cramer's Rule.

35. $$\begin{aligned} x + 2y - 4z &= -1 \\ 7x + 3y + 5z &= 26 \\ -2x - 6y + 7z &= -6 \end{aligned}$$

Show Solution

$(1, 3, 2)$

36. $$\begin{aligned} -5x + 2y - 4z &= -47 \\ 4x - 3y - z &= -94 \\ 3x - 3y + 2z &= 94 \end{aligned}$$

37. $$\begin{aligned} 4x + 5y - z &= -7 \\ -2x - 9y + 2z &= 8 \\ 5y + 7z &= 21 \end{aligned}$$

Show Solution

$(-1, 0, 3)$

38. $$\begin{aligned} 4x - 3y + 4z &= 10 \\ 5x - 2z &= -2 \\ 3x + 2y - 5z &= -9 \end{aligned}$$

39. $$\begin{aligned} 4x - 2y + 3z &= 6 \\ -6x + y &= -2 \\ 2x + 7y + 8z &= 24 \end{aligned}$$

Show Solution

$\left(\frac{1}{2}, 1, 2\right)$

40. $$\begin{aligned} 5x + 2y - z &= 1 \\ -7x - 8y + 3z &= 1.5 \\ 6x - 12y + z &= 7 \end{aligned}$$

41. $$\begin{aligned} 13x - 17y + 16z &= 73 \\ -11x + 15y + 17z &= 61 \\ 46x + 10y - 30z &= -18 \end{aligned}$$

Show Solution

$(2, 1, 4)$

42. $$\begin{aligned}-4x-3y-8z&=-7\\2x-9y+5z&=0.5\\5x-6y-5z&=-2\end{aligned}$$

43. $$\begin{aligned}4x-6y+8z&=10\\-2x+3y-4z&=-5\\x+y+z&=1\end{aligned}$$

Show Solution
Infinite solutions

44. $$\begin{aligned}4x-6y+8z&=10\\-2x+3y-4z&=-5\\12x+18y-24z&=-30\end{aligned}$$

Technology

For the following exercises, use the determinant function on a graphing utility.

45. $$\begin{vmatrix}1&0&8&9\\0&2&1&0\\1&0&3&0\\0&2&4&3\end{vmatrix}$$

Show Solution
24

46. $$\begin{vmatrix}1&0&2&1\\0&-9&1&3\\3&0&-2&-1\\0&1&1&-2\end{vmatrix}$$

47. $\begin{vmatrix} \frac{1}{2} & 1 & 7 & 4 \\ 0 & \frac{1}{2} & 100 & 5 \\ 0 & 0 & 2 & 2,000 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

Show Solution

1

48. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{vmatrix}$

Real-World Applications

For the following exercises, create a system of linear equations to describe the behavior. Then, calculate the determinant. Will there be a unique solution? If so, find the unique solution.

49. Two numbers add up to 56. One number is 20 less than the other.

Show Solution

Yes; 18, 38

50. Two numbers add up to 104. If you add two times the first number plus two times the second number, your total is 208

51. Three numbers add up to 106. The first number is 3 less than the second number. The third number is 4 more than the first number.

Show Solution

Yes; 33, 36, 37

52. Three numbers add to 216. The sum of the first two numbers is 112. The third number is 8 less than the first two numbers combined.

For the following exercises, create a system of linear equations to describe the behavior. Then, solve the system for all solutions using Cramer's Rule.

53. You invest \$10,000 into two accounts, which receive 8% interest and 5% interest. At the end of a year, you had \$10,710 in your combined accounts. How much was invested in each account?

Show Solution

\$7,000 in first account, \$3,000 in second account.

54. You invest \$80,000 into two accounts, \$22,000 in one account, and \$58,000 in the other account. At the end of one year, assuming simple interest, you have earned \$2,470 in interest. The second account receives half a percent less than twice the interest on the first account. What are the interest rates for your accounts?

55. A movie theater needs to know how many adult tickets and children tickets were sold out of the 1,200 total tickets. If children's tickets are \$5.95, adult tickets are \$11.15, and the total amount of revenue was \$12,756, how many children's tickets and adult tickets were sold?

Show Solution

120 children, 1,080 adult

56. A concert venue sells single tickets for \$40 each and couple's tickets for \$65. If the total revenue was \$18,090 and the 321 tickets were sold, how many single tickets and how many couple's tickets were sold?

57. You decide to paint your kitchen green. You create the color of paint by mixing yellow and blue paints. You cannot remember how many gallons of each color went into your mix, but you know there were 10 gal total. Additionally, you kept your receipt, and know the total amount spent was \$29.50. If each gallon of yellow costs \$2.59, and each gallon of blue costs \$3.19, how many gallons of each color go into your green mix?

Show Solution
4 gal yellow, 6 gal blue

58. You sold two types of scarves at a farmers' market and would like to know which one was more popular. The total number of scarves sold was 56, the yellow scarf cost \$10, and the purple scarf cost \$11. If you had total revenue of \$583, how many yellow scarves and how many purple scarves were sold?

59. Your garden produced two types of tomatoes, one green and one red. The red weigh 10 oz, and the green weigh 4 oz. You have 30 tomatoes, and a total weight of 13 lb, 14 oz. How many of each type of tomato do you have?

Show Solution
13 green tomatoes, 17 red tomatoes

60. At a market, the three most popular vegetables make up 53% of vegetable sales. Corn has 4% higher sales than broccoli, which has 5% more sales than onions. What percentage does each vegetable have in the market share?

61. At the same market, the three most popular fruits make up 37% of the total fruit sold. Strawberries sell twice as much as oranges, and kiwis sell one more percentage point than oranges. For each fruit, find the percentage of total fruit sold.

Show Solution
Strawberries 18%, oranges 9%, kiwi 10%

62. Three bands performed at a concert venue. The first band charged $15 per ticket, the second band charged $45 per ticket, and the final band charged $22 per ticket. There were 510 tickets sold, for a total of $12,700. If the first band had 40 more audience members than the second band, how many tickets were sold for each band?

63. A movie theatre sold tickets to three movies. The tickets to the first movie were $5, the tickets to the second movie were $11, and the third movie was $12. 100 tickets were sold to the first movie. The total number of tickets sold was 642, for a total revenue of $6,774. How many tickets for each movie were sold?

Show Solution
100 for movie 1, 230 for movie 2, 312 for movie 3

64. Men aged 20–29, 30–39, and 40–49 made up 78% of the population at a prison last year. This year, the same age groups made up 82.08% of the population. The 20–29 age group increased by 20%, the 30–39 age group increased by 2%, and the 40–49 age group decreased to $\frac{3}{4}$ of their previous population. Originally, the 30–39 age group had 2% more prisoners than the 20–29 age group. Determine the prison population percentage for each age group last year.

65. At a women's prison down the road, the total number of inmates aged 20–49 totaled 5,525. This year, the 20–29 age group increased by 10%, the 30–39 age group decreased by 20%, and the 40–49 age group doubled. There are now 6,040 prisoners. Originally, there were 500 more in the

30–39 age group than the 20–29 age group. Determine the prison population for each age group last year.

Show Solution
20–29: 2,100, 30–39: 2,600, 40–49: 825

For the following exercises, use this scenario: A health-conscious company decides to make a trail mix out of almonds, dried cranberries, and chocolate-covered cashews. The nutritional information for these items is shown in (Figure).

	Fat (g)	Protein (g)	Carbohydrates (g)
Almonds (10)	6	2	3
Cranberries (10)	0.02	0	8
Cashews (10)	7	3.5	5.5

66. For the special "low-carb"trail mix, there are 1,000 pieces of mix. The total number of carbohydrates is 425 g, and the total amount of fat is 570.2 g. If there are 200 more pieces of cashews than cranberries, how many of each item is in the trail mix?

67. For the "hiking" mix, there are 1,000 pieces in the mix, containing 390.8 g of fat, and 165 g of protein. If there is the same amount of almonds as cashews, how many of each item is in the trail mix?

Show Solution
300 almonds, 400 cranberries, 300 cashews

68. For the "energy-booster" mix, there are 1,000 pieces in the mix, containing 145 g of protein and 625 g of carbohydrates. If the number of almonds and cashews summed together is equivalent to the amount of cranberries, how many of each item is in the trail mix?

Review Exercises

Systems of Linear Equations: Two Variables

For the following exercises, determine whether the ordered pair is a solution to the system of equations.

1. $\begin{array}{r}3x-y=4\\ x+4y=-3\end{array}$ and $(-1,1)$

Show Solution

No

2. $\begin{array}{r}6x-2y=24\\ -3x+3y=18\end{array}$ and $(9,15)$

For the following exercises, use substitution to solve the system of equations.

3. $\begin{array}{r}10x+5y=-5\\ 3x-2y=-12\end{array}$

Show Solution

$(-2,3)$

4. $\begin{array}{r}\frac{4}{7}x+\frac{1}{5}y=\frac{43}{70}\\ \frac{5}{6}x-\frac{1}{3}y=-\frac{2}{3}\end{array}$

5. $\begin{array}{r}5x+6y=14\\ 4x+8y=8\end{array}$

Show Solution
$(4, -1)$

For the following exercises, use addition to solve the system of equations.

6. $$\begin{aligned} 3x + 2y &= -7 \\ 2x + 4y &= 6 \end{aligned}$$

7. $$\begin{aligned} 3x + 4y &= 2 \\ 9x + 12y &= 3 \end{aligned}$$

Show Solution
No solutions exist.

8. $$\begin{aligned} 8x + 4y &= 2 \\ 6x - 5y &= 0.7 \end{aligned}$$

For the following exercises, write a system of equations to solve each problem. Solve the system of equations.

9. A factory has a cost of production $C(x) = 150x + 15{,}000$ and a revenue function $R(x) = 200x$. What is the break-even point?

Show Solution
$(300, 60, 000)$

10. A performer charges $C(x) = 50x + 10{,}000$, where x is the total number of attendees at a show. The venue charges $75 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

Show Solution

$(400, 30, 000)$

Systems of Linear Equations: Three Variables

For the following exercises, solve the system of three equations using substitution or addition.

11. $$\begin{aligned} 0.5x - 0.5y &= 10 \\ -0.2y + 0.2x &= 4 \\ 0.1x + 0.1z &= 2 \end{aligned}$$

Show Solution

$(10, -10, 10)$

12. $$\begin{aligned} 5x + 3y - z &= 5 \\ 3x - 2y + 4z &= 13 \\ 4x + 3y + 5z &= 22 \end{aligned}$$

13. $$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + 2z &= 1 \\ 3x + 3y &= 2 \end{aligned}$$

Show Solution

No solutions exist.

14. $$\begin{aligned} 2x - 3y + z &= -1 \\ x + y + z &= -4 \\ 4x + 2y - 3z &= 33 \end{aligned}$$

15. $$\begin{aligned} 3x + 2y - z &= -10 \\ x - y + 2z &= 7 \\ -x + 3y + z &= -2 \end{aligned}$$

Show Solution

$(-1, -2, 3)$

16. $$\begin{aligned} 3x + 4z &= -11 \\ x - 2y &= 5 \\ 4y - z &= -10 \end{aligned}$$

17. $$\begin{aligned} 2x - 3y + z &= 0 \\ 2x + 4y - 3z &= 0 \\ 6x - 2y - z &= 0 \end{aligned}$$

Show Solution

$\left(x, \frac{8x}{5}, \frac{14x}{5}\right)$

18. $$\begin{aligned} 6x - 4y - 2z &= 2 \\ 3x + 2y - 5z &= 4 \\ 6y - 7z &= 5 \end{aligned}$$

For the following exercises, write a system of equations to solve each problem. Solve the system of equations.

19. Three odd numbers sum up to 61. The smaller is one-third the larger and the middle number is 16 less than the larger. What are the three numbers?

Show Solution

11, 17, 33

20. A local theatre sells out for their show. They sell all 500 tickets for a total purse of $8,070.00. The tickets were priced at $15 for students, $12 for children, and $18 for adults. If the band sold three times as many adult tickets as children's tickets, how many of each type was sold?

Solving Systems with Cramer's Rule

For the following exercises, find the determinant.

21. $\begin{vmatrix} 100 & 0 \\ 0 & 0 \end{vmatrix}$

Show Solution

0

22. $\begin{vmatrix} 0.2 & -0.6 \\ 0.7 & -1.1 \end{vmatrix}$

23. $\begin{vmatrix} -1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

Show Solution

6

24. $\begin{vmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{vmatrix}$

For the following exercises, use Cramer's Rule to solve the linear systems of equations.

25. $$\begin{aligned} 4x - 2y &= 23 \\ -5x - 10y &= -35 \end{aligned}$$

Show Solution

$\left(6, \frac{1}{2}\right)$

26. $$\begin{aligned} 0.2x - 0.1y &= 0 \\ -0.3x + 0.3y &= 2.5 \end{aligned}$$

27. $$\begin{aligned} -0.5x + 0.1y &= 0.3 \\ -0.25x + 0.05y &= 0.15 \end{aligned}$$

Show Solution

$(x, 5x + 3)$

28. $$\begin{aligned} x + 6y + 3z &= 4 \\ 2x + y + 2z &= 3 \\ 3x - 2y + z &= 0 \end{aligned}$$

29. $$\begin{aligned} 4x - 3y + 5z &= -\tfrac{5}{2} \\ 7x - 9y - 3z &= \tfrac{3}{2} \\ x - 5y - 5z &= \tfrac{5}{2} \end{aligned}$$

Show Solution

$\left(0, 0, -\frac{1}{2}\right)$

30. $$\begin{aligned} \frac{3}{10}x - \frac{1}{5}y - \frac{3}{10}z &= -\frac{1}{50} \\ \frac{1}{10}x - \frac{1}{10}y - \frac{1}{2}z &= -\frac{9}{50} \\ \frac{2}{5}x - \frac{1}{2}y - \frac{3}{5}z &= -\frac{1}{5} \end{aligned}$$

Glossary

Cramer's Rule
: a method for solving systems of equations that have the same number of equations as variables using determinants

determinant
: a number calculated using the entries of a square matrix that determines such information as whether there is a solution to a system of equations

CHAPTER 3: RIGHT ANGLE TRIGONOMETRY

CHAPTER 3.1: INTRODUCTION TO THE UNIT CIRCLE: SINE AND COSINE FUNCTIONS

Figure 1. The tide rises and falls at regular, predictable intervals. (credit: Andrea Schaffer, Flickr)

Life is dense with phenomena that repeat in regular intervals. Each day, for example, the tides rise and fall in response to the gravitational pull of the moon. Similarly, the progression from day to night occurs as a result of Earth's rotation, and the pattern of the seasons repeats in response to Earth's revolution around the sun. Outside of nature, many stocks that mirror a company's profits are influenced by changes in the economic business cycle.

In mathematics, a function that repeats its values in regular intervals is known as a periodic function. The graphs of such functions show a general shape reflective of a pattern that keeps repeating. This means the graph of the function has the same output at exactly the same place in every cycle. And this translates to all the cycles of the function having exactly the same length. So, if we know all the details of one full cycle of a true periodic function, then we know the state of the function's outputs at all times, future and past. In this chapter, we will investigate various examples of periodic functions.

CHAPTER 3.2: ANGLES

Learning Objectives

In this section you will:

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a ray. A ray is a directed line segment. It consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its endpoint and any other point on it. The ray in (Figure) can be named as ray EF, or in symbol form EF .

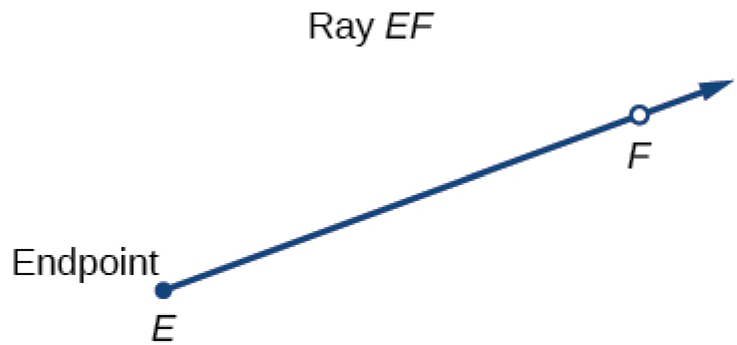


Figure 1.

An angle is the union of two rays having a common endpoint. The endpoint is called the vertex of the angle, and the two rays are the sides of the angle. The angle in (Figure) is formed from ED and EF . Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

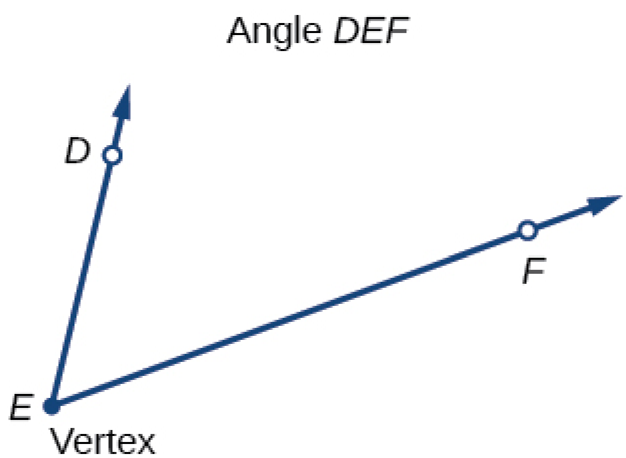


Figure 2.

Greek letters are often used as variables for the measure of an angle. (Figure) is a list of Greek letters commonly used to represent angles, and a sample angle is shown in (Figure).

θ	ϕ **or** φ	α	β	γ
theta	phi	alpha	beta	gamma

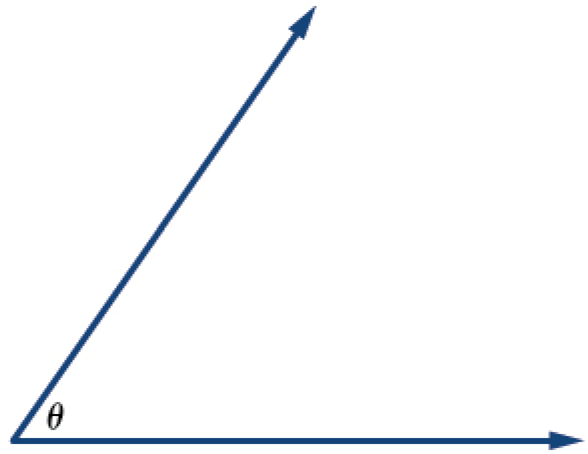


Figure 3. Angle theta, shown as $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the initial side**,** and the rotated ray is the terminal side. In order to identify the different sides, we indicate the rotation with a small arrow close to the vertex as in (Figure).

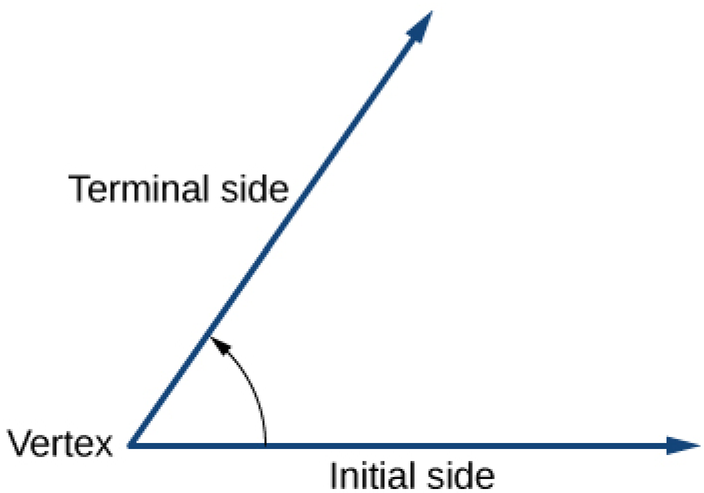


Figure 4.

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The measure of an angle is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One degree is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol $^\circ$. For example, $90 \text{ degrees} = 90^\circ$.

To formalize our work, we will begin by drawing angles on an x–y coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in standard position if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See (Figure).

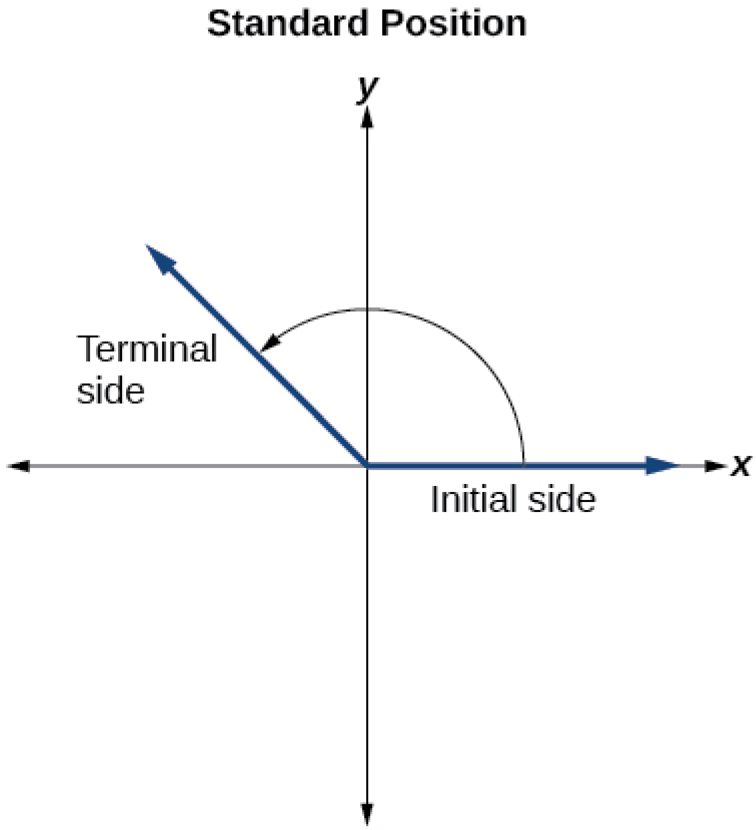


Figure 5.

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a positive angle. If the angle is measured in a clockwise direction, the angle is said to be a negative angle.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by $360°$. For example, to draw a $90°$ angle, we calculate that $\frac{90°}{360°} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a $360°$ angle, we calculate that $\frac{360°}{360°} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case, the initial side and the terminal side overlap. See (Figure).

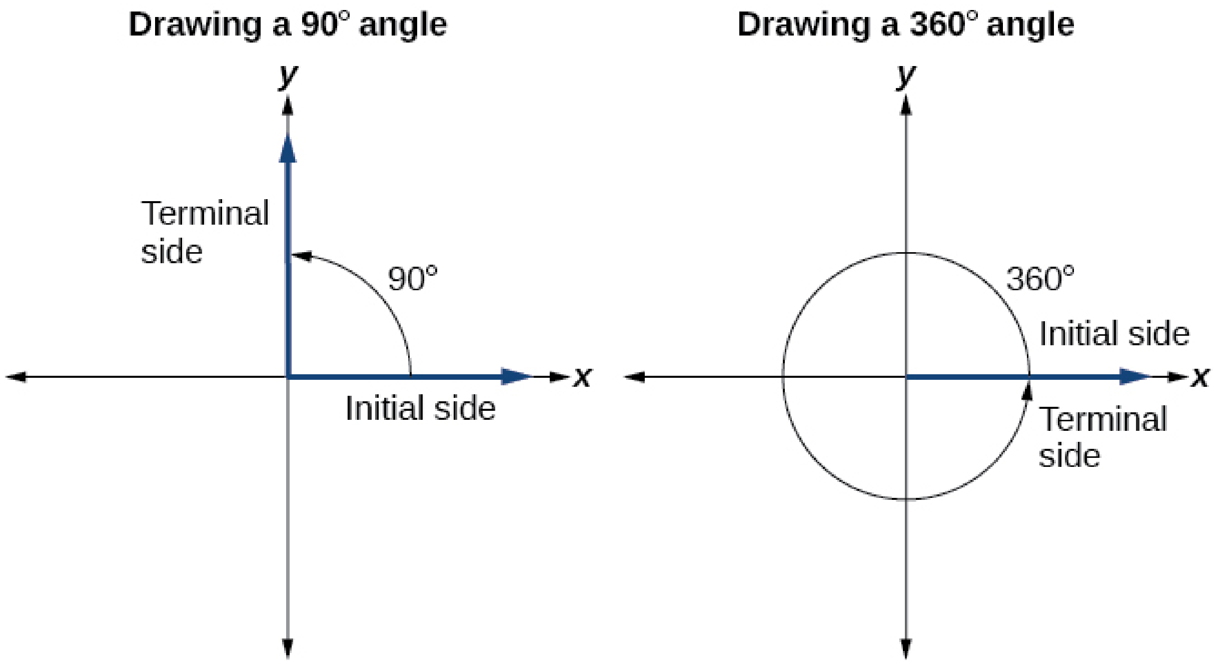


Figure 6.

Since we define an angle in standard position by its terminal side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of $0°$, $90°$, $180°$, $270°$ or $360°$. See (Figure).

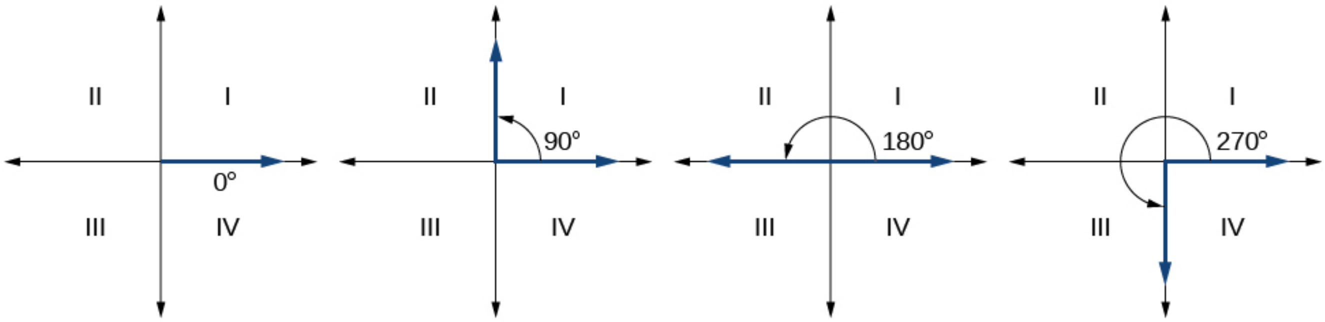


Figure 7. Quadrantal angles have a terminal side that lies along an axis. Examples are shown.

Quadrantal Angles

An angle is a quadrantal angle if its terminal side lies on an axis, including $0°$, , $90°$, $180°$, $270°$ or $360°$.

How To

Given an angle measure in degrees, draw the angle in standard position.

1. Express the angle measure as a fraction of $360°$.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Drawing an Angle in Standard Position Measured in Degrees

a. Sketch an angle of $30°$ in standard position.

b. Sketch an angle of $-135°$ in standard position.

Show Solution

a. Divide the angle measure by $360°$.

$\frac{30°}{360°} = \frac{1}{12}$

To rewrite the fraction in a more familiar fraction, we can recognize that

$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right)$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at $30°$, as in (Figure).

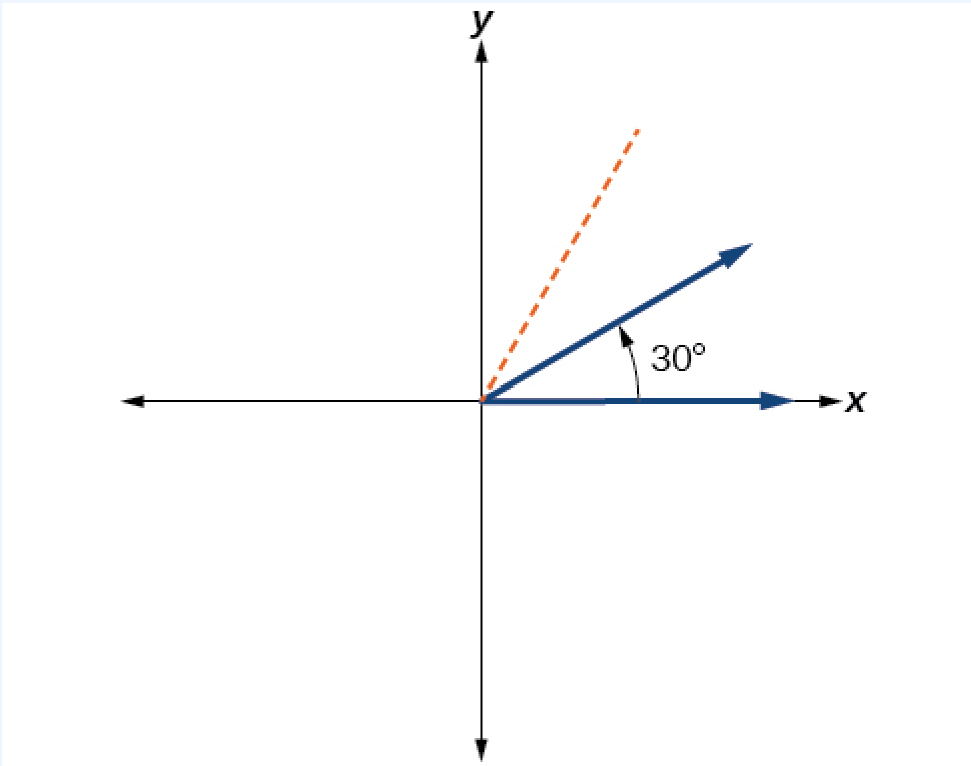


Figure 8.

b. Divide the angle measure by $360°$.

$\frac{-135°}{360°} = -\frac{3}{8}$

In this case, we can recognize that

$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right)$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in (Figure).

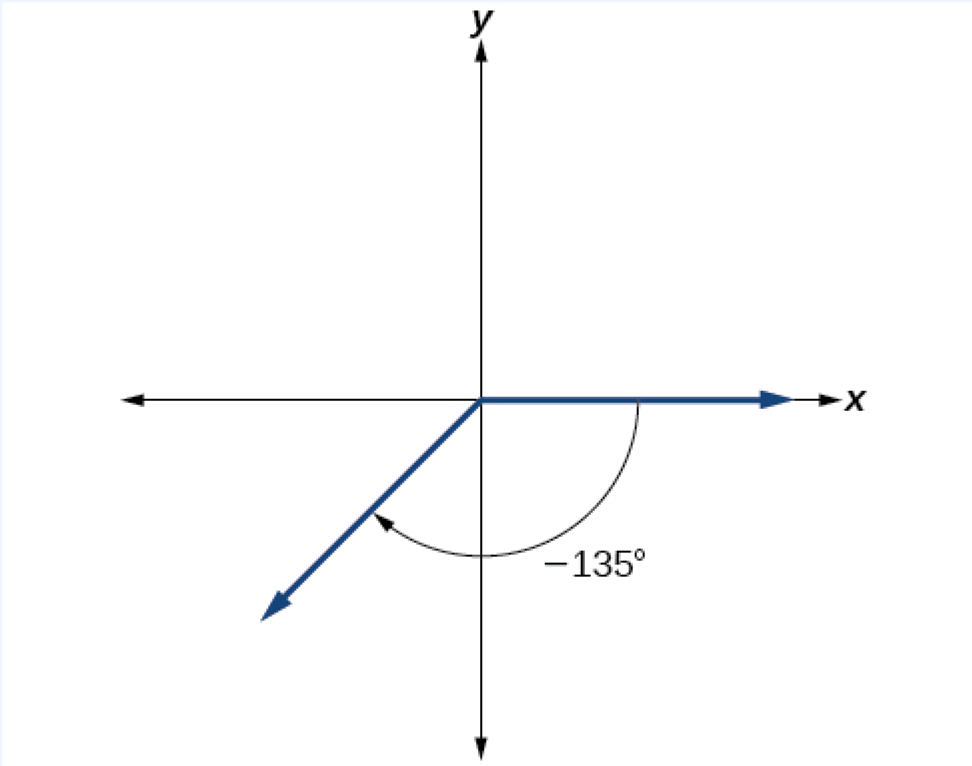
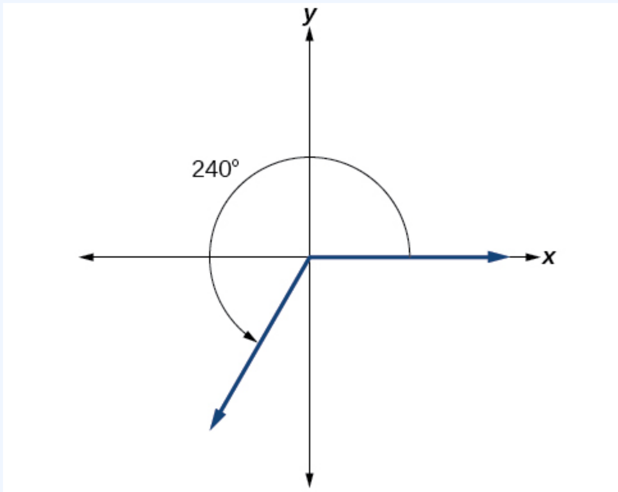


Figure 9.

Try It

Show an angle of $240°$ on a circle in standard position.

Show Solution



Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An arc may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the circumference of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference, which is always 2π, to the radius, regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in (Figure).

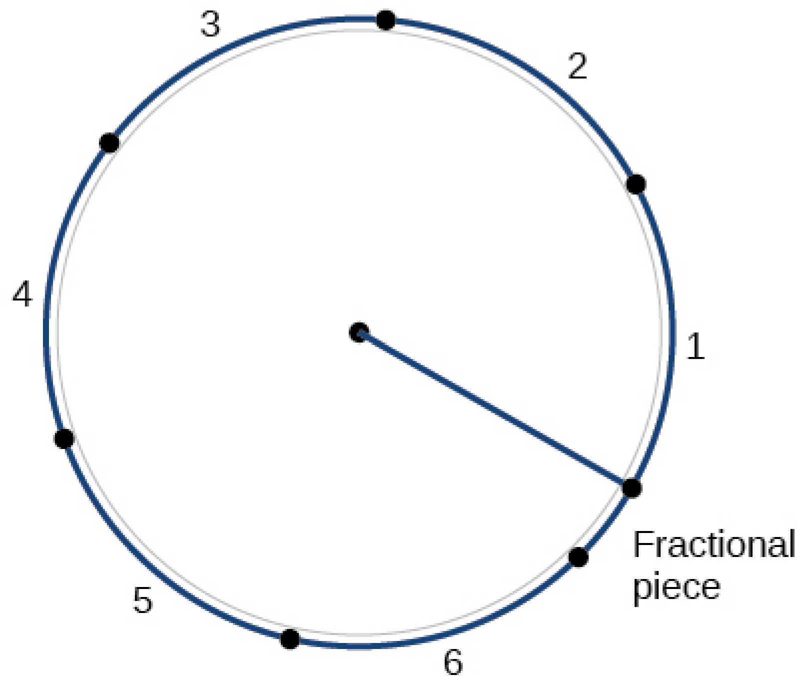


Figure 10.

This brings us to our new angle measure. One radian is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians.

$$\begin{aligned} 2\pi \text{ radians} &= 360° \\ \pi \text{ radians} &= \frac{360°}{2} = 180° \\ 1 \text{ radian} &= \frac{180°}{\pi} \approx 57.3° \end{aligned}$$

See (Figure). Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel.

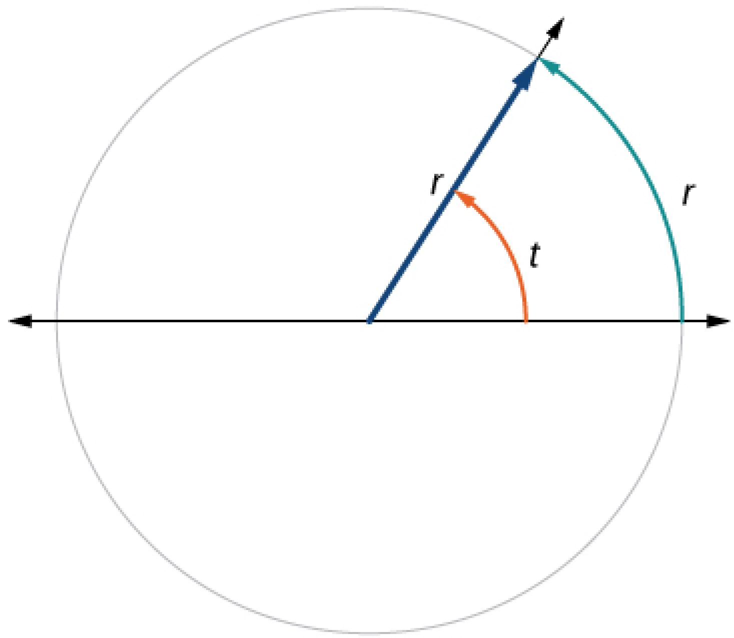


Figure 11. The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An arc length s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the radian measure, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s to the radius r. See (Figure).

$$\begin{array}{lll} s & = & r\theta \\ \theta & = & \frac{s}{r} \end{array}$$

If $s = r$, then $\theta = \frac{r}{r} = \ 1 \text{ radian.}$

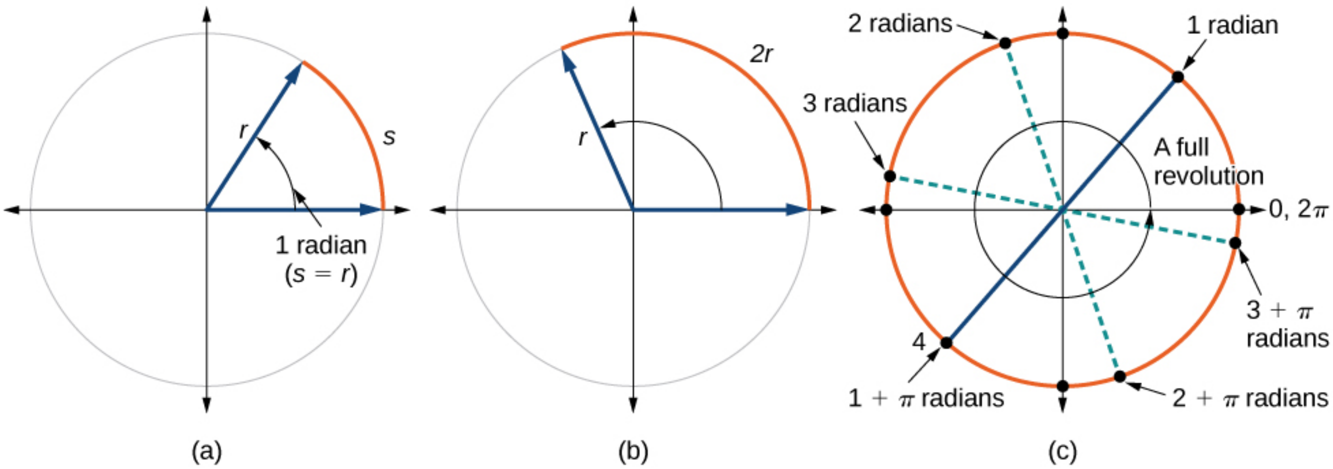


Figure 12. (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π, or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$, where r is the radius. The smaller circle then has circumference $2\pi\left(2\right) = 4\pi$ and the larger has circumference $2\pi\left(3\right) = 6\pi$. Now we draw a $45°$ angle on the two circles, as in (Figure).

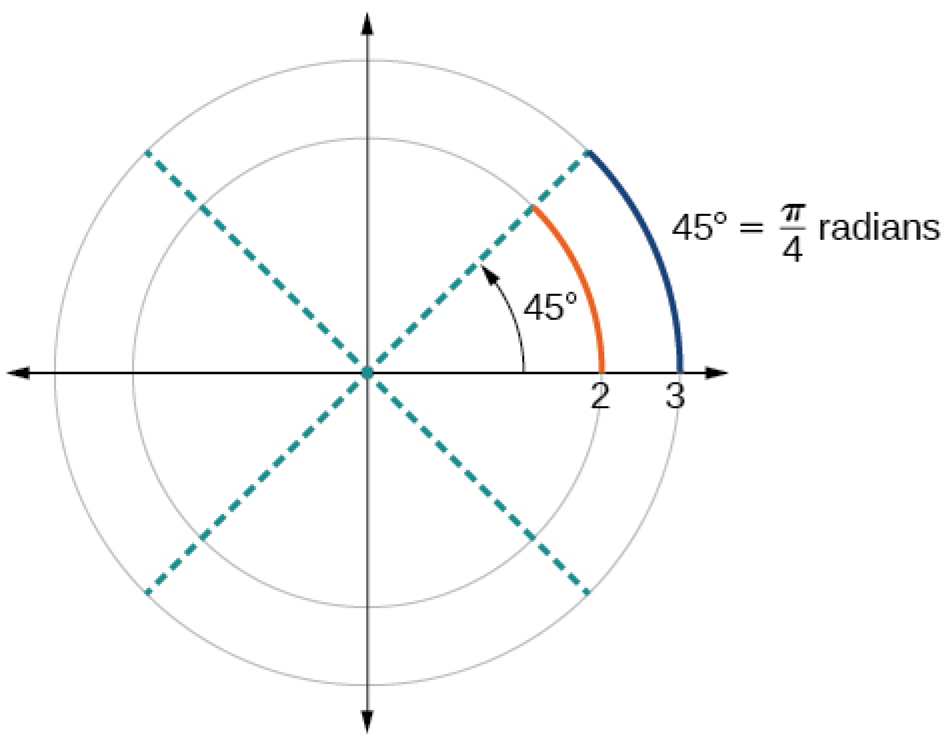


Figure 13. A $45°$ angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi$$
$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

Radians

One radian is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360) equals 2π radians. A half revolution $(180°)$ is equivalent to π radians.

The radian measure of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

A measure of 1 radian looks to be about 60. **Is that correct?**

Yes. It is approximately $57.3°$. *Because* 2π *radians equals* $360°$, 1 *radian equals* $\frac{360°}{2\pi} \approx 57.3°$.

Using Radians

Because radian measure is the ratio of two lengths, it is a unitless measure. For example, in (Figure), suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the unit circle (a circle with radius 1), we know that 1 rotation equals 360 degrees, $360°$. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{array}{lllll} 1 \text{ rotation} & = & 360° & = & 2\pi \text{ radians} \\ \frac{1}{2} \text{ rotation} & = & 180° & = & \pi \text{ radians} \\ \frac{1}{4} \text{ rotation} & = & 90° & = & \frac{\pi}{2} \text{ radians} \end{array}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in (Figure). Memorizing these angles will be very useful as we study the properties associated with angles.

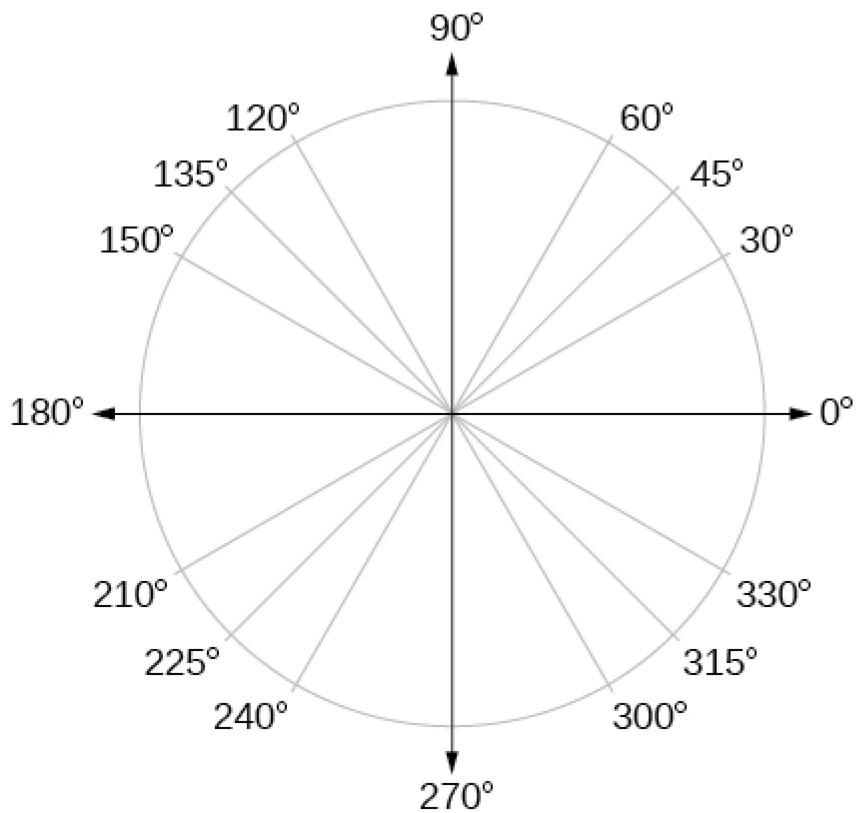


Figure 14. Commonly encountered angles measured in degrees

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in (Figure), which are shown in (Figure). Be sure you can verify each of these measures.

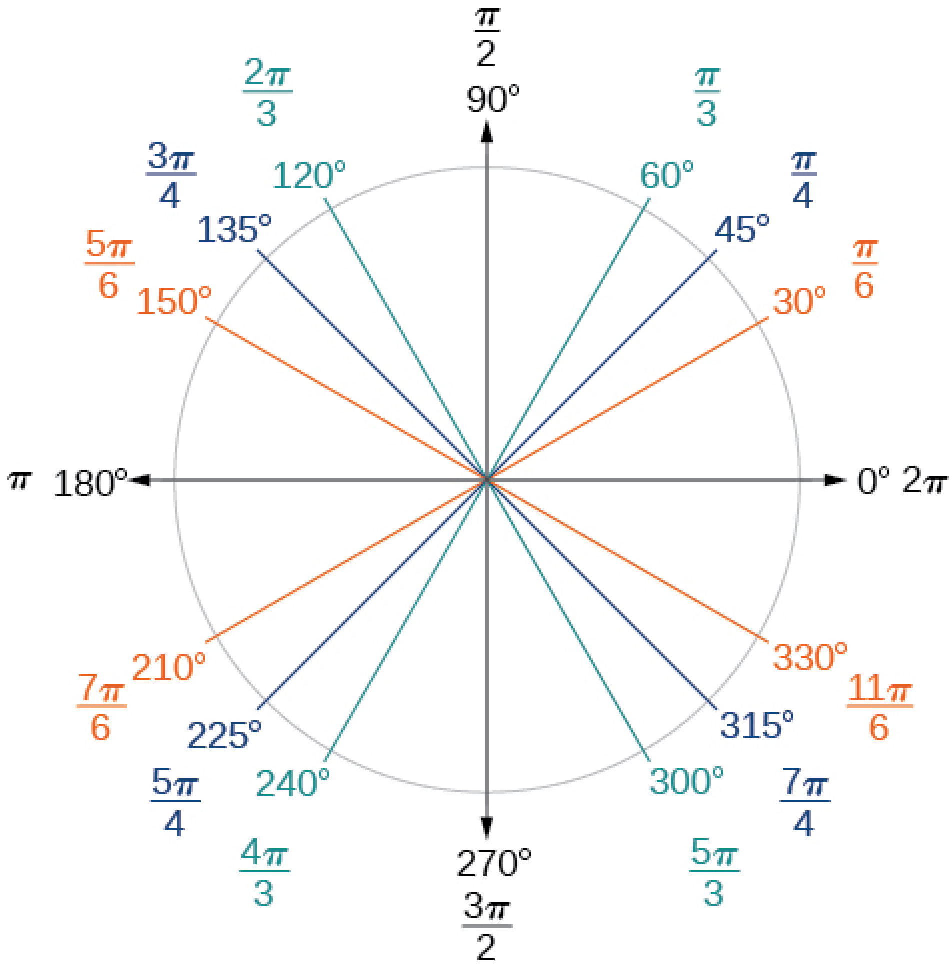


Figure 15. Commonly encountered angles measured in radians

Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Show Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1\ \text{rotation} = 2\pi r$$

So,

$$\begin{aligned} s &= \frac{1}{3}(2\pi r) \\ &= \frac{2\pi r}{3} \end{aligned}$$

The radian measure would be the arc length divided by the radius.

$$\begin{aligned} \text{radian measure} &= \frac{\frac{2\pi r}{3}}{r} \\ &= \frac{2\pi r}{3r} \\ &= \frac{2\pi}{3} \end{aligned}$$

Try It

Find the radian measure of three-fourths of a full rotation.

Show Solution

$\frac{3\pi}{2}$

Converting Between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion where θ is the measure of the angle in degrees and θ_R is the measure of the angle in radians.

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{\text{Radians}}{\pi}$$

Converting between Radians and Degrees

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

Converting Radians to Degrees

Convert each radian measure to degrees.

a. $\frac{\pi}{6}$
b. 3

Show Solution

Because we are given radians and we want degrees, we should set up a proportion and solve it.

a. We use the proportion, substituting the given information.

$$\begin{aligned} \frac{\theta}{180} &= \frac{\theta_R}{\pi} \\ \frac{\theta}{180} &= \frac{\frac{\pi}{6}}{\pi} \\ \theta &= \frac{180}{6} \\ \theta &= 30^\circ \end{aligned}$$

b. We use the proportion, substituting the given information.

$$\begin{aligned} \frac{\theta}{180} &= \frac{\theta_R}{\pi} \\ \frac{\theta}{180} &= \frac{3}{\pi} \\ \theta &= \frac{3(180)}{\pi} \\ \theta &\approx 172^\circ \end{aligned}$$

Try It

Convert $-\frac{3\pi}{4}$ radians to degrees.

Show Solution
$-135°$

Converting Degrees to Radians

Convert 15 degrees to radians.

Show Solution
In this example, we start with degrees and want radians, so we again set up a proportion, but we substitute the given information into a different part of the proportion.

$$\begin{aligned}\frac{\theta}{180} &= \frac{\theta_R}{\pi}\\ \frac{15}{180} &= \frac{\theta_R}{\pi}\\ \frac{15\pi}{180} &= \theta_R\\ \frac{\pi}{12} &= \theta_R\end{aligned}$$

Analysis

Another way to think about this problem is by remembering that $30° = \frac{\pi}{6}$. Because $15° = \frac{1}{2}(30°)$, we can find that $\frac{1}{2}\left(\frac{\pi}{6}\right)$ is $\frac{\pi}{12}$.

Try It

Convert 126 to radians.

Show Solution
$\frac{7\pi}{10}$

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of $0°$ to $360°$, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at (Figure). The angle of $140°$ is a positive angle, measured counterclockwise. The angle of $-220°$ is a negative angle, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are coterminal angles. Every angle greater than $360°$ or less than $0°$ is coterminal with an angle between $0°$ and $360°$, and it is often more convenient to find the coterminal angle within the range of $0°$ to $360°$ than to work with an angle that is outside that range.

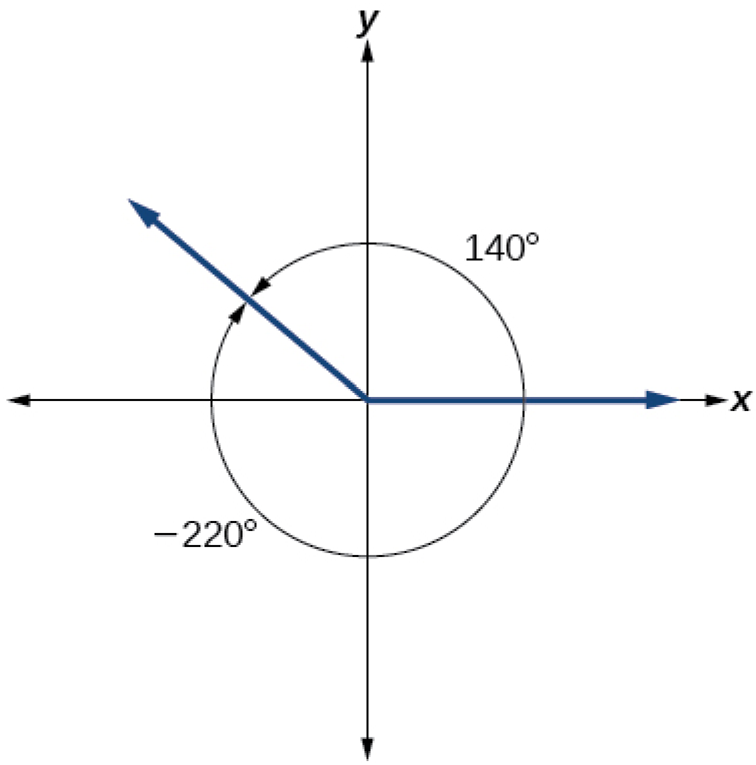


Figure 16. An angle of $140°$ and an angle of $-220°$ are coterminal angles.

Any angle has infinitely many coterminal angles because each time we add $360°$ to that angle—or subtract $360°$ from it—the resulting value has a terminal side in the same location. For example, $100°$ and $460°$ are coterminal for this reason, as is $-260°$.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See (Figure) for examples of reference angles for angles in different quadrants.

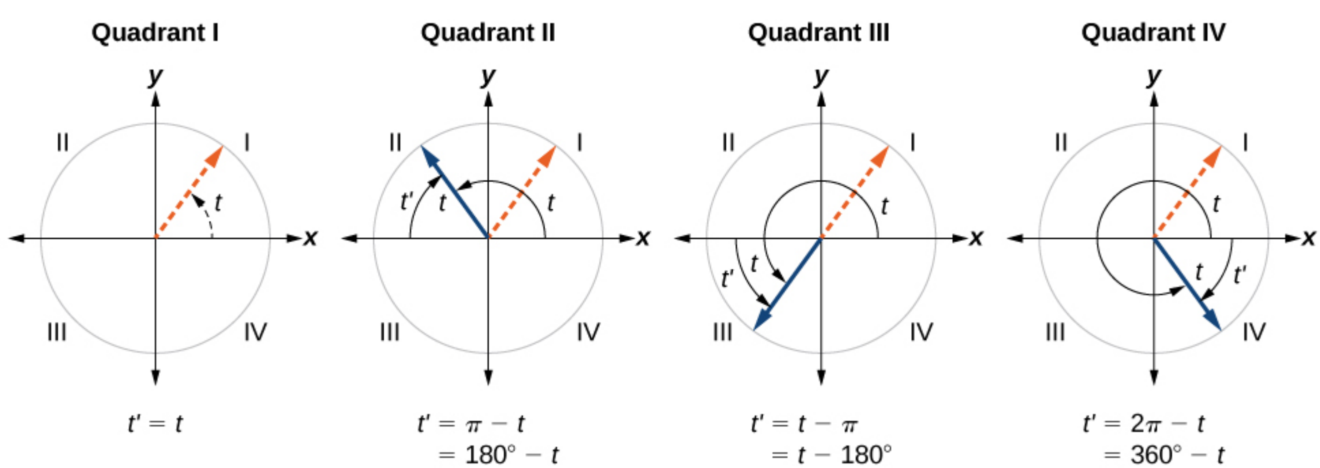


Figure 17.

Coterminal and Reference Angles

Coterminal angles are two angles in standard position that have the same terminal side.

An angle's reference angle is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

How To

Given an angle greater than $360°$, find a coterminal angle between $0°$ and $360°$

1. Subtract $360°$ from the given angle.
2. If the result is still greater than $360°$, subtract $360°$ again till the result is between $0°$ and $360°$.
3. The resulting angle is coterminal with the original angle.

Finding an Angle Coterminal with an Angle of Measure Greater Than $360°$

Find the least positive angle θ that is coterminal with an angle measuring $800°$, where $0° \leq \theta < 360°$.

Show Solution

An angle with measure $800°$ is coterminal with an angle with measure $800 - 360 = 440°$, but $440°$ is still greater than $360°$, so we subtract $360°$ again to find another coterminal angle: $440 - 360 = 80°$.

The angle $\theta = 80°$ is coterminal with $800°$. To put it another way, $800°$ equals $80°$ plus two full rotations, as shown in (Figure).

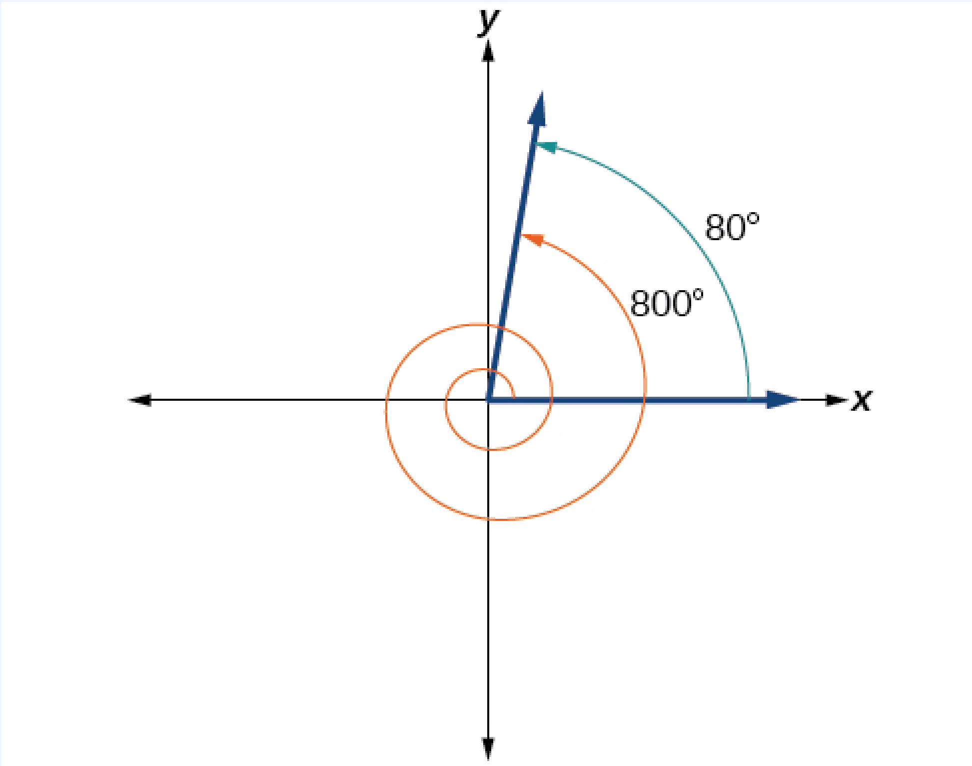


Figure 18.

Try It

Find an angle α that is coterminal with an angle measuring $870,$ where $0 \leq \alpha < 360.$

Show Solution

$\alpha = 150°$

How To

Given an angle with measure less than $0°$, find a coterminal angle having a measure between $0°$ and $360°$.

1. Add $360°$ to the given angle.
2. If the result is still less than $0°$, add $360°$ again until the result is between $0°$ and $360°$.
3. The resulting angle is coterminal with the original angle.

Finding an Angle Coterminal with an Angle Measuring Less Than 0

Show the angle with measure $-45°$ on a circle and find a positive coterminal angle α such that $0° \leq \alpha < 360°$.

Show Solution

Since $45°$ is half of $90°$ we can start at the positive horizontal axis and measure clockwise half of a $90°$ angle.

Because we can find coterminal angles by adding or subtracting a full rotation of $360°$, we can find a positive coterminal angle here by adding $360°$.

$-45° + 360° = 315°$

We can then show the angle on a circle, as in (Figure).

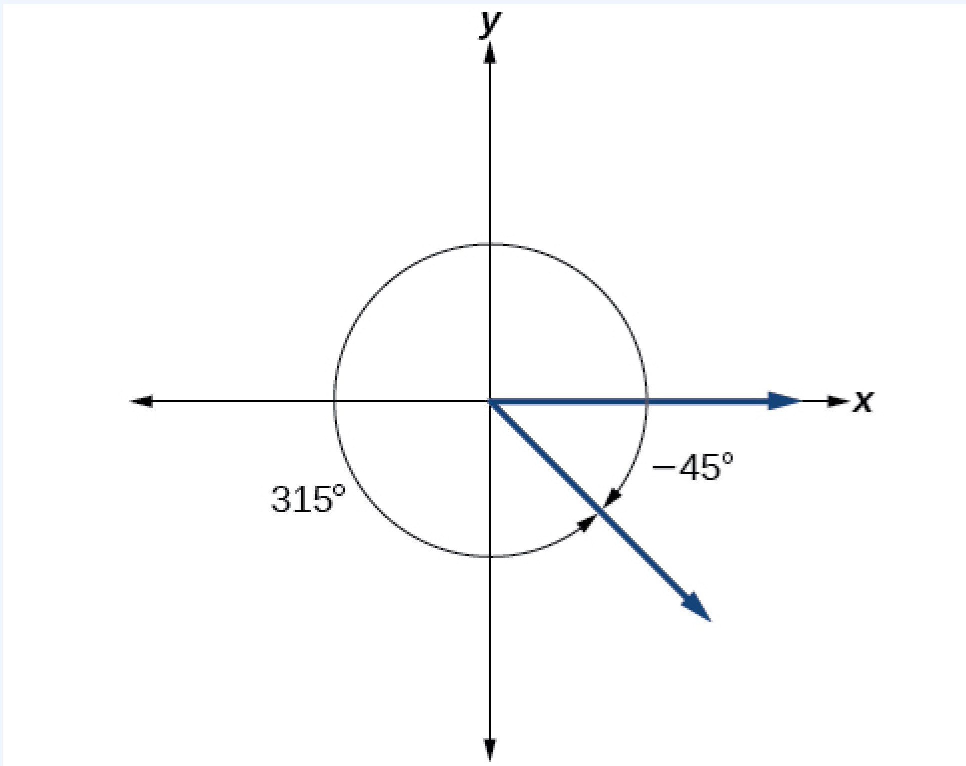


Figure 19.

Try It

Find an angle β that is coterminal with an angle measuring $-300°$ such that $0° \leq \beta < 360°.$

Show Solution
$\beta = 60°$

Finding Coterminal Angles Measured in Radians

We can find coterminal angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

How To

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

Show Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\begin{aligned}\frac{19\pi}{4} - 2\pi &= \frac{19\pi}{4} - \frac{8\pi}{4} \\ &= \frac{11\pi}{4}\end{aligned}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation.

$$\begin{aligned}\frac{11\pi}{4} - 2\pi &= \frac{11\pi}{4} - \frac{8\pi}{4} \\ &= \frac{3\pi}{4}\end{aligned}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in (Figure).

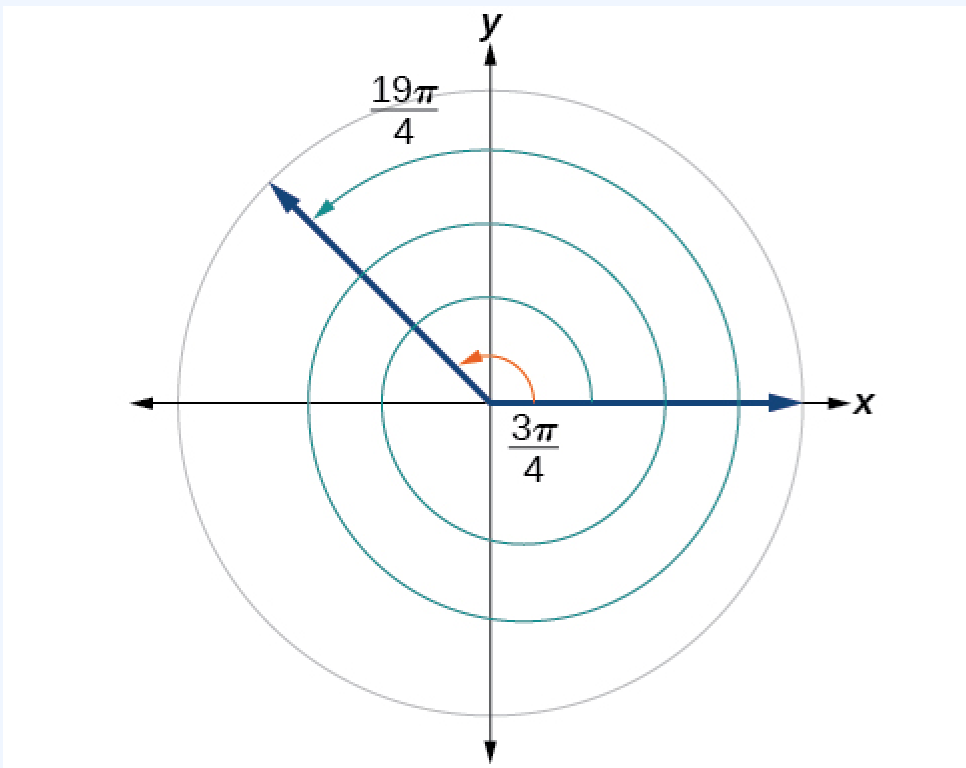


Figure 20.

Try It

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Show Solution

$\frac{7\pi}{6}$

Determining the Length of an Arc

Recall that the radian measure θ of an angle was defined as the ratio of the arc length s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

Arc Length on a Circle

In a circle of radius *r*, the length of an arc s subtended by an angle with measure θ in radians, shown in (Figure), is

$$s = r\theta$$

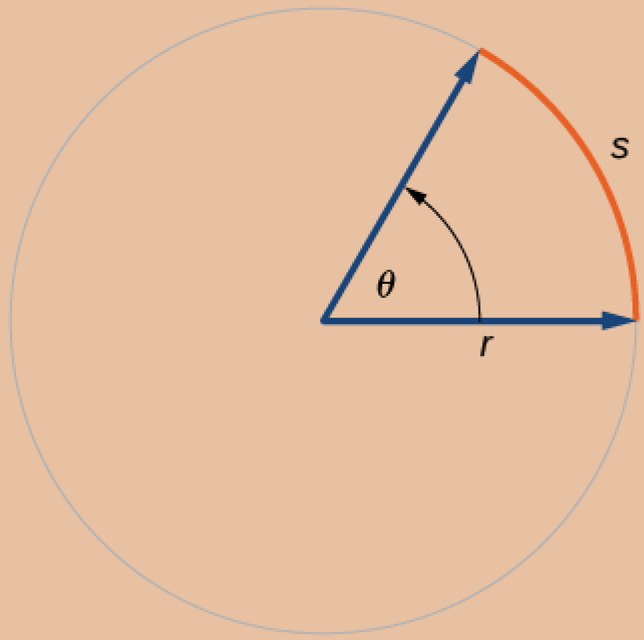


Figure 21.

How To

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r $\theta : s = r\theta$.

Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

a. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
b. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Show Solution

a. Let's begin by finding the circumference of Mercury's orbit.
$$\begin{aligned} C &= 2\pi r \\ &= 2\pi\,(36\text{ million miles}) \\ &\approx 226\text{ million miles} \end{aligned}$$
Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled.
$$(0.0114)\,226\text{ million miles} = 2.58\text{ million miles}$$
b. Now, we convert to radians.
$$\begin{aligned} \text{radian} &= \frac{\text{arclength}}{\text{radius}} \\ &= \frac{2.58\text{ million miles}}{36\text{ million miles}} \\ &= 0.0717 \end{aligned}$$

Try It

Find the arc length along a circle of radius 10 units subtended by an angle of $215°$.

Show Solution
$\frac{215\pi}{18} = 37.525\text{ units}$

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a sector of a circle. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A=\pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\begin{aligned}\text{Area of sector} &= \left(\frac{\theta}{2\pi}\right)\pi r^2\\ &= \frac{\theta\pi r^2}{2\pi}\\ &= \frac{1}{2}\theta r^2\end{aligned}$$

Area of a Sector

The area of a sector of a circle with radius r subtended by an angle θ, measured in radians, is

$A=\frac{1}{2}\theta r^2$

See (Figure).

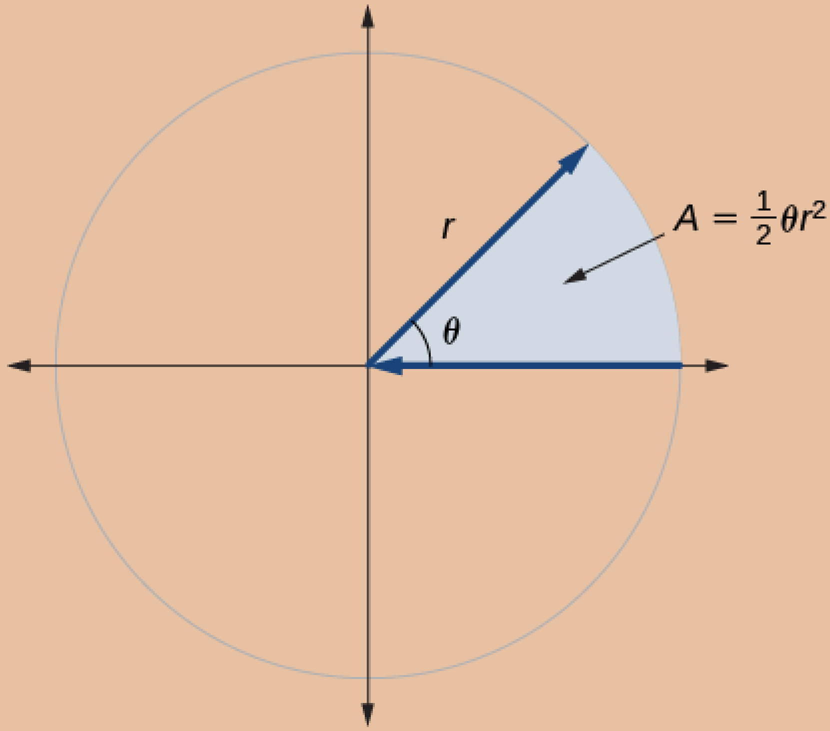


Figure 22. The area of the sector equals half the square of the radius times the central angle measured in radians.

How To

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in (Figure). What is the area of the sector of grass the sprinkler waters?

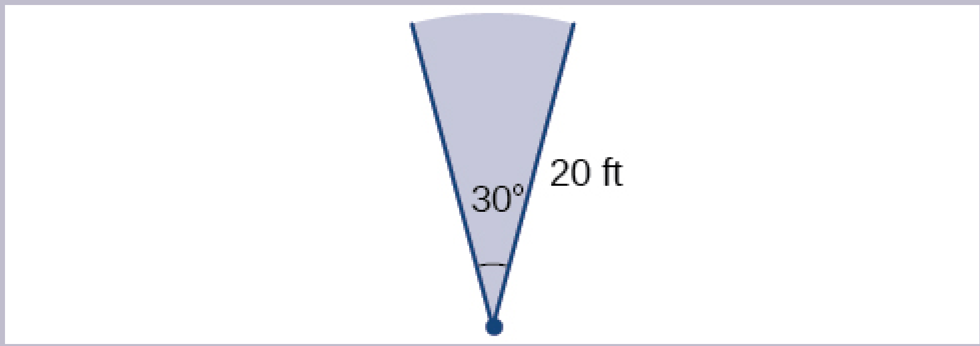


Figure 23. The sprinkler sprays 20 ft within an arc of $30°$.

Show Solution

First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$\begin{aligned} 30 \text{ degrees} &= 30 \cdot \tfrac{\pi}{180} \\ &= \tfrac{\pi}{6} \text{ radians} \end{aligned}$$

The area of the sector is then

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}\left(\tfrac{\pi}{6}\right)(20)^2 \\ &\approx 104.72 \end{aligned}$$

So the area is about 104.72 ft^2.

Try It

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Show Solution

1.88

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its displacement) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}}$ = 90 degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$\begin{aligned} s &= r\theta \\ &= r\omega t \end{aligned}$$

Substituting this into the linear speed equation gives:

$$\begin{aligned} v &= \frac{s}{t} \\ &= \frac{r\omega t}{t} \\ &= r\omega \end{aligned}$$

Angular and Linear Speed

As a point moves along a circle of radius r, its angular speed, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t}$$

The linear speed, v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

How To

Given the amount of angle rotation and the time elapsed, calculate the angular speed.

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Finding Angular Speed

A water wheel, shown in (Figure), completes 1 rotation every 5 seconds. Find the angular speed in radians per second.

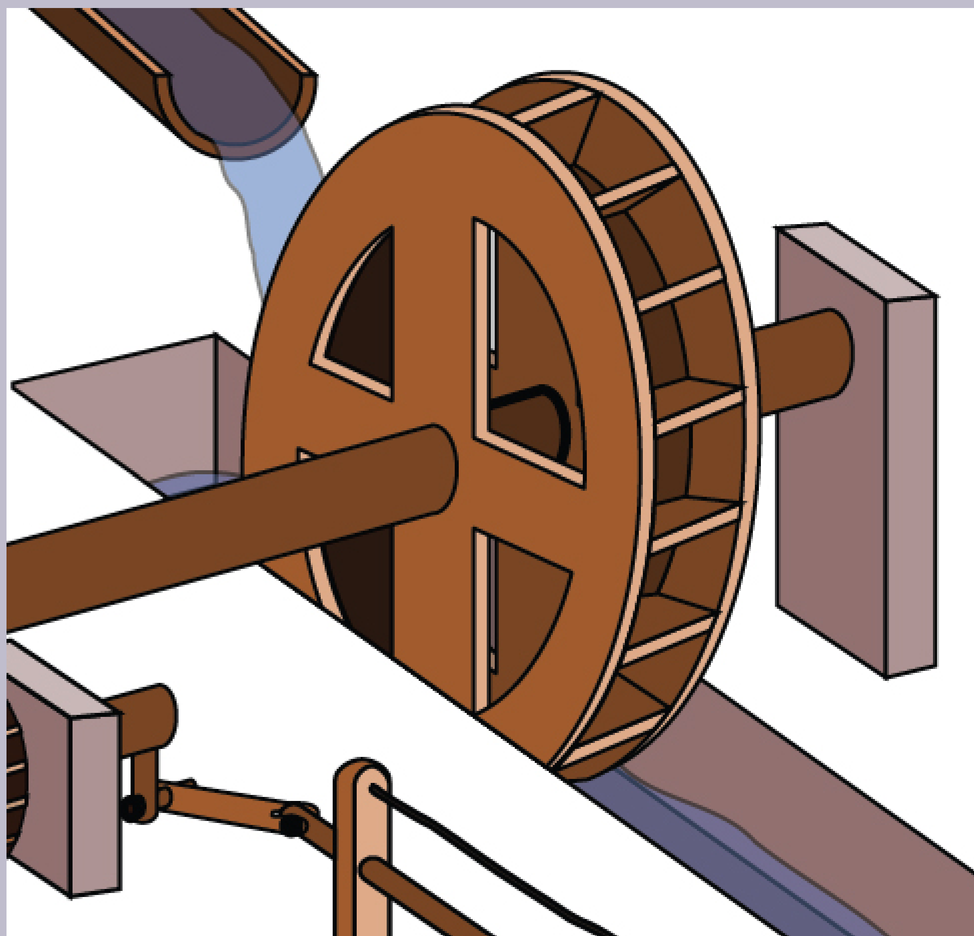


Figure 24.

Show Solution

The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

Try It

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Show Solution
$\frac{-3\pi}{2}$ rad/s

How To

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed.

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Show Solution

Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180\frac{\cancel{\text{rotations}}}{\text{minute}}\cdot\frac{2\pi\text{ radians}}{\cancel{\text{rotation}}}=360\pi\frac{\text{radians}}{\text{minute}}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$\begin{aligned} v &= (14\text{ inches})\left(360\pi\frac{\text{radians}}{\text{minute}}\right) \\ &= 5040\pi\frac{\text{inches}}{\text{minute}} \end{aligned}$$

Remember that radians are a unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi\frac{\cancel{\text{inches}}}{\cancel{\text{minute}}}\cdot\frac{1\ \cancel{\text{feet}}}{12\ \cancel{\text{inches}}}\cdot\frac{1\text{ mile}}{5280\ \cancel{\text{feet}}}\cdot\frac{60\ \cancel{\text{minutes}}}{1\text{ hour}}\approx 14.99\text{ miles per hour (mph)}$$

Try It

A satellite is rotating around Earth at 0.25 radian per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Show Solution

1655 kilometers per hour

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position
- Angle of Rotation
- Coterminal Angles
- Determining Coterminal Angles
- Positive and Negative Coterminal Angles
- Radian Measure
- Coterminal Angles in Radians
- Arc Length and Area of a Sector

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See (Figure).
- In addition to degrees, the measure of an angle can be described in radians. See (Figure).
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta_R}{\pi}$. See (Figure) and (Figure).
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting $360°$ or 2π. See (Figure) and (Figure).

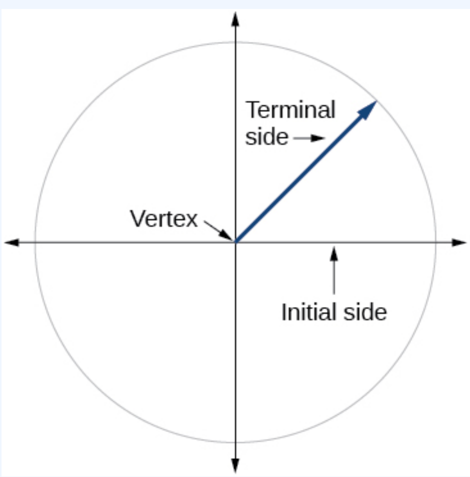
- Coterminal angles can be found using radians just as they are for degrees. See (Figure).
- The length of a circular arc is a fraction of the circumference of the entire circle. See (Figure).
- The area of sector is a fraction of the area of the entire circle. See (Figure).
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See (Figure).
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See (Figure).

Section Exercises

Verbal

1. Draw an angle in standard position. Label the vertex, initial side, and terminal side.

Show Solution



2. Explain why there are an infinite number of angles that are coterminal to a certain angle.

3. State what a positive or negative angle signifies, and explain how to draw each.

Show Solution

Whether the angle is positive or negative determines the direction. A positive angle is drawn in the counterclockwise direction, and a negative angle is drawn in the clockwise direction.

4. How does radian measure of an angle compare to the degree measure? Include an explanation of 1 radian in your paragraph.

5. Explain the differences between linear speed and angular speed when describing motion along a circular path.

Show Solution

Linear speed is a measurement found by calculating distance of an arc compared to time. Angular speed is a measurement found by calculating the angle of an arc compared to time.

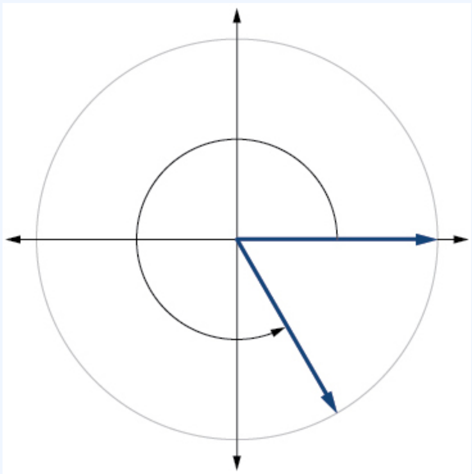
Graphical

For the following exercises, draw an angle in standard position with the given measure.

6. $30°$

7. $300°$

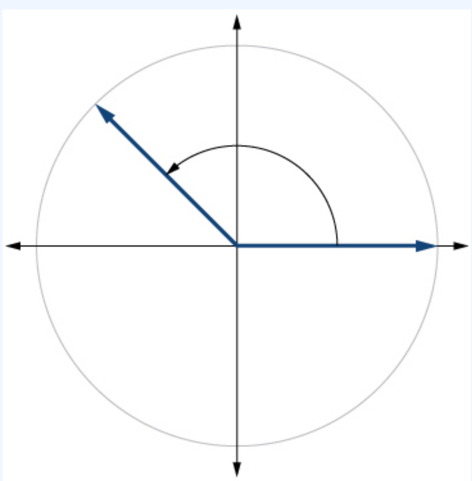
Show Solution



8. $-80°$

9. $135°$

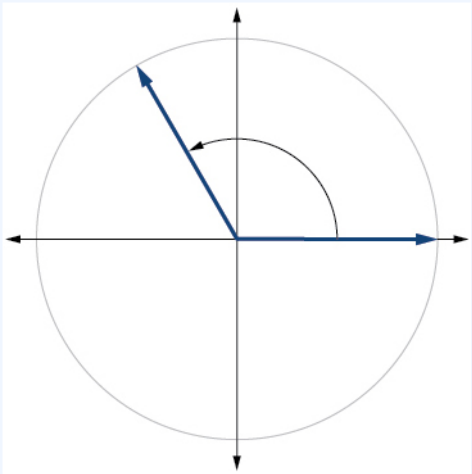
Show Solution



10. $-150°$

11. $\frac{2\pi}{3}$

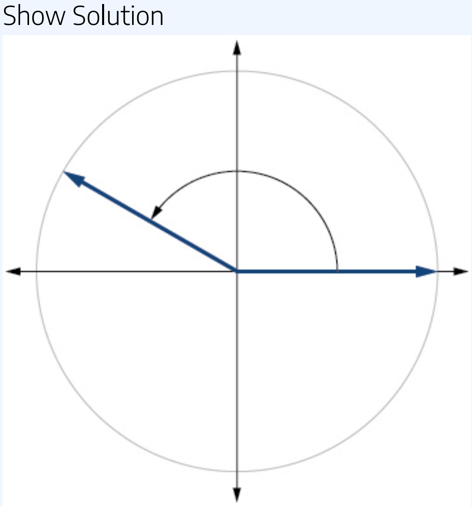
Show Solution



12. $\frac{7\pi}{4}$

13. $\frac{5\pi}{6}$

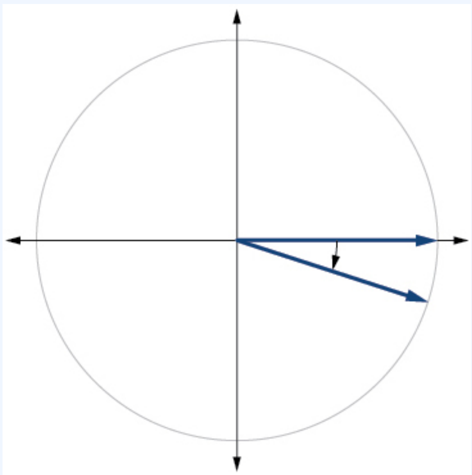
Show Solution



14. $\frac{\pi}{2}$

15. $-\frac{\pi}{10}$

Show Solution

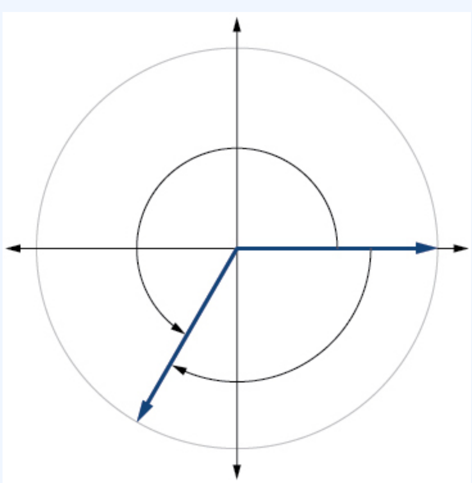


16. $415°$

17. $-120°$

Show Solution

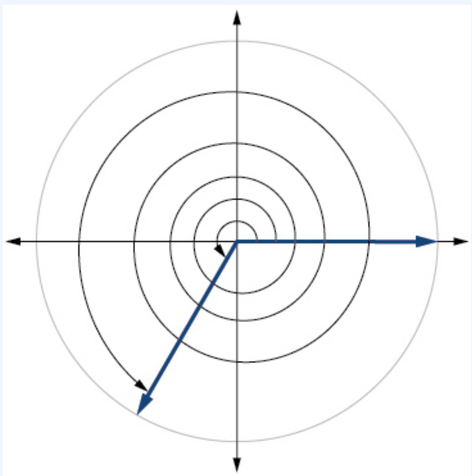
$240°$



18. $-315°$

19. $\frac{22\pi}{3}$

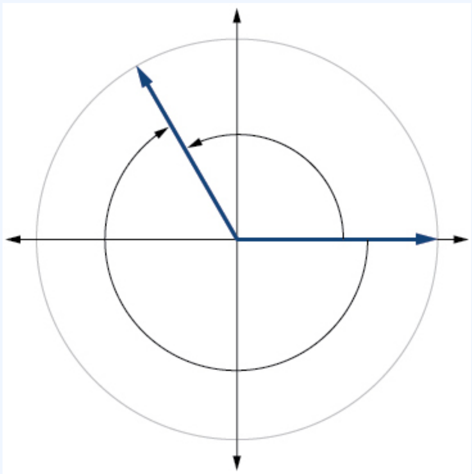
Show Solution
$\frac{4\pi}{3}$



20. $-\frac{\pi}{6}$

21. $-\frac{4\pi}{3}$

Show Solution
$\frac{2\pi}{3}$



For the following exercises, refer to (Figure). Round to two decimal places.

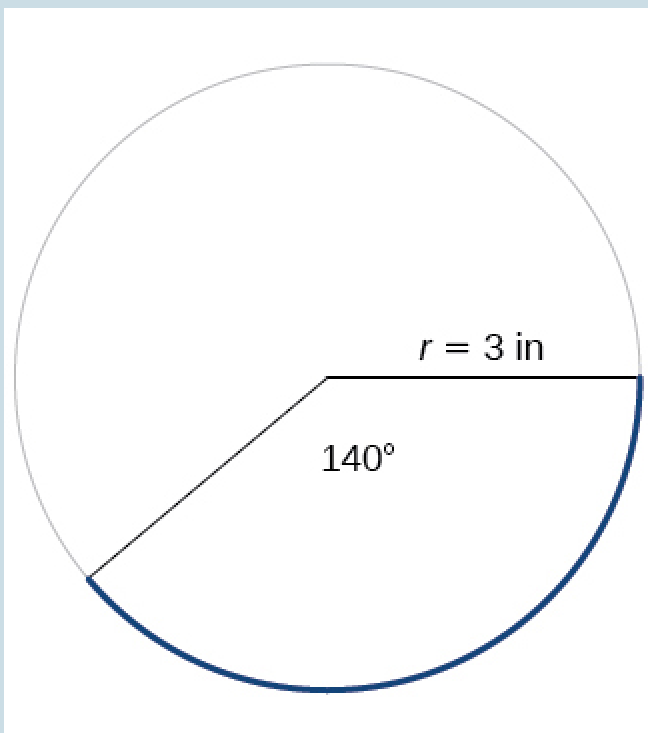


Figure 25.

22. Find the arc length.

23. Find the area of the sector.

Show Solution
$\frac{7\pi}{2} \approx 11.00 \text{ in}^2$

For the following exercises, refer to (Figure). Round to two decimal places.

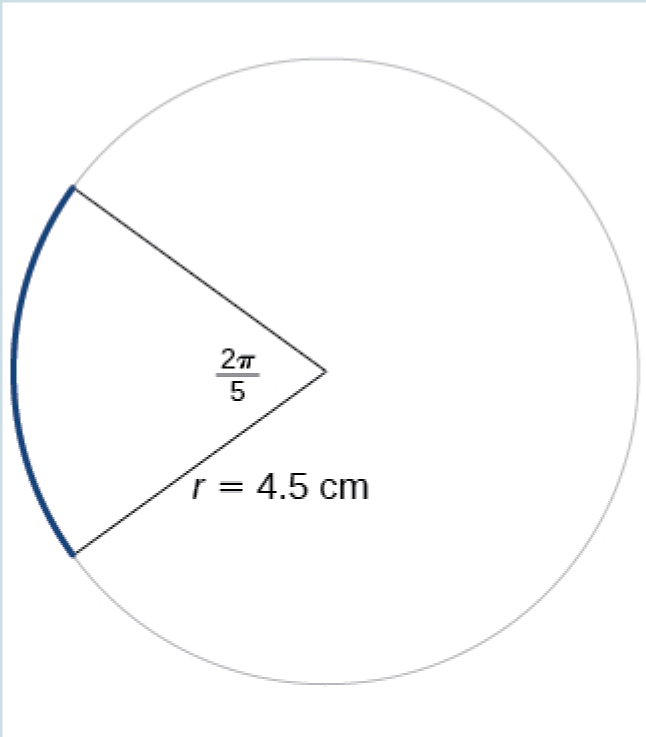


Figure 26.

24. Find the arc length.

25. Find the area of the sector.

Show Solution
$\frac{81\pi}{20} \approx 12.72 \text{ cm}^2$

Algebraic

For the following exercises, convert angles in radians to degrees.

26. $\frac{3\pi}{4}$ radians

27. $\frac{\pi}{9}$ radians

Show Solution
$20°$

28. $-\frac{5\pi}{4}$ radians

29. $\frac{\pi}{3}$ radians

Show Solution
$60°$

30. $-\frac{7\pi}{3}$ radians

31. $-\frac{5\pi}{12}$ radians

Show Solution
$-75°$

32. $\frac{11\pi}{6}$ radians

For the following exercises, convert angles in degrees to radians.

33. $90°$

Show Solution
$\frac{\pi}{2}$ radians

34. $100°$

35. $-540°$

Show Solution
-3π radians

36. $-120°$

37. $180°$

Show Solution
π radians

38. $-315°$

39. $150°$

Show Solution
$\frac{5\pi}{6}$ radians

For the following exercises, use the given information to find the length of a circular arc. Round to two decimal places.

40. Find the length of the arc of a circle of radius 12 inches subtended by a central angle of $\frac{\pi}{4}$. radians.

41. Find the length of the arc of a circle of radius 5.02 miles subtended by the central angle of $\frac{\pi}{3}$.

Show Solution
$\frac{5.02\pi}{3} \approx 5.26$ miles

42. Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of $\frac{5\pi}{6}$.

43. Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of $50°$.

Show Solution
$\frac{25\pi}{9} \approx 8.73$ centimeters

44. Find the length of the arc of a circle of radius 5 inches subtended by the central angle of $220°$.

45. Find the length of the arc of a circle of diameter 12 meters subtended by the central angle is $63°$.

Show Solution
$\frac{21\pi}{10} \approx 6.60$ meters

For the following exercises, use the given information to find the area of the sector. Round to four decimal places.

46. A sector of a circle has a central angle of $45°$ and a radius 6 cm.

47. A sector of a circle has a central angle of $30°$ and a radius of 20 cm.

Show Solution
104.7198 cm2

48. A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians.

49. A sector of a circle with radius of 0.7 inches and an angle of π radians.

Show Solution
0.7697 in2

For the following exercises, find the angle between 0 and 360 that is coterminal to the given angle.

50. $-40°$

51. $-110°$

Show Solution
$250°$

52. $700°$

53. $1400°$

Show Solution
$320°$

For the following exercises, find the angle between 0 and 2π in radians that is coterminal to the given angle.

54. $-\frac{\pi}{9}$

55. $\frac{10\pi}{3}$

Show Solution
$\frac{4\pi}{3}$

56. $\frac{13\pi}{6}$

57. $\frac{44\pi}{9}$

Show Solution
$\frac{8\pi}{9}$

Real-World Applications

58. A truck with 32-inch diameter wheels is traveling at 60 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

59. A bicycle with 24-inch diameter wheels is traveling at 15 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

Show Solution
1320 rad/min 210.085 RPM

60. A wheel of radius 8 inches is rotating 15^{\circ}/s. What is the linear speed v, the angular speed in RPM, and the angular speed in rad/s?

61. A wheel of radius 14 inches is rotating 0.5 rad/s. What is the linear speed v, the angular speed in RPM, and the angular speed in deg/s?

Show Solution
7 in./s, 4.77 RPM , 28.65 deg/s

62. A CD has diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

63. When being burned in a writable CD-R drive, the angular speed of a CD is often much faster than when playing audio, but the angular speed still varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4800 RPM (revolutions per minute). Find the linear speed if the CD has diameter of 120 millimeters.

Show Solution
$1,809,557.37 \text{ mm/min} = 30.16 \text{ m/s}$

64. A person is standing on the equator of Earth (radius 3960 miles). What are his linear and angular speeds?

65. Find the distance along an arc on the surface of Earth that subtends a central angle of 5 minutes

$\left(1 \text{ minute} = \frac{1}{60} \text{ degree}\right)$. The radius of Earth is 3960 miles.

Show Solution
5.76 miles

Find the distance along an arc on the surface of Earth that subtends a central angle of 7 minutes

66. $\left(1 \text{ minute} = \frac{1}{60} \text{ degree}\right)$. The radius of Earth is 3960 miles.

67. Consider a clock with an hour hand and minute hand. What is the measure of the angle the minute hand traces in 20 minutes?

Show Solution
$120°$

Extensions

68. Two cities have the same longitude. The latitude of city A is 9.00 degrees north and the latitude of city B is 30.00 degree north. Assume the radius of the earth is 3960 miles. Find the distance between the two cities.

69. A city is located at 40 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

Show Solution
794 miles per hour

70. A city is located at 75 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

71. Find the linear speed of the moon if the average distance between the earth and moon is 239,000 miles, assuming the orbit of the moon is circular and requires about 28 days. Express answer in miles per hour.

Show Solution
2,234 miles per hour

72. A bicycle has wheels 28 inches in diameter. A tachometer determines that the wheels are

rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is travelling down the road.

73. A car travels 3 miles. Its tires make 2640 revolutions. What is the radius of a tire in inches?

Show Solution
11.5 inches

74. A wheel on a tractor has a 24-inch diameter. How many revolutions does the wheel make if the tractor travels 4 miles?

Show Solution
3361 revolutions

Glossary

angle
: the union of two rays having a common endpoint

angular speed
: the angle through which a rotating object travels in a unit of time

arc length
: the length of the curve formed by an arc

area of a sector
: area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$. multiplied by the area of the entire circle

coterminal angles
: description of positive and negative angles in standard position sharing the same terminal

side

degree
: a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
: the side of an angle from which rotation begins

linear speed
: the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle
: the amount of rotation from the initial side to the terminal side

negative angle
: description of an angle measured clockwise from the positive x-axis

positive angle
: description of an angle measured counterclockwise from the positive x-axis

quadrantal angle
: an angle whose terminal side lies on an axis

radian measure
: the ratio of the arc length formed by an angle divided by the radius of the circle

radian
: the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray
: one point on a line and all points extending in one direction from that point; one side of an angle

reference angle
: the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position
: the position of an angle having the vertex at the origin and the initial side along the positive x-axis

terminal side
: the side of an angle at which rotation ends

vertex
: the common endpoint of two rays that form an angle

CHAPTER 3.3: RIGHT TRIANGLE TRIGONOMETRY

Learning Objectives

In this section you will:

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°\left(\frac{\pi}{6}\right)$, $45°\left(\frac{\pi}{4}\right)$, and $60°\left(\frac{\pi}{3}\right)$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

Using Right Triangles to Evaluate Trigonometric Functions

(Figure) shows a right triangle with a vertical side of length y and a horizontal side has length x. Notice that the triangle is inscribed in a circle of radius 1. Such a circle, with a center at the origin and a radius of 1, is known as a unit circle.

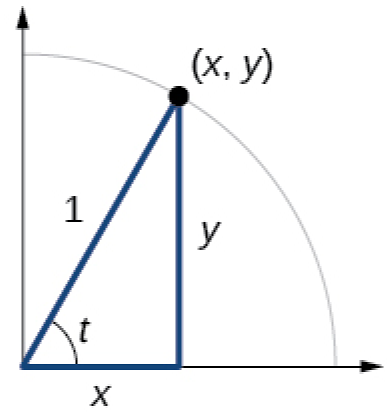


Figure 1.

We can define the trigonometric functions in terms an angle t and the lengths of the sides of the triangle. The adjacent side is the side closest to the angle, x. (Adjacent means "next to.") The opposite side is the side across from the angle, y. The hypotenuse is the side of the triangle opposite the right angle, 1. These sides are labeled in (Figure).

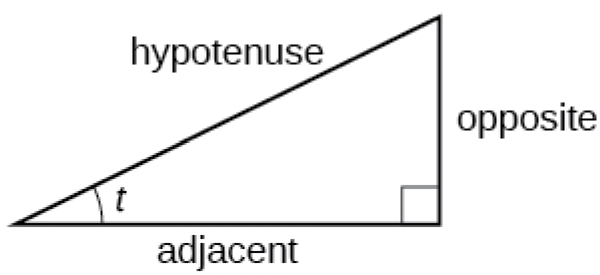


Figure 2. The sides of a right triangle in relation to angle t

Given a right triangle with an acute angle of t, the first three trigonometric functions are listed.

$$\begin{aligned} \text{Sine} \qquad \sin t &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \text{Cosine} \qquad \cos t &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \text{Tangent} \qquad \tan t &= \frac{\text{opposite}}{\text{adjacent}} \end{aligned}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "**S**ine is **o**pposite over **h**ypotenuse, **C**osine is **a**djacent over **h**ypotenuse, **T**angent is **o**pposite over **a**djacent."

For the triangle shown in (Figure), we have the following.

$$\begin{aligned} \sin t &= \frac{y}{1} \\ \cos t &= \frac{x}{1} \\ \tan t &= \frac{y}{x} \end{aligned}$$

How To

Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle.

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent as the ratio of the opposite side to the adjacent side.

Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in (Figure), find the value of $\cos \alpha$.

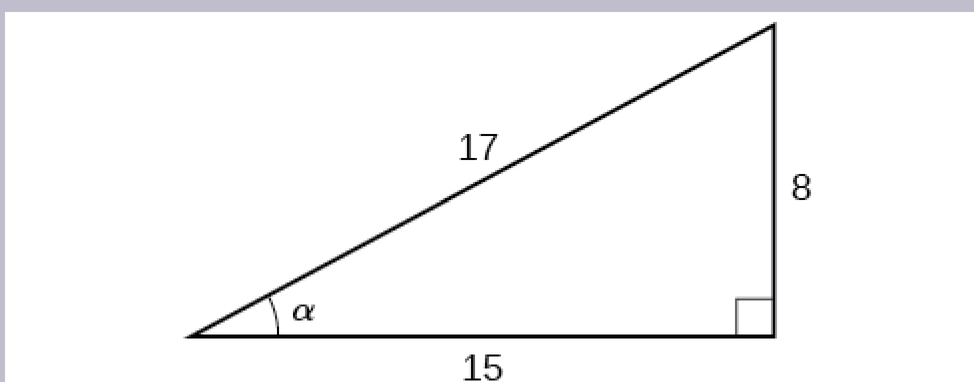


Figure 3.

Show Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17.

$$\begin{aligned} \cos(\alpha) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{15}{17} \end{aligned}$$

Try It

Given the triangle shown in (Figure), find the value of $\sin t$.

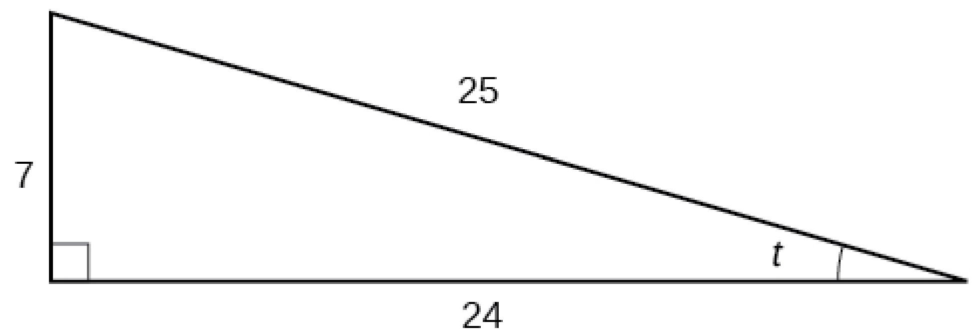


Figure 4.

Show Solution

$\frac{7}{25}$

Reciprocal Functions

In addition to sine, cosine, and tangent, there are three more functions. These too are defined in terms of the sides of the triangle.

$$\begin{aligned} &\text{Secant} & \sec t &= \frac{\text{hypotenuse}}{\text{adjacent}} \\ &\text{Cosecant} & \csc t &= \frac{\text{hypotenuse}}{\text{opposite}} \\ &\text{Cotangent} & \cot t &= \frac{\text{adjacent}}{\text{opposite}} \end{aligned}$$

Take another look at these definitions. These functions are the reciprocals of the first three functions.

$$\begin{aligned} \sin t &= \frac{1}{\csc t} & \csc t &= \frac{1}{\sin t} \\ \cos t &= \frac{1}{\sec t} & \sec t &= \frac{1}{\cos t} \\ \tan t &= \frac{1}{\cot t} & \cot t &= \frac{1}{\tan t} \end{aligned}$$

When working with right triangles, keep in mind that the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in (Figure). The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

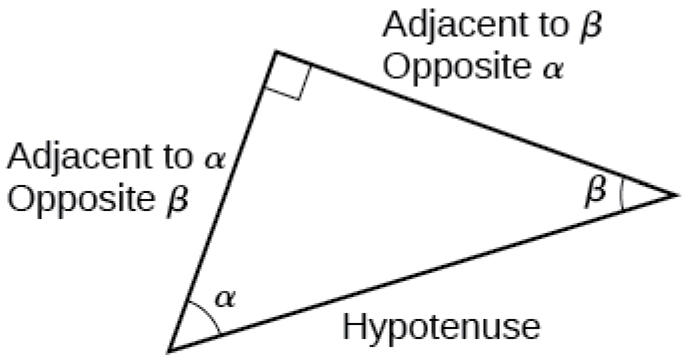


Figure 5. The side adjacent to one angle is opposite the other angle.

Many problems ask for all six trigonometric functions for a given angle in a triangle. A possible strategy to use is to find the sine, cosine, and tangent of the angles first. Then, find the other trigonometric functions easily using the reciprocals.

How To

Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles.

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in (Figure), evaluate $\sin\alpha, \cos\alpha, \tan\alpha, \sec\alpha, \csc\alpha,$ and $\cot\alpha.$

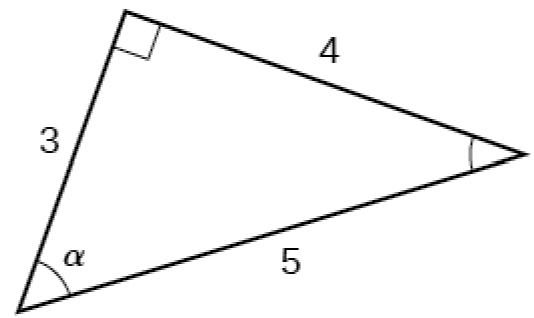


Figure 6.

Show Solution

$$\begin{aligned}
\sin\ \alpha &= \frac{\text{opposite } \alpha}{\text{hypotenuse}} &= \frac{4}{5}\\
\cos\ \alpha &= \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} &= \frac{3}{5}\\
\tan\ \alpha &= \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} &= \frac{4}{3}\\
\sec\ \alpha &= \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} &= \frac{5}{3}\\
\csc\ \alpha &= \frac{\text{hypotenuse}}{\text{opposite } \alpha} &= \frac{5}{4}\\
\cot\ \alpha &= \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} &= \frac{3}{4}
\end{aligned}$$

Analysis

Another approach would have been to find sine, cosine, and tangent first. Then find their reciprocals to determine the other functions.

$$\sec\ \alpha = \frac{1}{\cos\ \alpha} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$
$$\csc\ \alpha = \frac{1}{\csc\ \alpha} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$
$$\cot\ \alpha = \frac{1}{\tan\ \alpha} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

Try It

Using the triangle shown in (Figure),evaluate $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

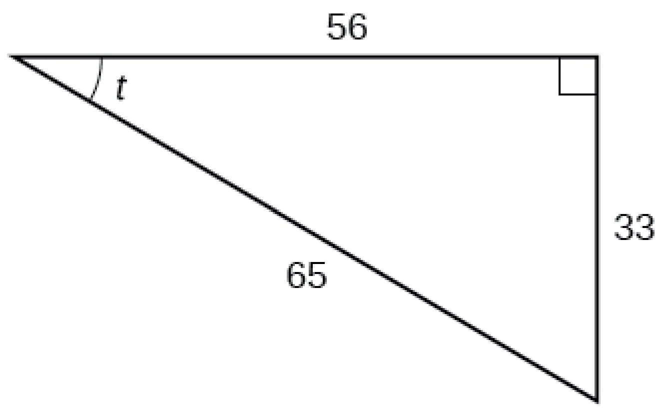


Figure 7.

Show Solution

$$\begin{aligned} \sin\ t &= \tfrac{33}{65}, & \cos\ t &= \tfrac{56}{65}, & \tan\ t &= \tfrac{33}{56}, \\ \sec\ t &= \tfrac{65}{56}, & \csc\ t &= \tfrac{65}{33}, & \cot\ t &= \tfrac{56}{33} \end{aligned}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

It is helpful to evaluate the trigonometric functions as they relate to the special angles—multiples of $30°$, $60°$, and $45°$. Remember, however, that when dealing with right triangles, we are limited to angles between $0°$

Suppose we have a $30°, 60°, 90°$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45°, 45°, 90°$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in (Figure).

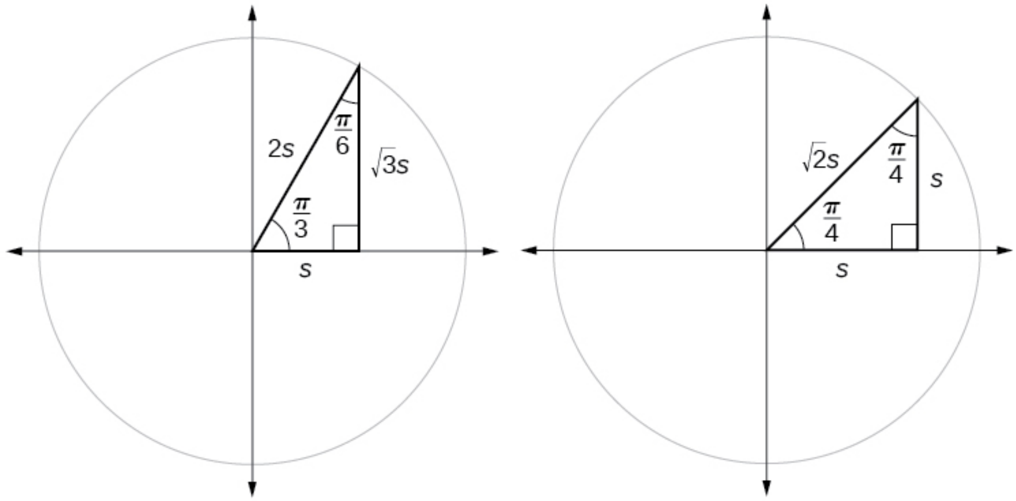


Figure 8. Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

How To

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in (Figure) for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Show Solution

$$\begin{aligned}
\sin\left(\frac{\pi}{3}\right) &= \frac{\text{opp}}{\text{hyp}} &&= \frac{\sqrt{3}s}{2s} &&= \frac{\sqrt{3}}{2} \\
\cos\left(\frac{\pi}{3}\right) &= \frac{\text{adj}}{\text{hyp}} &&= \frac{s}{2s} &&= \frac{1}{2} \\
\tan\left(\frac{\pi}{3}\right) &= \frac{\text{opp}}{\text{adj}} &&= \frac{\sqrt{3}s}{s} &&= \sqrt{3} \\
\sec\left(\frac{\pi}{3}\right) &= \frac{\text{hyp}}{\text{adj}} &&= \frac{2s}{s} &&= 2 \\
\csc\left(\frac{\pi}{3}\right) &= \frac{\text{hyp}}{\text{opp}} &&= \frac{2s}{\sqrt{3}s} &&= \frac{2}{\sqrt{3}} &&= \frac{2\sqrt{3}}{3} \\
\cot\left(\frac{\pi}{3}\right) &= \frac{\text{adj}}{\text{opp}} &&= \frac{s}{\sqrt{3}s} &&= \frac{1}{\sqrt{3}} &&= \frac{\sqrt{3}}{3}
\end{aligned}$$

Try It

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$, using side lengths.

Show Solution

$$\begin{aligned}
\sin\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \tan\left(\frac{\pi}{4}\right) = 1, \\
\sec\left(\frac{\pi}{4}\right) &= \sqrt{2}, \csc\left(\frac{\pi}{4}\right) = \sqrt{2}, \cot\left(\frac{\pi}{4}\right) = 1
\end{aligned}$$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles, we notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$.

$$\begin{aligned}
\sin\frac{\pi}{3} &= \cos\frac{\pi}{6} &&= \frac{\sqrt{3}s}{2s} &&= \frac{\sqrt{3}}{2} \\
\sin\frac{\pi}{6} &= \cos\frac{\pi}{3} &&= \frac{s}{2s} &&= \frac{1}{2}
\end{aligned}$$

See (Figure).

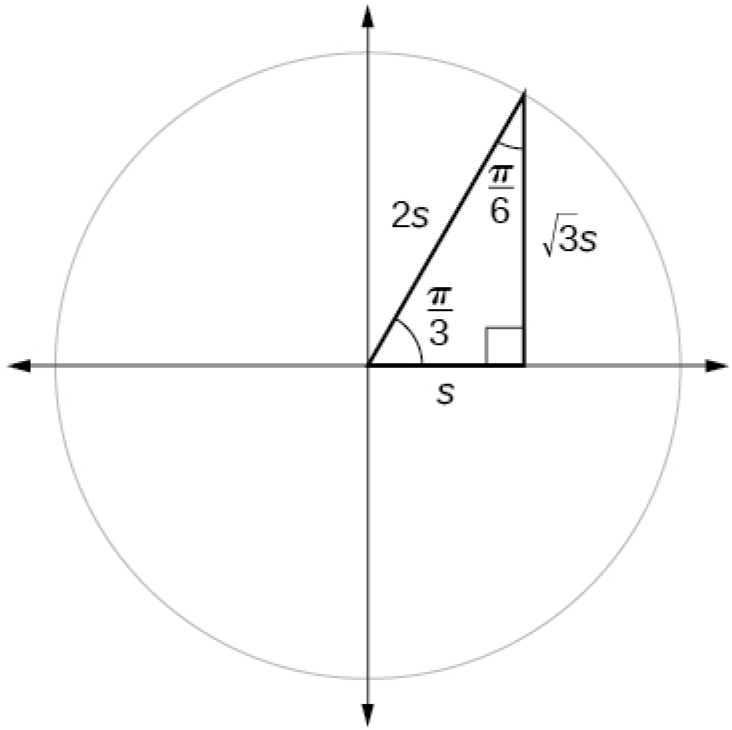


Figure 9. The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from (Figure), the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{6}\right)$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{6}\right)$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in (Figure).

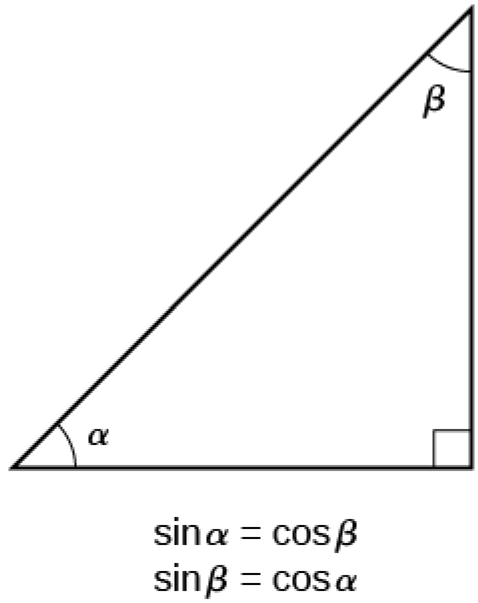


Figure 10. Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a given angle t, $\cos\ t = \frac{5}{13}$, then $\sin\left(\frac{\pi}{2} - t\right) = \frac{5}{13}$ as well.

Cofunction Identities

The cofunction identities in radians are listed in (Figure).

$$\begin{array}{ll} \cos t = \sin\left(\frac{\pi}{2} - t\right) & \sin t = \cos\left(\frac{\pi}{2} - t\right) \\ \tan t = \cot\left(\frac{\pi}{2} - t\right) & \cot t = \tan\left(\frac{\pi}{2} - t\right) \\ \sec t = \csc\left(\frac{\pi}{2} - t\right) & \csc t = \sec\left(\frac{\pi}{2} - t\right) \end{array}$$

How To

Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $\cos \left(\frac{\pi}{2} - t\right)$.

Show Solution

According to the cofunction identities for sine and cosine, we have the following.

$$\sin t = \cos \left(\frac{\pi}{2} - t\right)$$

So

$$\cos \left(\frac{\pi}{2} - t\right) = \frac{5}{12}$$

Try It

If $\csc \left(\frac{\pi}{6}\right) = 2$, find $\sec \left(\frac{\pi}{3}\right)$.

Show Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

How To

Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides.

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in (Figure).

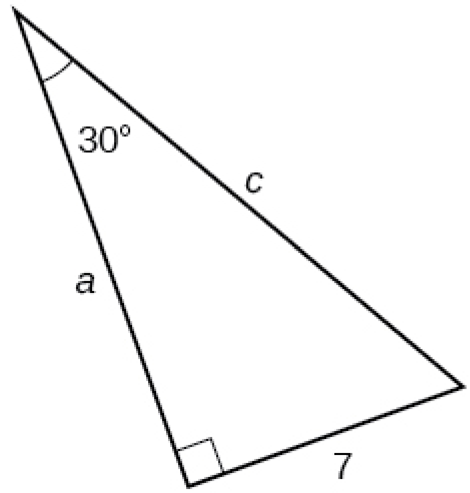


Figure 11.

Show Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30°) = \frac{7}{a}$$

We rearrange to solve for a.

$$\begin{aligned} a &= \frac{7}{\tan(30°)} \\ &\approx 12.1 \end{aligned}$$

We can use the sine to find the hypotenuse.

$$\sin(30°) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$\begin{aligned} c &= \frac{7}{\sin(30°)} \\ &= 14 \end{aligned}$$

Try It

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Show Solution
$\text{adjacent} = 10; \text{opposite} = 10\sqrt{3};$ missing angle is $\frac{\pi}{6}$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The angle of elevation of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height.

Similarly, we can form a triangle from the top of a tall object by looking downward. The angle of depression of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See (Figure).

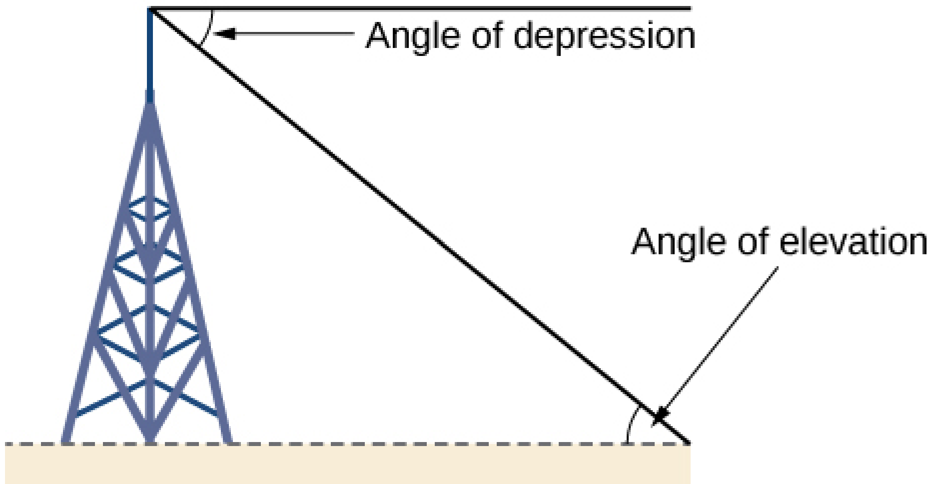


Figure 12.

How To

Given a tall object, measure its height indirectly.

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of $57°$ between a line of sight to the top of the tree and the ground, as shown in (Figure). Find the height of the tree.

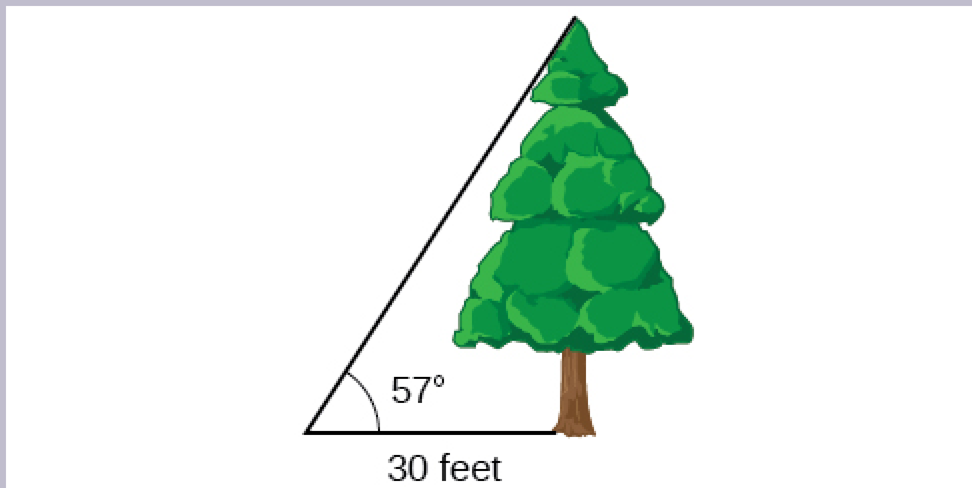


Figure 13.

Show Solution

We know that the angle of elevation is $57°$ and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of $57°$, letting h be the unknown height.

$$\begin{array}{rcll} \tan\,\theta & = & \frac{\text{opposite}}{\text{adjacent}} & \\ \tan\left(57°\right) & = & \frac{h}{30} & \text{Solve for } h. \\ h & = & 30\tan\left(57°\right) & \text{Multiply.} \\ h & \approx & 46.2 & \text{Use a calculator.} \end{array}$$

The tree is approximately 46 feet tall.

Try It

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Show Solution
About 52 ft

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Key Equations

Trigonometric Functions	Sine	$\sin\ t=\frac{\text{opposite}}{\text{hypotenuse}}$
	Cosine	$\cos\ t=\frac{\text{adjacent}}{\text{hypotenuse}}$
	Tangent	$\tan\ t=\frac{\text{opposite}}{\text{adjacent}}$
	Secant	$\sec\ t=\frac{\text{hypotenuse}}{\text{adjacent}}$
	Cosecant	$\csc\ t=\frac{\text{hypotenuse}}{\text{opposite}}$
	Cotangent	$\cot\ t=\frac{\text{adjacent}}{\text{opposite}}$
Reciprocal Trigonometric Functions	$\sin\ t=\frac{1}{\csc\ t}$	$\csc\ t=\frac{1}{\sin\ t}$
	$\cos\ t=\frac{1}{\sec\ t}$	$\sec\ t=\frac{1}{\cos\ t}$
	$\tan\ t=\frac{1}{\cot\ t}$	$\cot\ t=\frac{1}{\tan\ t}$
Cofunction Identities	$\cos\ t=\sin\left(\frac{\pi}{2}-t\right)$	
	$\sin\ t=\cos\left(\frac{\pi}{2}-t\right)$	
	$\tan\ t=\cot\left(\frac{\pi}{2}-t\right)$	
	$\cot\ t=\tan\left(\frac{\pi}{2}-t\right)$	
	$\sec\ t=\csc\left(\frac{\pi}{2}-t\right)$	

Key Concepts

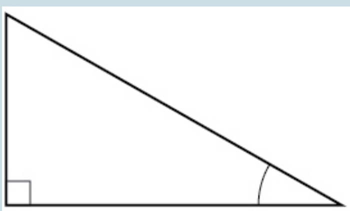
- We can define trigonometric functions as ratios of the side lengths of a right triangle. See (Figure).
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See (Figure).
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See (Figure).
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See (Figure).
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See (Figure).
- Right-triangle trigonometry facilitates the measurement of inaccessible heights and distances.

- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See (Figure).

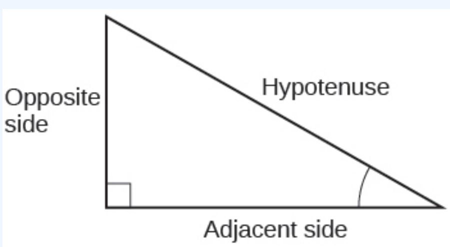
Section Exercises

Verbal

1. For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.



Show Solution



2. When a right triangle with a hypotenuse of 1 is placed in a circle of radius 1, which sides of the triangle correspond to the *x*– and *y*-coordinates?

3. The tangent of an angle compares which sides of the right triangle?

Show Solution

The tangent of an angle is the ratio of the opposite side to the adjacent side.

4. What is the relationship between the two acute angles in a right triangle?

5. Explain the cofunction identity.

Show Solution

For example, the sine of an angle is equal to the cosine of its complement; the cosine of an angle is equal to the sine of its complement.

Algebraic

For the following exercises, use cofunctions of complementary angles.

6. $\cos\left(34°\right) = \sin\left(\_\_\_°\right)$

7. $\cos\left(\frac{\pi}{3}\right) = \sin\left(\_\_\_\right)$

Show Solution

$\frac{\pi}{6}$

8. $\csc\left(21\right) = \sec\left(\_\_\_\right)$

9. $\tan\left(\frac{\pi}{4}\right) = \cot\left(\_\_\_\right)$

Show Solution

$\frac{\pi}{4}$

For the following exercises, find the lengths of the missing sides if side a is opposite angle A, side b is opposite angle B, and side c is the hypotenuse.

10. $\cos B = \frac{4}{5}, a = 10$

11. $\sin B = \frac{1}{2}, a = 20$

Show Solution
$b = \frac{20\sqrt{3}}{3}, c = \frac{40\sqrt{3}}{3}$

12. $\tan A = \frac{5}{12}, b = 6$

13. $\tan A = 100, b = 100$

Show Solution
$a = 10,000, c = 10,00.5$

14. $\sin B = \frac{1}{\sqrt{3}}, a = 2$

15. $a = 5, \measuredangle A = 60$

Show Solution
$b = \frac{5\sqrt{3}}{3}, c = \frac{10\sqrt{3}}{3}$

16. $c = 12, \measuredangle A = 45$

Graphical

For the following exercises, use (Figure) to evaluate each trigonometric function of angle A.

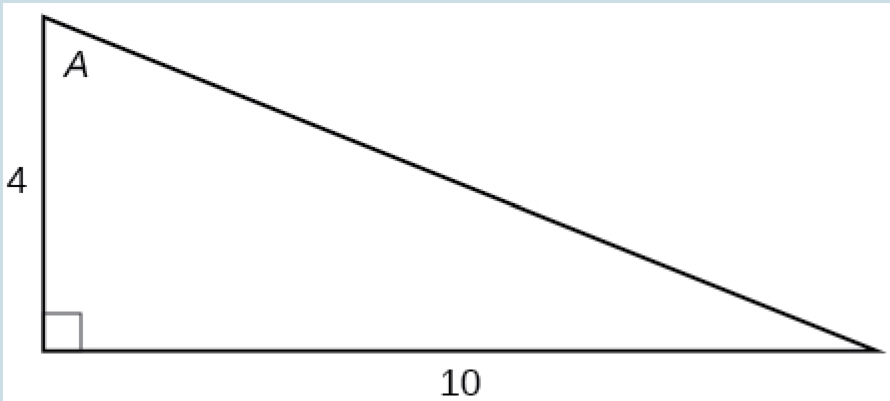


Figure 14.

17. $\sin A$

Show Solution
$\frac{5\sqrt{29}}{29}$

18. $\cos A$

19. $\tan A$

Show Solution
$\frac{5}{2}$

20. $\csc A$

21. $\sec A$

Show Solution
$\frac{\sqrt{29}}{2}$

22. $\cot A$

For the following exercises, use (Figure) to evaluate each trigonometric function of angle A.

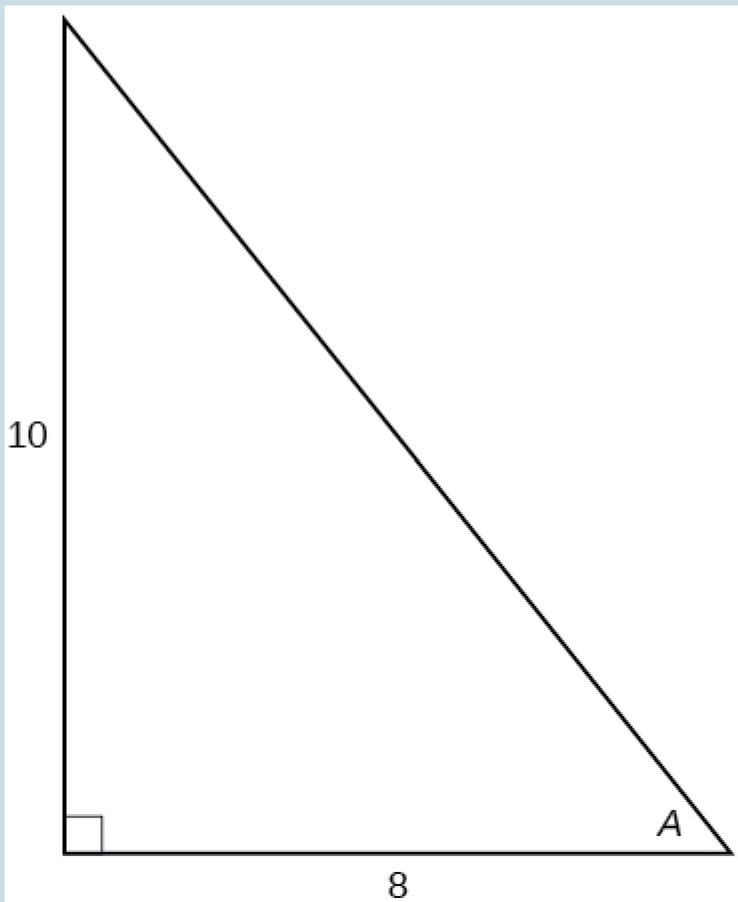


Figure 15.

23. $\sin A$

Show Solution
$\frac{5\sqrt{41}}{41}$

24. $\cos A$

25. $\tan A$

Show Solution
$\frac{5}{4}$

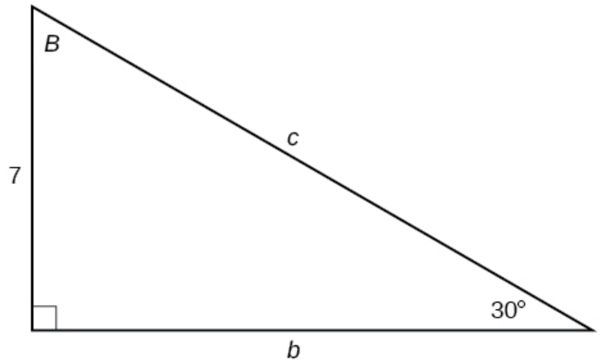
26. $\csc A$

27. $\sec A$

Show Solution
$\frac{\sqrt{41}}{4}$

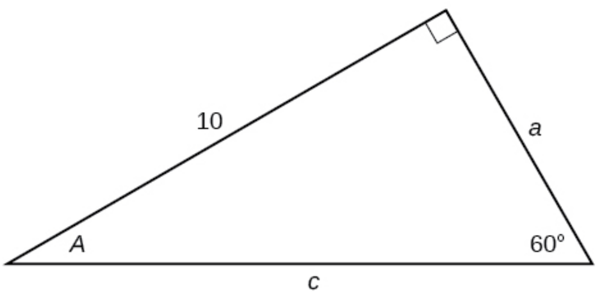
28. $\cot A$

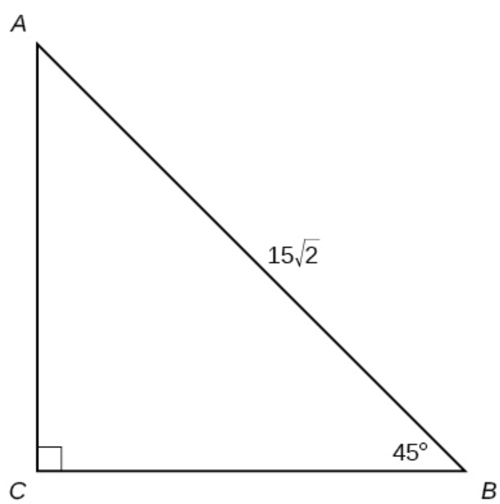
For the following exercises, solve for the unknown sides of the given triangle.

29. 

Show Solution

$c=14, b=7\sqrt{3}$

30. 

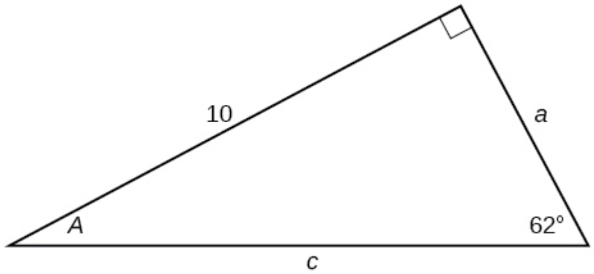
31. 

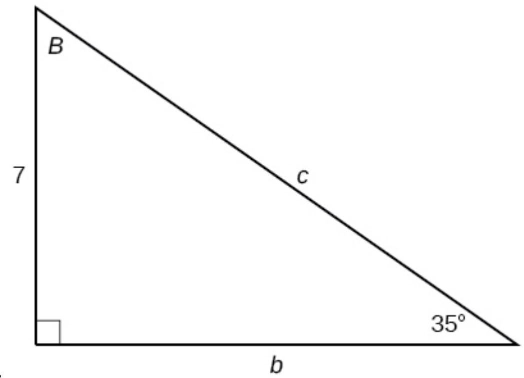
Show Solution

$a=15, b=15$

Technology

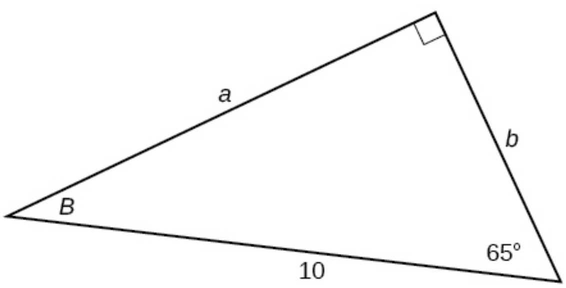
For the following exercises, use a calculator to find the length of each side to four decimal places.

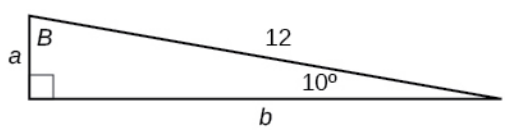
32. 

33. 

Show Solution

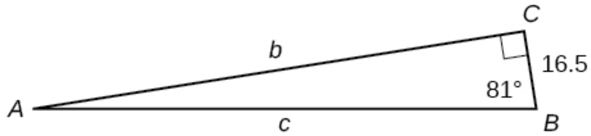
$b = 9.9970, c = 12.2041$

34. 

35. 

Show Solution

$a = 2.0838, b = 11.8177$

36. 

37. $b = 15, \measuredangle B = 15°$

Show Solution

$a = 55.9808, c = 57.9555$

38. $c = 200, \measuredangle B = 5°$

39. $c = 50, \measuredangle B = 21°$

Show Solution

$a = 46.6790, b = 17.9184$

40. $a = 30, \measuredangle A = 27°$

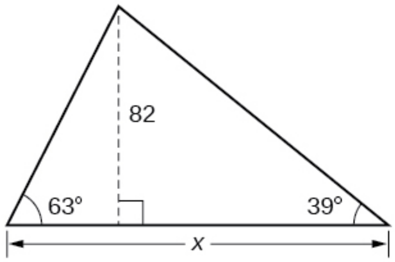
41. $b = 3.5, \measuredangle A = 78°$

Show Solution

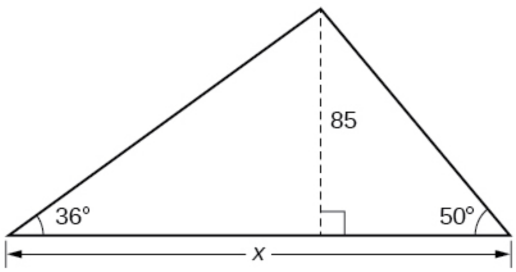
$a = 16.4662, c = 16.8341$

Extensions

42. Find x.

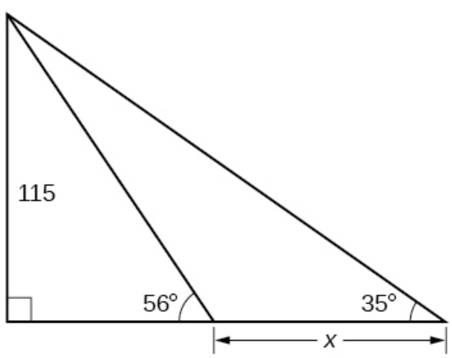


43. Find x.

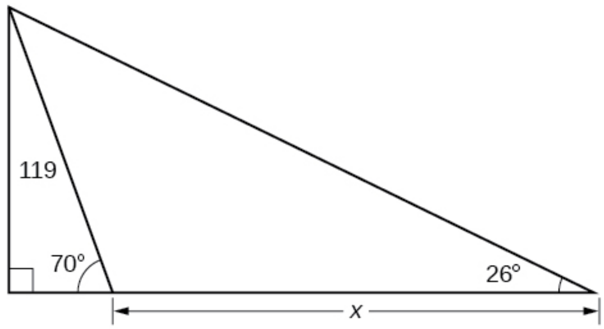


Show Solution
188.3159

44. Find x.



45. Find x.



Show Solution
200.6737

46. A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is $36°$, and that the angle of depression to the bottom of the tower is $23°$. How tall is the tower?

47. A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is $43°$, and that the angle of depression to the bottom of the tower is $31°$. How tall is the tower?

Show Solution
498.3471 ft

48. A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is $15°$, and that the angle of depression to the bottom of the monument is $2°$. How far is the person from the monument?

49. A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is $18°$, and that the angle of depression to the bottom of the monument is $3°$. How far is the person from the monument?

Show Solution

1060.09 ft

50. There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be $40°$. From the same location, the angle of elevation to the top of the antenna is measured to be $43°$. Find the height of the antenna.

51. There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be $36°$. From the same location, the angle of elevation to the top of the lightning rod is measured to be $38°$. Find the height of the lightning rod.

Show Solution

27.372 ft

Real-World Applications

52. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is $80°$. How high does the ladder reach up the side of the building?

53. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is $80°$. How high does the ladder reach up the side of the building?

Show Solution

22.6506 ft

54. The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

55. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

Show Solution
368.7633 ft

56. Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be $60°$, how far from the base of the tree am I?

Glossary

adjacent side
: in a right triangle, the side between a given angle and the right angle

angle of depression
: the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation
: the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side
: in a right triangle, the side most distant from a given angle

hypotenuse
: the side of a right triangle opposite the right angle

unit circle
: a circle with a center at $(0, 0)$ and radius 1

CHAPTER 3.4: UNIT CIRCLE

Learning Objectives

In this section you will:

- Find function values for the sine and cosine of $30^\circ \text{ or } \left(\frac{\pi}{6}\right)$, $45^\circ \text{ or } \left(\frac{\pi}{4}\right)$, and $60^\circ \text{ or } \left(\frac{\pi}{3}\right)$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Figure 1. The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.

Finding Trigonometric Functions Using the Unit Circle

We have already defined the trigonometric functions in terms of right triangles. In this section, we will redefine them in terms of the unit circle. Recall that a unit circle is a circle centered at the origin with radius 1, as shown in (Figure). The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a unit circle, $s = t$.

The *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos\ t$ and $y = \sin\ t$.

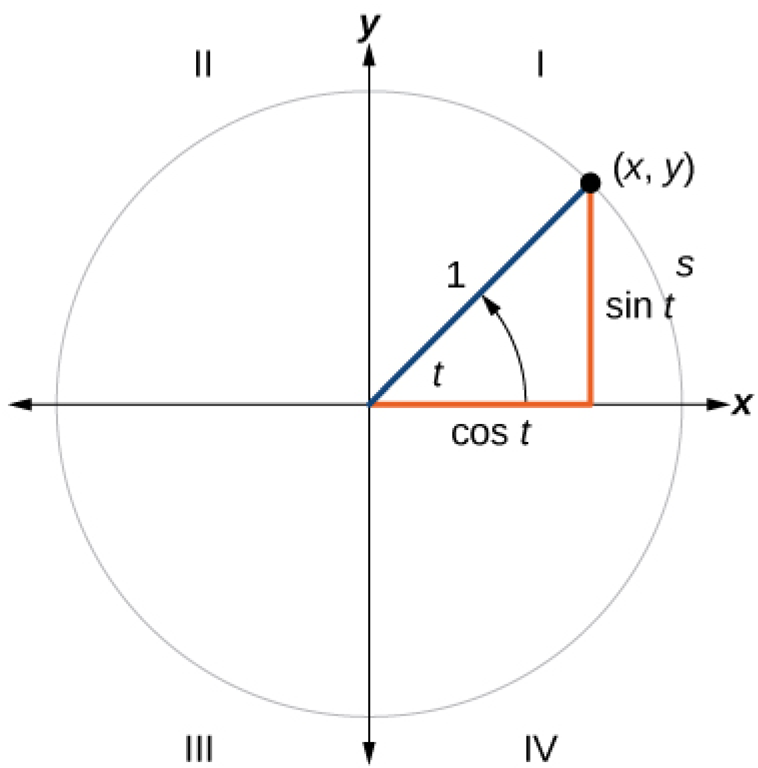


Figure 2.

Unit Circle

A unit circle has a center at $(0, 0)$ and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions from the Unit Circle

The sine function relates a real number t to the *y*-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the *y*-value of the endpoint on the unit circle of an arc of length t. In (Figure), the sine is equal to y. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the *y*-coordinate of the corresponding point on the unit circle.

The cosine function of an angle t equals the *x*-value of the endpoint on the unit circle of an arc of length t. In (Figure), the cosine is equal to x.

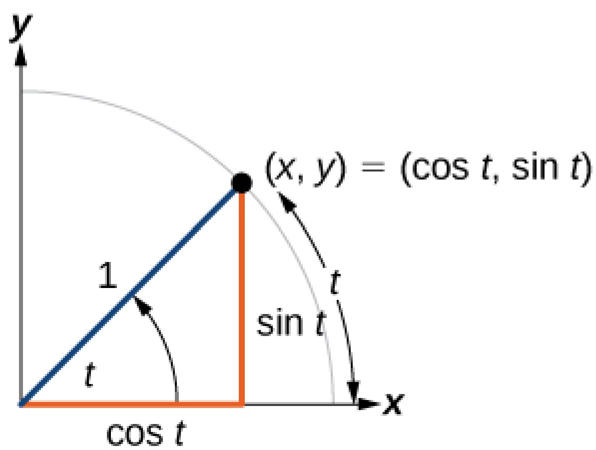


Figure 3.

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin (t)$ and $\cos t$ is the same as $\cos (t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos (t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

Sine and Cosine Functions

If t is a real number and a point (x, y) on the unit circle corresponds to a central angle t, then

$$\cos t = x$$
$$\sin t = y$$

How To

Given a point *P* (x, y) on the unit circle corresponding to an angle of t, find the sine and cosine.

1. The sine of t is equal to the *y*-coordinate of point $P: \sin\ t = y.$
2. The cosine of t is equal to the *x*-coordinate of point $P: \text{cos}t = x.$

Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in (Figure). Find $\cos\left(t\right)$ and $\sin\left(t\right)$.

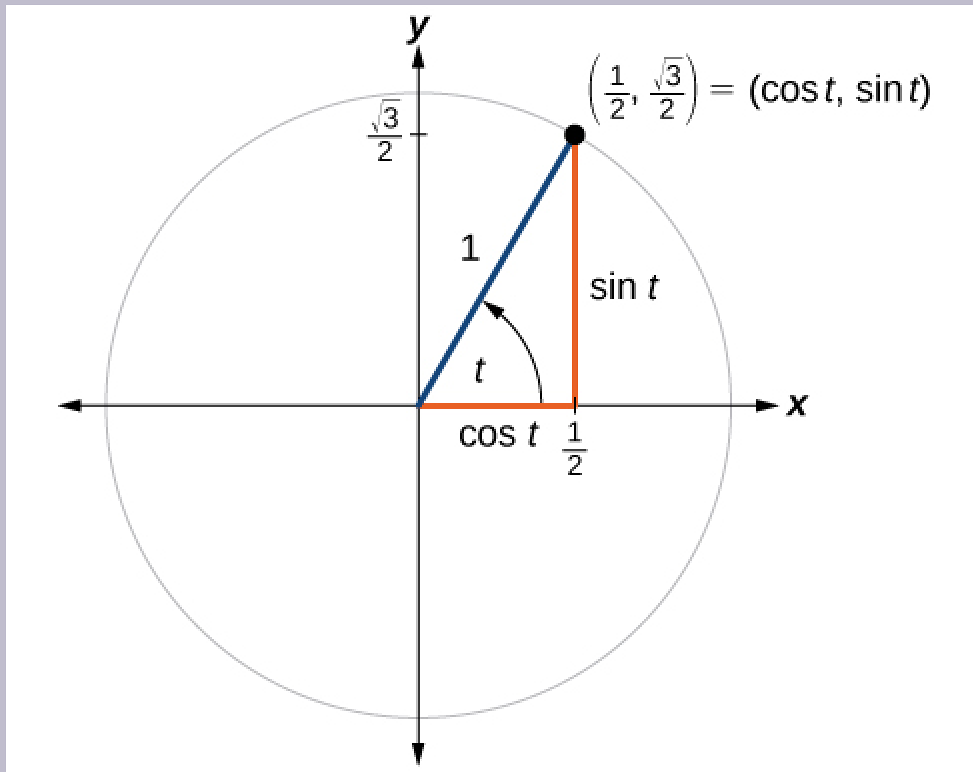


Figure 4.

Show Solution

We know that $\cos t$ is the *x*-coordinate of the corresponding point on the unit circle and $\sin t$ is the *y*-coordinate of the corresponding point on the unit circle. So:

$$\begin{aligned} x &= \cos t &= \tfrac{1}{2} \\ y &= \sin t &= \tfrac{\sqrt{3}}{2} \end{aligned}$$

Try It

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in (Figure). Find $\cos t$ and $\sin t$.

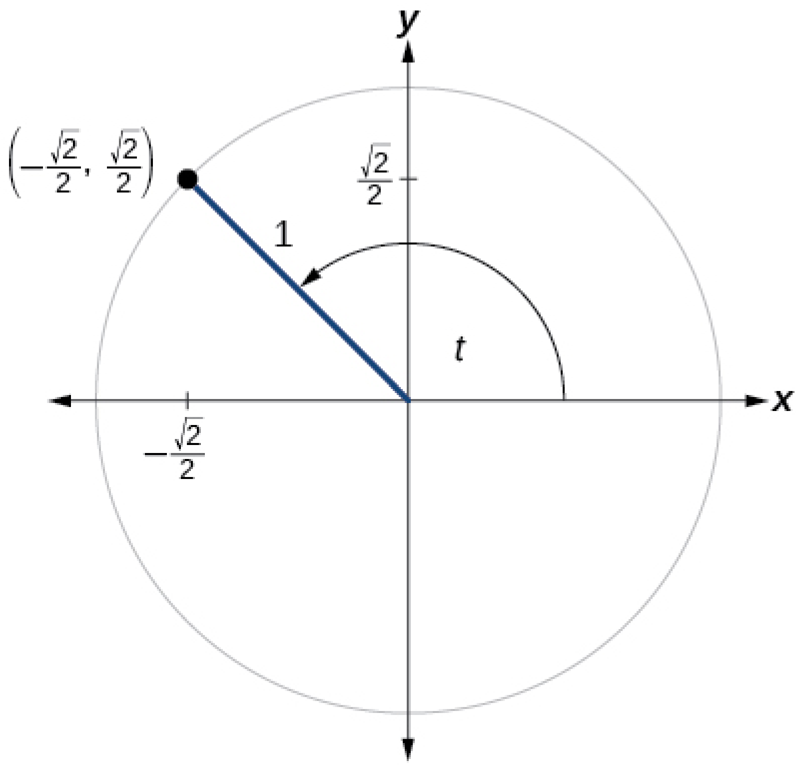


Figure 5.

Show Solution

$\cos\left(t\right) = -\frac{\sqrt{2}}{2}, \sin\left(t\right) = \frac{\sqrt{2}}{2}$

Finding Sines and Cosines of Angles on an Axis

For quadrantal angles, the corresponding point on the unit circle falls on the *x*- or *y*-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Calculating Sines and Cosines along an Axis

Find $\cos\left(90°\right)$ and $\sin\left(90°\right)$.

Show Solution

Moving $90°$ counterclockwise around the unit circle from the positive *x*-axis brings us to the top of the circle, where the (x, y) coordinates are $\left(0, 1\right)$, as shown in (Figure).

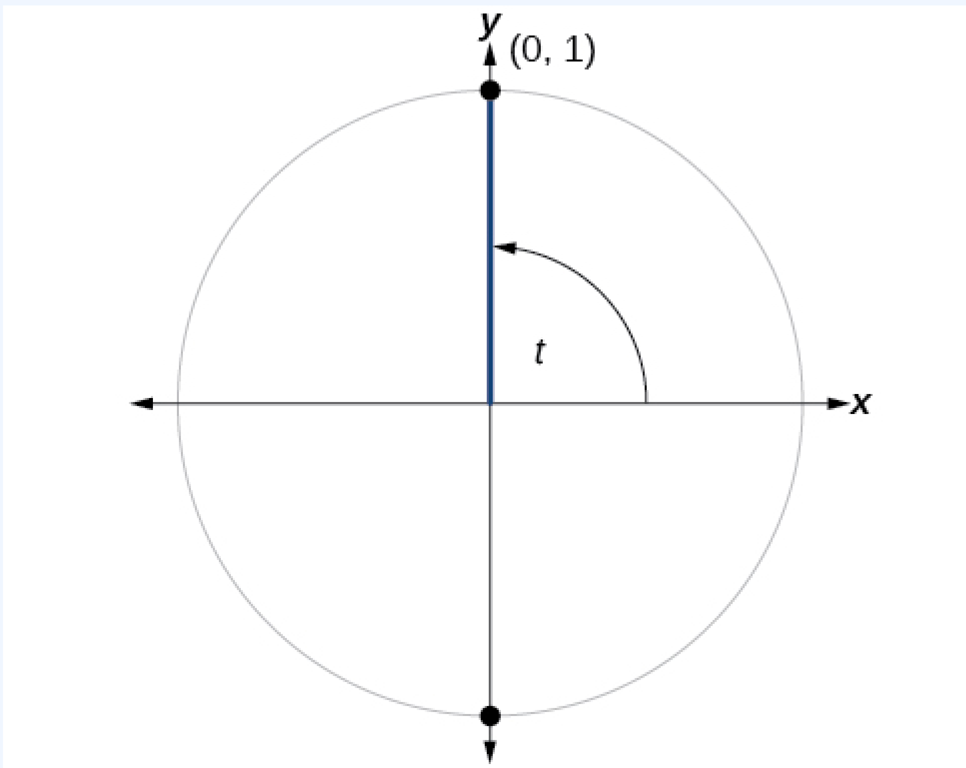


Figure 6.

We can then use our definitions of cosine and sine.

$$\begin{array}{llll} x & = \cos\,t & = \cos\left(90°\right) & = 0 \\ y & = \sin\,t & = \sin\left(90°\right) & = 1 \end{array}$$

The cosine of $90°$ is 0; the sine of $90°$ is 1.

Try It

Find cosine and sine of the angle π.

Show Solution

$\cos(\pi) = -1, \sin(\pi) = 0$

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the Pythagorean Identity. See (Figure).

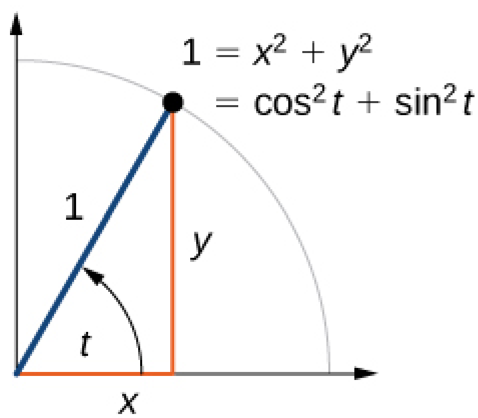


Figure 7.

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

Pythagorean Identity

The Pythagorean Identity states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1$$

How To

Given the sine of some angle t and its quadrant location, find the cosine of t.

1. Substitute the known value of $\sin t$ into the Pythagorean Identity.
2. Solve for $\cos t$.
3. Choose the solution with the appropriate sign for the *x*-values in the quadrant where t is located.

Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Show Solution

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See (Figure).

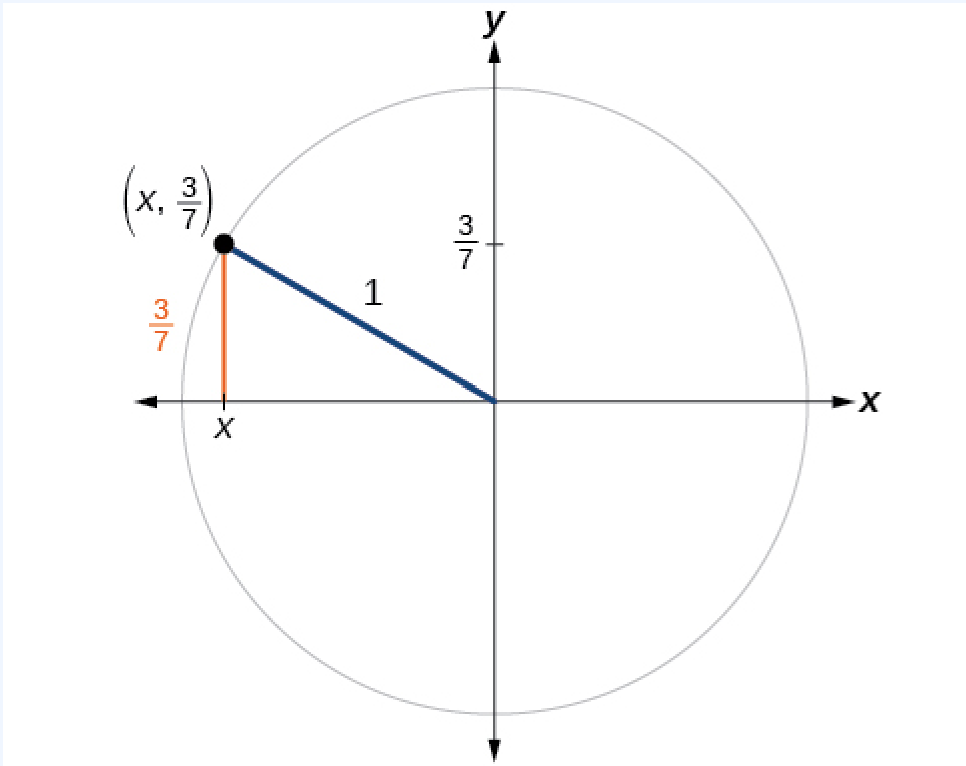


Figure 8.

Substituting the known value for sine into the Pythagorean Identity,

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \cos^2(t) + \frac{9}{49} &= 1 \\ \cos^2(t) &= \frac{40}{49} \\ \cos(t) &= \sqrt{\frac{40}{49}} = \frac{\sqrt{40}}{7} = \frac{2\sqrt{10}}{7} \end{aligned}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative.

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

Try It

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Show Solution

$\sin(t) = -\frac{7}{25}$

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees, and we found their sines and cosines using right triangles. We can also calculate sines and cosines of the special angles using the Pythagorean Identity.

Finding Sines and Cosines of 45 Angles

First, we will look at angles of $45°$ or $\frac{\pi}{4}$, as shown in (Figure). A $45° - 45° - 90°$ triangle is an isosceles triangle, so the *x*- and *y*-coordinates of the corresponding point on the circle are the same. Because the *x*- and *y*-values are the same, the sine and cosine values will also be equal.

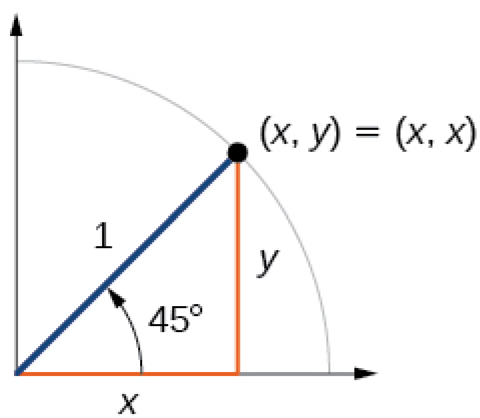


Figure 9.

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first quadrantal angle. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1 so the right triangle formed below the line $y = x$ has sides x and y $(y = x)$, and radius = 1. See (Figure).

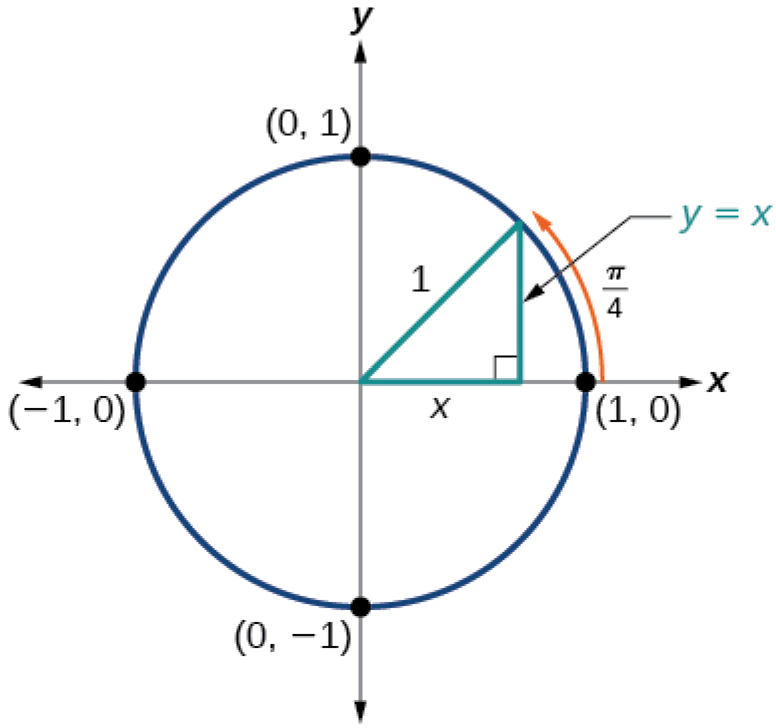


Figure 10.

From the Pythagorean Theorem we get
$x^2 + y^2 = 1$

We can then substitute $y = x$.
$x^2 + x^2 = 1$

Next we combine like terms.
$2x^2 = 1$

And solving for x, we get

$$\begin{aligned} x^2 &= \frac{1}{2} \\ x &= \frac{1}{\sqrt{2}} \end{aligned}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$\begin{aligned}(x, y) &= (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ x &= \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}} \\ \cos t &= \frac{1}{\sqrt{2}}, \sin t = \frac{1}{\sqrt{2}}\end{aligned}$$

If we then rationalize the denominators, we get

$$\begin{aligned}\cos t &= \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \\ \sin t &= \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of $45°$ are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of $30°$ and $60°$ Angles

Next, we will find the cosine and sine at an angle of $30°$, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of $30°$, and another at an angle of $-30°$, as shown in (Figure). If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be $60°$, as shown in (Figure).

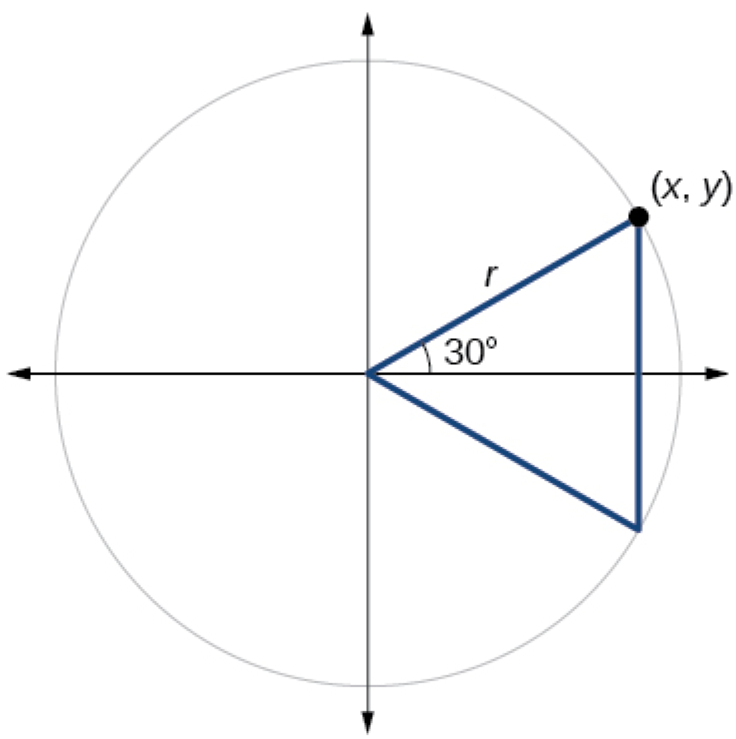


Figure 12.

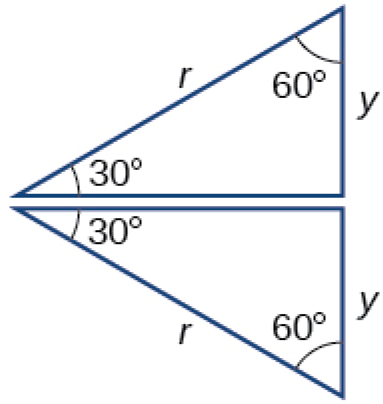


Figure 13.

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}r$$

And since $r = 1$ in our unit circle,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1)\\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\begin{aligned}\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) &= 1\\ \cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 &= 1\\ \cos^2\left(\frac{\pi}{6}\right) &= \frac{3}{4} && \text{Use the square root property.}\\ \cos\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} && \text{Since } y \text{ is positive, choose the positive root.}\end{aligned}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of $30°$ are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in (Figure). Angle A has measure $60°$. At point B, we draw an angle ABC with measure of $60°$. We know the angles in a triangle sum to $180°$, so the measure of angle C is also $60°$. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

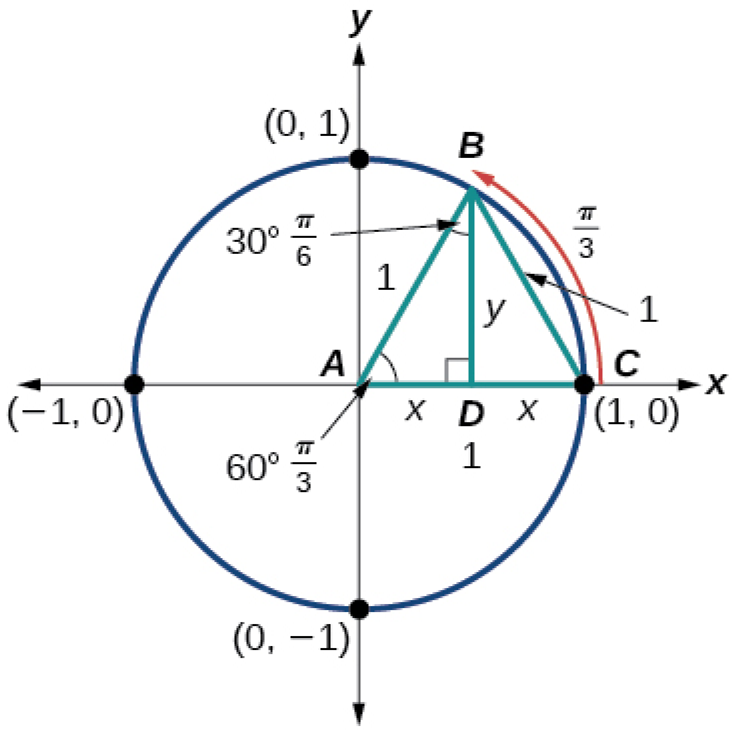


Figure 13.

The measure of angle ABD is 30^{\circ}. Angle ABC is double angle ABD, so its measure is 60^{\circ}. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the *x*-coordinate of point B, which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2 + y^2 = 1$$

Substituting $x = \frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

Solving for y, we get

$$\begin{aligned} \frac{1}{4} + y^2 &= 1 \\ y^2 &= 1 - \frac{1}{4} \\ y^2 &= \frac{3}{4} \\ y &= \frac{\sqrt{3}}{2} \end{aligned}$$

Since $t = \frac{\pi}{3}$ has the terminal side in quadrant I where the *y*-coordinate is positive, we choose $y = \frac{\sqrt{3}}{2}$, the positive value.

At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of $60°$ are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$\begin{aligned}(x,y) &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\\ x &= \frac{1}{2}, y = \frac{\sqrt{3}}{2}\\ \cos\ t &= \frac{1}{2}, \sin\ t = \frac{\sqrt{3}}{2}\end{aligned}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. (Figure) summarizes these values.

Angle	0	$\frac{\pi}{6}$, or $30°$	$\frac{\pi}{4}$, or $45°$	$\frac{\pi}{3}$, or $60°$	$\frac{\pi}{2}$, or $90°$
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

(Figure) shows the common angles in the first quadrant of the unit circle.

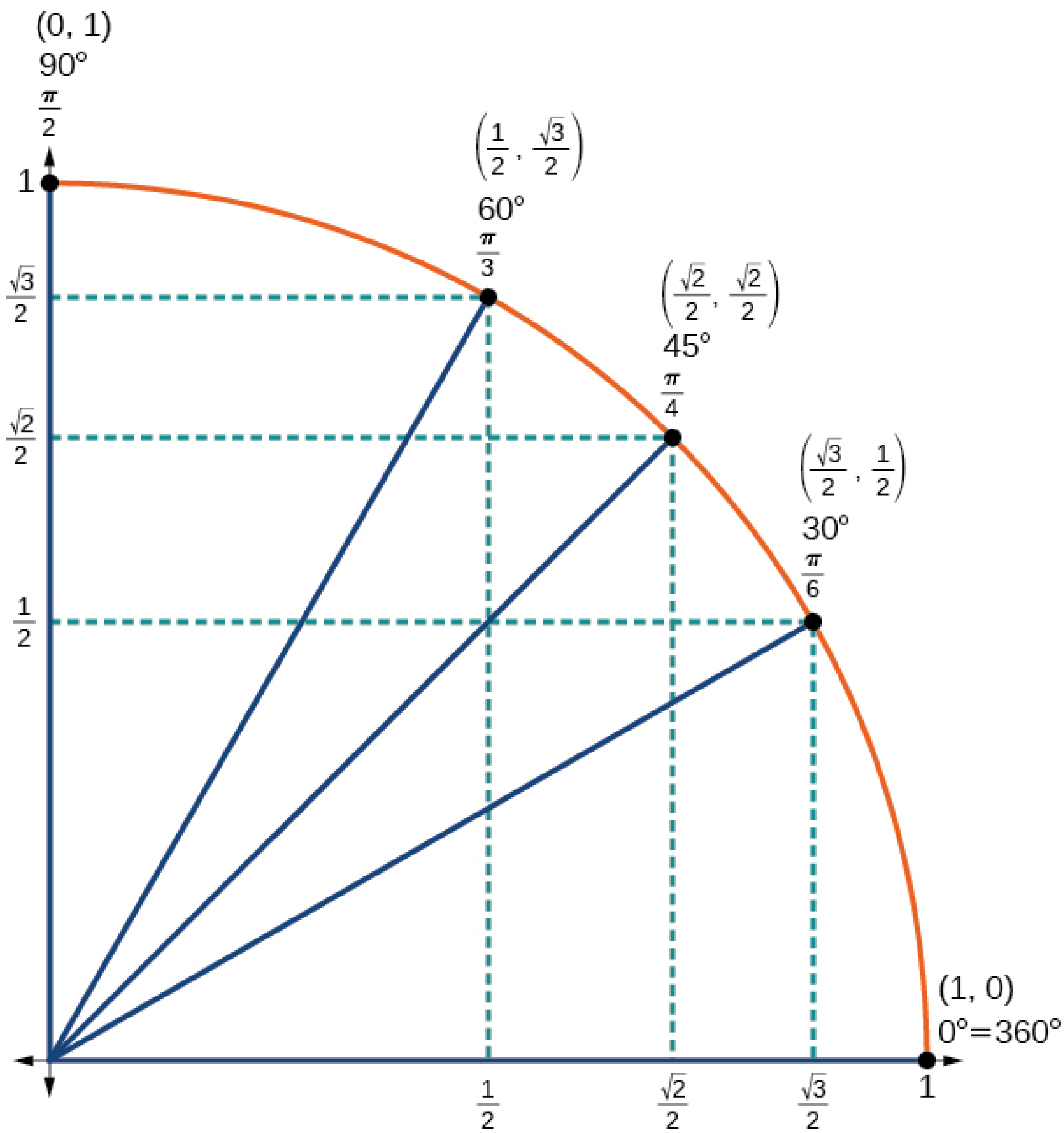


Figure 14.

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the special angles, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To

Given an angle in radians, use a graphing calculator to find the cosine.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Show Solution

Enter the following keystrokes:

$$\text{COS}(\ 5\ \times\ \pi\ \div\ 3\ )\ \text{ENTER}$$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of $20°$, for example, by including the conversion factor to radians as part of the input:

$$\text{SIN}(\ 20\ \times\ \pi\ \div\ 180\ )\ \text{ENTER}$$

Try It

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Show Solution
approximately 0.866025403

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of $x, y,$ and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive *x*-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the unit circle, as shown in (Figure). The bounds of the *x*-coordinate are $[-1, 1]$. The bounds of the *y*-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

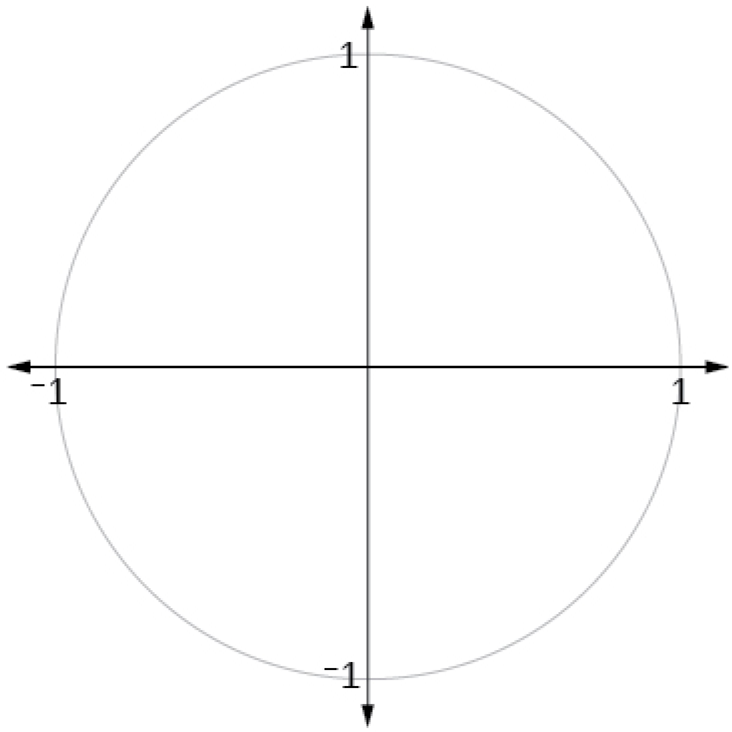


Figure 15.

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the *y*-coordinate on the unit circle, the other angle with the same sine will share the same *y*-value, but have the opposite *x*-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same *x*-value but will have the opposite *y*-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in (Figure), angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\begin{array}{ccc}\sin\left(t\right)=\sin\left(\alpha\right) & \text{and} & \cos\left(t\right)=-\cos\left(\alpha\right)\\ \sin\left(t\right)=-\sin\left(\beta\right) & \text{and} & \cos\left(t\right)=\cos\left(\beta\right)\end{array}$$

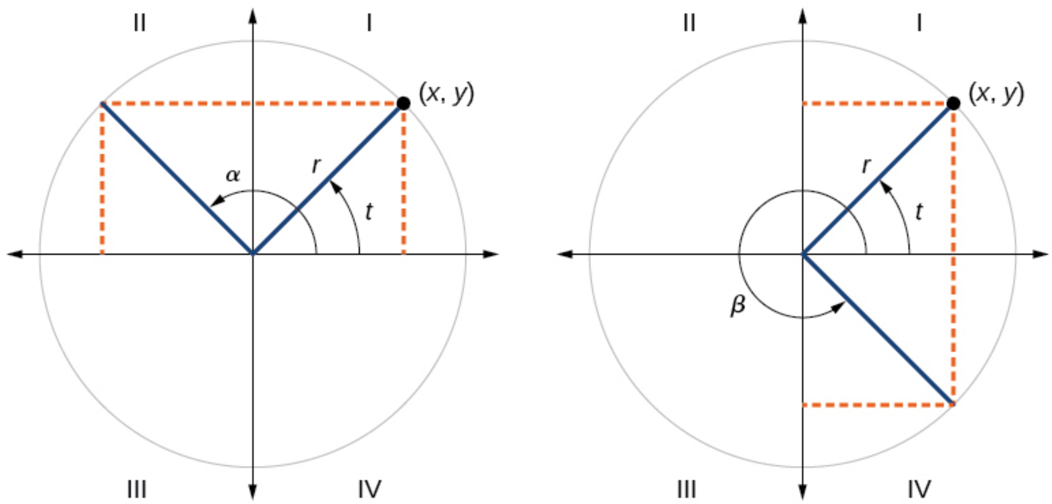


Figure 16.

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and $90°$, or 0 and $\frac{\pi}{2}$ radians. As we can see from (Figure), for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

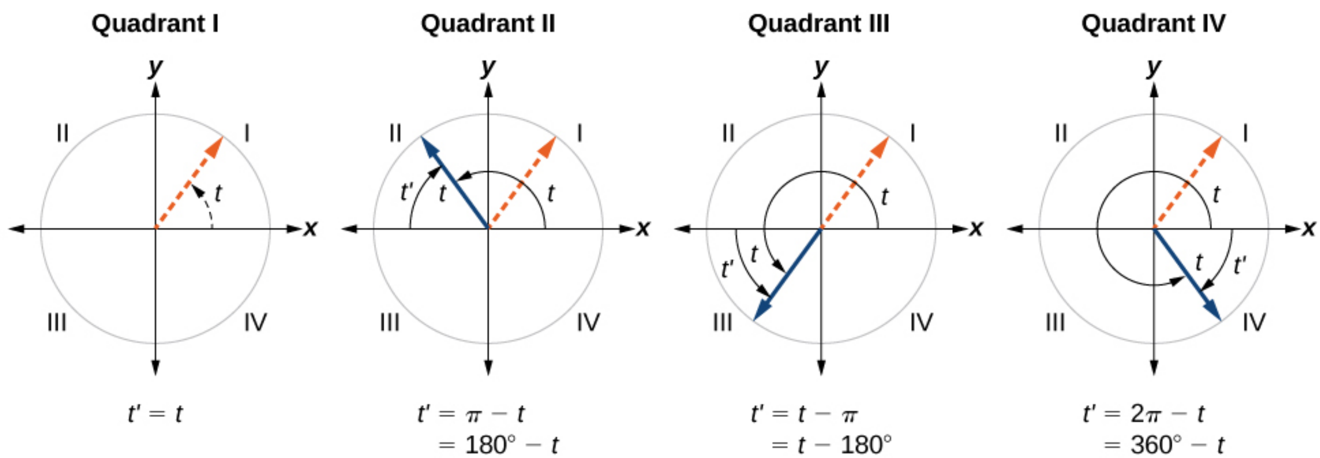


Figure 17.

How To

Given an angle between 0 and 2π, find its reference angle.

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180° - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360° - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Finding a Reference Angle

Find the reference angle of $225°$ as shown in (Figure).

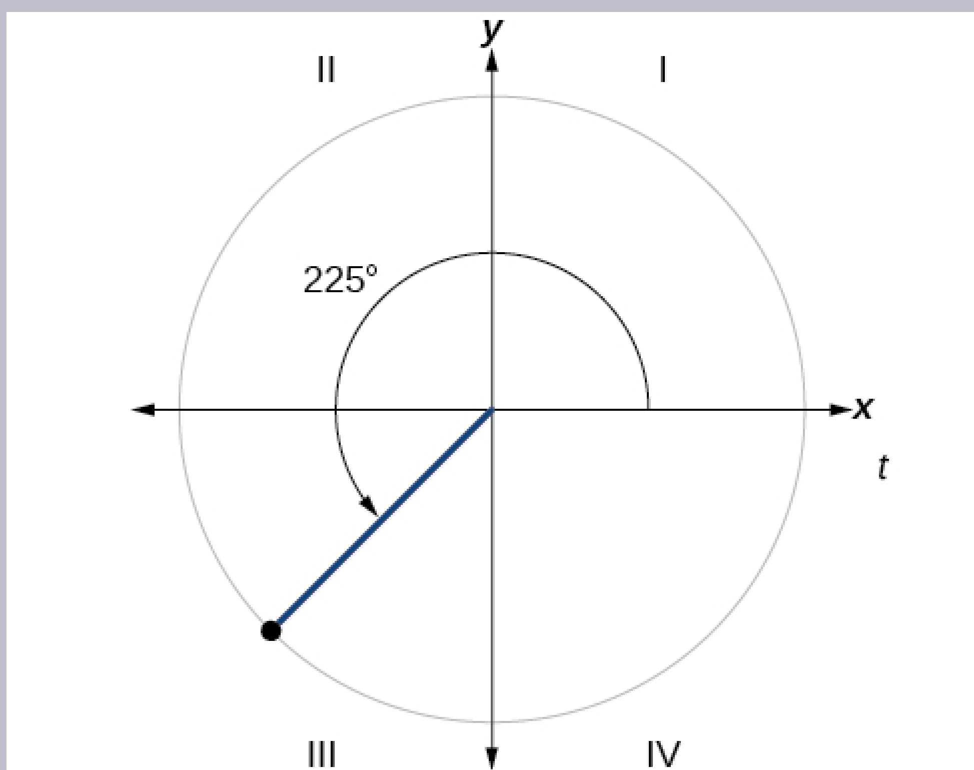


Figure 17.

Show Solution

Because $225°$ is in the third quadrant, the reference angle is

$$|(180° - 225°)| = |-45°| = 45°$$

Try It

Find the reference angle of $\frac{5\pi}{3}$.

Show Solution

$\frac{\pi}{3}$

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a negative angle. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the reference angle of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

Using Reference Angles to Find Cosine and Sine

Angles have cosines and sines with the same absolute value as their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To

Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Using Reference Angles to Find Sine and Cosine

a. Using a reference angle, find the exact value of $\cos(150°)$ and $\sin(150°)$.
b. Using the reference angle, find $\cos\frac{5\pi}{4}$ and $\sin\frac{5\pi}{4}$.

Show Solution

a. $150°$ is located in the second quadrant. The angle it makes with the *x*-axis is $180° - 150° = 30°$, so the reference angle is $30°$.
This tells us that $150°$ has the same sine and cosine values as $30°$, except for the sign.

$\cos\left(30°\right) = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad \sin\left(30°\right) = \frac{1}{2}$

Since $150°$ is in the second quadrant, the *x*-coordinate of the point on the circle is negative, so the cosine value is negative. The *y*-coordinate is positive, so the sine value is positive.

$\cos\left(150°\right) = -\frac{\sqrt{3}}{2} \qquad \text{and} \qquad \sin\left(150°\right) = \frac{1}{2}$

b. $\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \qquad \text{and} \qquad \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

Try It

a. Use the reference angle of $315°$ to find $\cos\left(315°\right)$ and $\sin\left(315°\right)$.
b. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$.

Show Solution

a. $\cos\left(315°\right) = \frac{\sqrt{2}}{2}, \sin\left(315°\right) = \frac{-\sqrt{2}}{2}$
b. $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in (Figure). Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

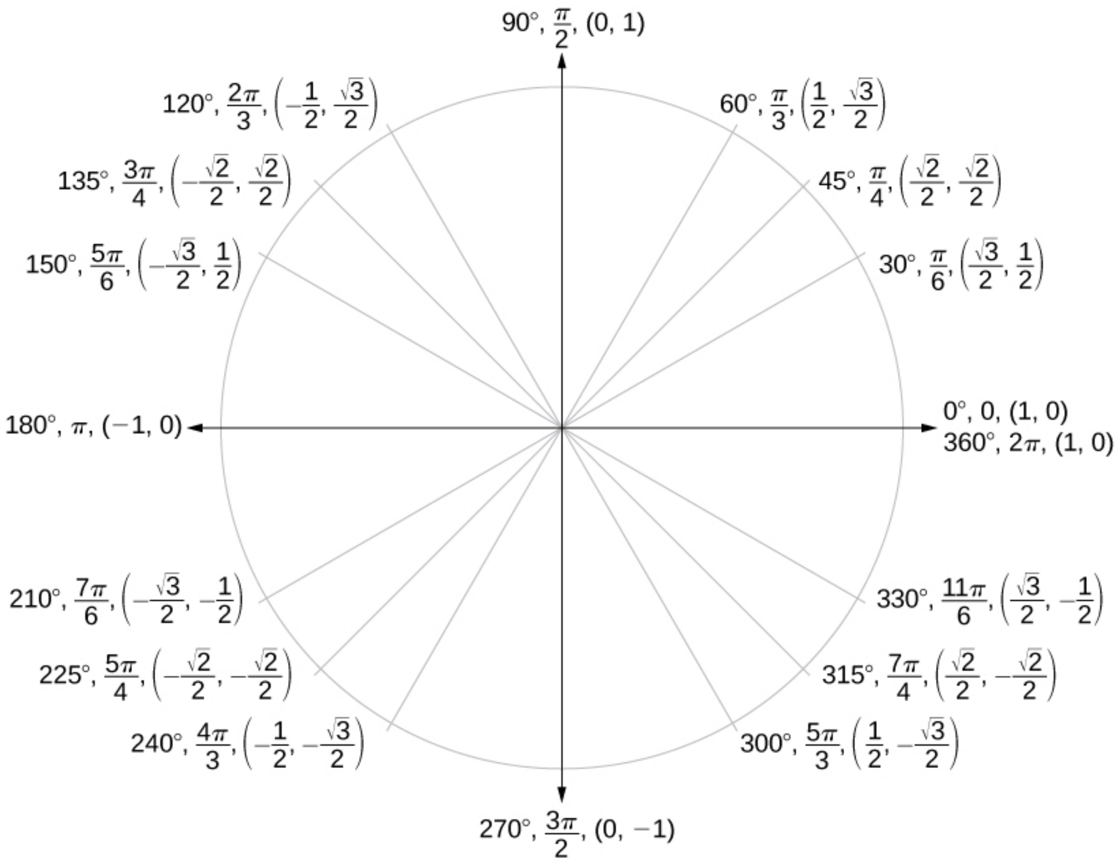


Figure 19. Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the identities

$$x = \cos\ t$$
$$y = \sin\ t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the reference angle, and give them the signs corresponding to the *y*– and *x*-values of the quadrant.

How To

Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point.

1. Find the reference angle by measuring the smallest angle to the *x*-axis.

2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Show Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the *x*-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$

Next, we will find the cosine and sine of the reference angle.

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

We must determine the appropriate signs for *x* and *y* in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\begin{aligned} \cos\left(\frac{7\pi}{6}\right) &= -\frac{\sqrt{3}}{2} \\ \sin\left(\frac{7\pi}{6}\right) &= -\frac{1}{2} \end{aligned}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\theta$ and $y = \sin\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Show Solution

$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle
- Sine and Cosine from the Unit
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four
- Trigonometric Functions Using Reference Angles

Key Equations

Cosine	$\cos t = x$
Sine	$\sin t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the *y*-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the *x*-value of the

endpoint. See (Figure).

- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See (Figure).
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See (Figure).
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See (Figure).
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the *x*– and *y*-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See (Figure).
- Reference angles can be used to find the sine and cosine of the original angle. See (Figure).
- Reference angles can also be used to find the coordinates of a point on a circle. See (Figure).

Section Exercises

Verbal

1. Describe the unit circle.

Show Solution
The unit circle is a circle of radius 1 centered at the origin.

2. What do the *x*- and *y*-coordinates of the points on the unit circle represent?

3. Discuss the difference between a coterminal angle and a reference angle.

Show Solution

Coterminal angles are angles that share the same terminal side. A reference angle is the size of the smallest acute angle, t, formed by the terminal side of the angle t and the horizontal axis.

4. Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

5. Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

Show Solution

The sine values are equal.

Algebraic

For the following exercises, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

6. $\sin(t) < 0$ and $\cos(t) < 0$

7. $\sin(t) > 0$ and $\cos(t) > 0$

Show Solution

I

8. $\sin(t) > 0$ and $\cos(t) < 0$

9. $\sin(t) > 0$ and $\cos(t) > 0$

Show Solution

IV

For the following exercises, find the exact value of each trigonometric function.

10. $\sin \frac{\pi}{2}$

11. $\sin \frac{\pi}{3}$

Show Solution

$\frac{\sqrt{3}}{2}$

12. $\cos \frac{\pi}{2}$

13. $\cos \frac{\pi}{3}$

Show Solution

$\frac{1}{2}$

14. $\sin \frac{\pi}{4}$

15. $\cos \frac{\pi}{4}$

Show Solution

$\frac{\sqrt{2}}{2}$

16. $\sin \frac{\pi}{6}$

17. $\sin \pi$

Show Solution

0

18. $\sin \frac{3\pi}{2}$

19. $\cos \pi$

Show Solution

-1

20. $\cos 0$

21. $\cos \frac{\pi}{6}$

Show Solution

$\frac{\sqrt{3}}{2}$

22. $\sin 0$

Numeric

For the following exercises, state the reference angle for the given angle.

23. $240°$

Show Solution

$60°$

24. $-170°$

25. $100°$

Show Solution

$80°$

26. $-315°$

27. $135°$

Show Solution

$45°$

28. $\frac{5\pi}{4}$

29. $\frac{2\pi}{3}$

Show Solution

$\frac{\pi}{3}$

30. $\frac{5\pi}{6}$

31. $\frac{-11\pi}{3}$

Show Solution
$\frac{\pi}{3}$

32. $\frac{-7\pi}{4}$

33. $\frac{-\pi}{8}$

Show Solution
$\frac{\pi}{8}$

For the following exercises, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

40. $225°$

41. $300°$

Show Solution
$60°$, Quadrant IV, $\sin\left(300°\right)=-\frac{\sqrt{3}}{2}, \cos\left(300°\right)=\frac{1}{2}$

42. $320°$

43. $135°$

Show Solution
45, Quadrant II, $\sin\left(135\right)=\frac{\sqrt{2}}{2}, \cos\left(135\right)=-\frac{\sqrt{2}}{2}$

44. $210°$

45. $120°$

Show Solution

$60°$, Quadrant II, $\sin(120°) = \frac{\sqrt{3}}{2}$, $\cos(120°) = -\frac{1}{2}$

46. $250°$

47. $150°$

Show Solution

$30°$, Quadrant II, $\sin(150°) = \frac{1}{2}$, $\cos(150°) = -\frac{\sqrt{3}}{2}$

48. $\frac{5\pi}{4}$

49. $\frac{7\pi}{6}$

Show Solution

$\frac{\pi}{6}$, Quadrant III, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$, $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

50. $\frac{5\pi}{3}$

51. $\frac{3\pi}{4}$

Show Solution

$\frac{\pi}{4}$, Quadrant II, $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\cos\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{2}}{2}$

52. $\frac{4\pi}{3}$

53. $\frac{2\pi}{3}$

Show Solution
$\frac{\pi}{3}$, Quadrant II, $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

54. $\frac{5\pi}{6}$

55. $\frac{7\pi}{4}$

Show Solution
$\frac{\pi}{4}$, Quadrant IV, $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$

For the following exercises, find the requested value.

56. If $\cos(t) = \frac{1}{7}$ and t is in the fourth quadrant, find $\sin(t)$.

57. If $\cos(t) = \frac{2}{9}$ and t is in the first quadrant, find $\sin(t)$.

Show Solution
$\frac{\sqrt{77}}{9}$

58. If $\sin(t) = \frac{3}{8}$ and t is in the second quadrant, find $\cos(t)$.

59. If $\sin(t) = -\frac{1}{4}$ and t is in the third quadrant, find $\cos(t)$.

Show Solution

$-\frac{\sqrt{15}}{4}$

60. Find the coordinates of the point on a circle with radius 15 corresponding to an angle of $220°$.

61. Find the coordinates of the point on a circle with radius 20 corresponding to an angle of $120°$.

Show Solution

$(-10, 10\sqrt{3})$

62. Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

63. Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9}$.

Show Solution

$(-2.778,\ 15.757)$

64. State the domain of the sine and cosine functions.

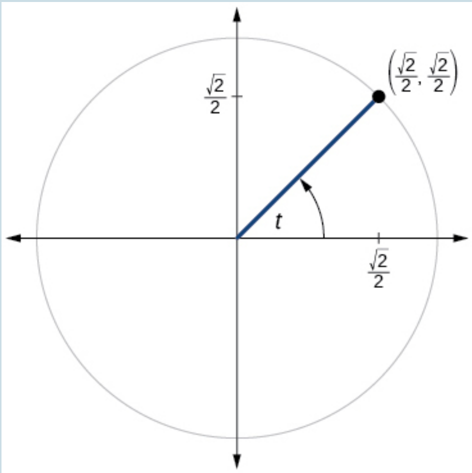
65. State the range of the sine and cosine functions.

Show Solution

$[-1,\ 1]$

Graphical

For the following exercises, use the given point on the unit circle to find the value of the sine and cosine of t.

66. 

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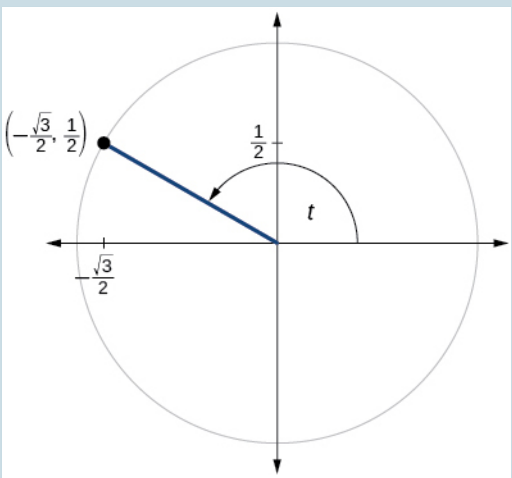
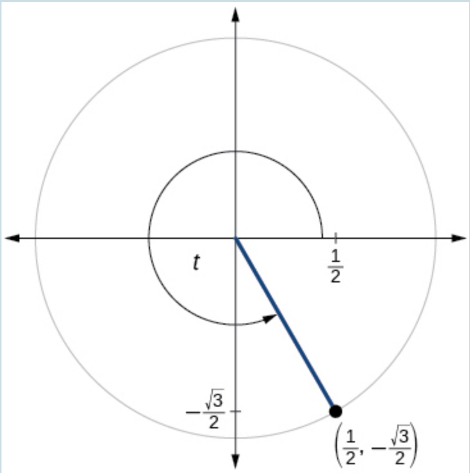
67. 

Show Solution

$\sin t = \frac{1}{2}, \cos t = -\frac{\sqrt{3}}{2}$

68. 

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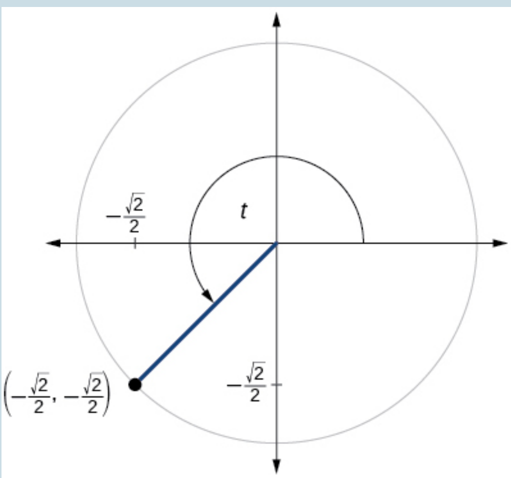
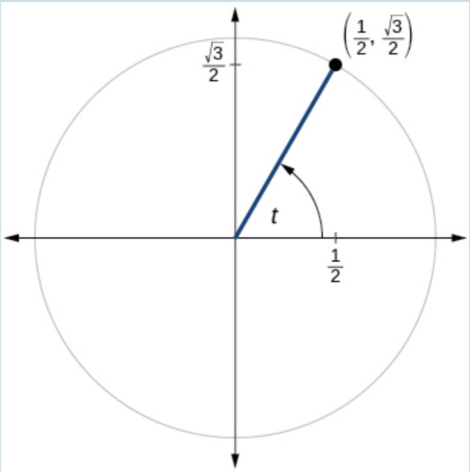
69. 

Show Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = -\frac{\sqrt{2}}{2}$

70. 

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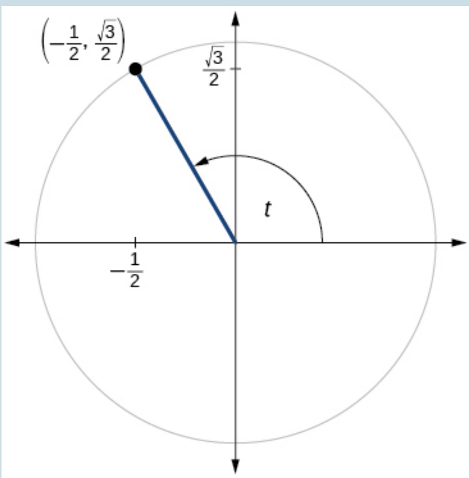
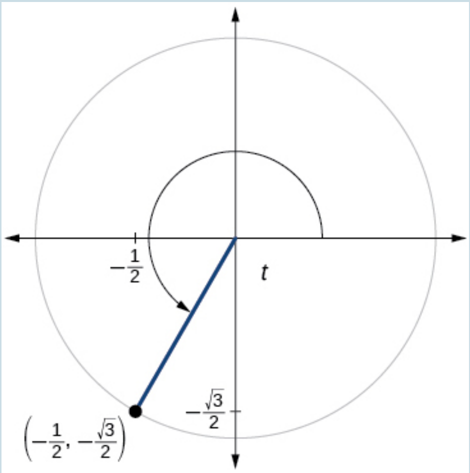
71. 

Show Solution

$\sin t = \frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}$

72. 

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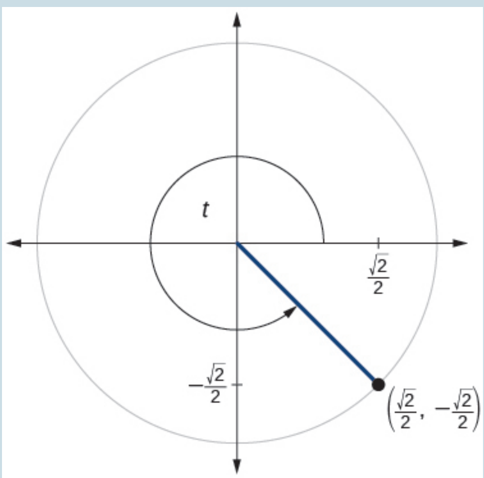
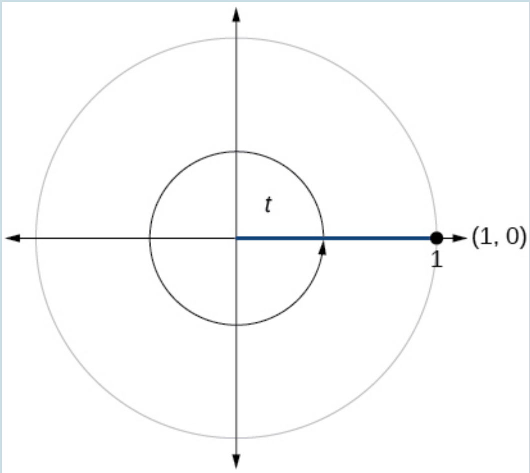
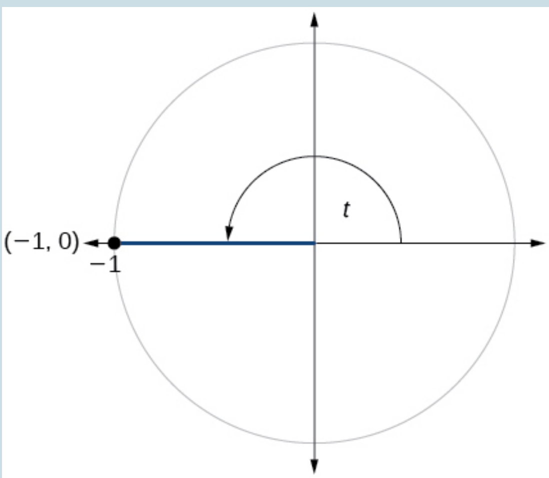
73. 

Show Solution

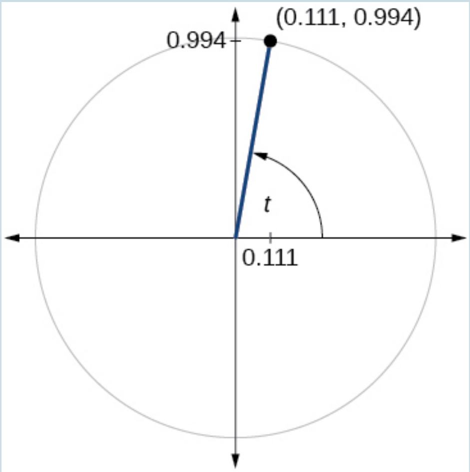
$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}$

74. 

75. 

Show Solution

$\sin t = 0, \cos t = -1$

76. 

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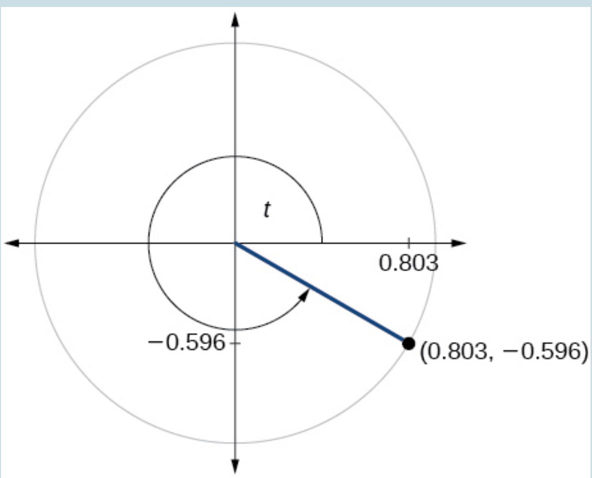
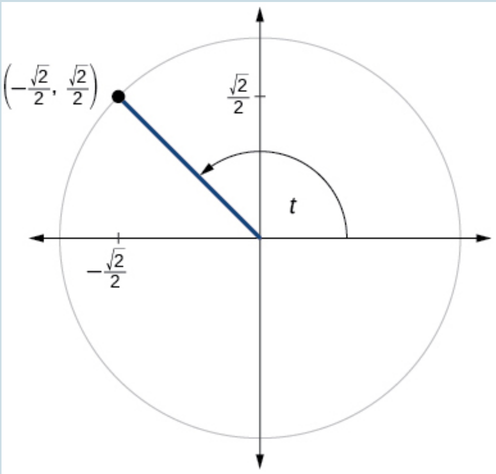
77. 

Show Solution

$\sin t = -0.596, \cos t = 0.803$

78. 

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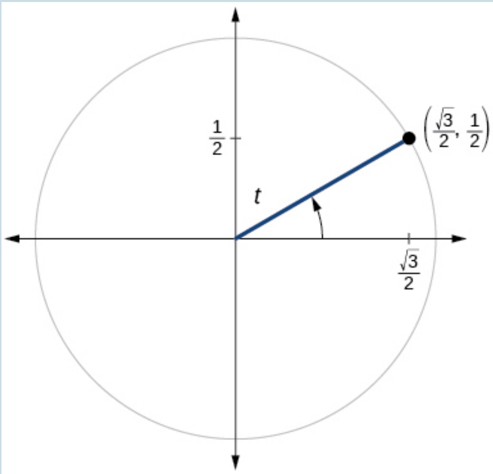
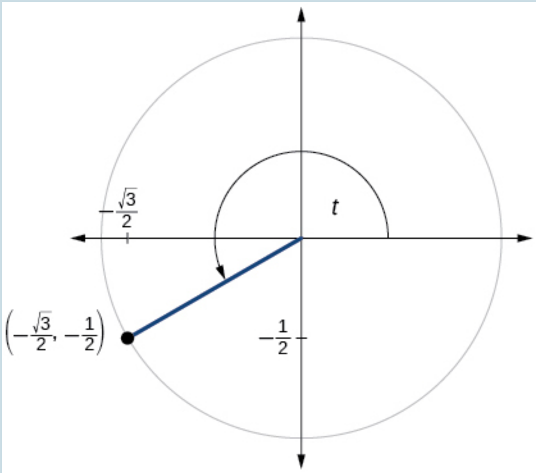
79. 

Show Solution

$\sin t = \frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$

80. 

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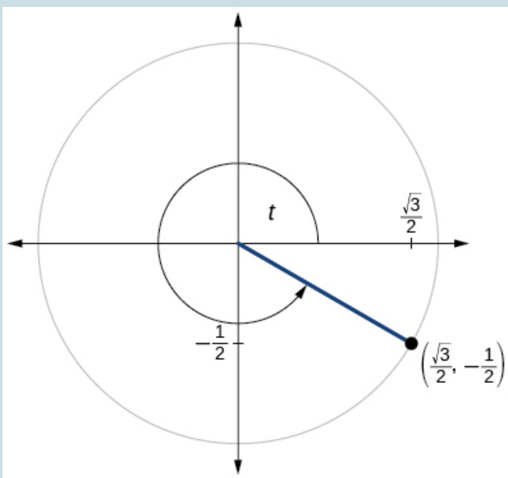
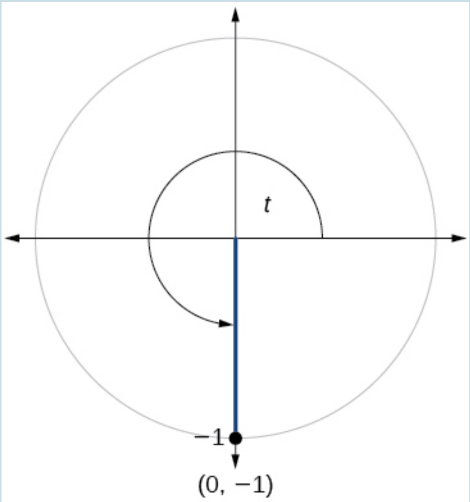
81. 

Show Solution

$\sin t = -\frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$

82. 

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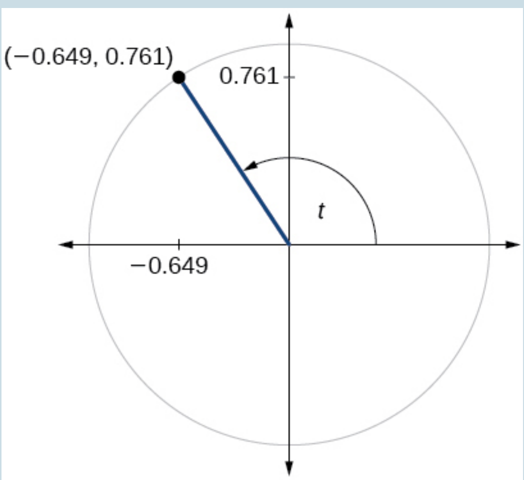
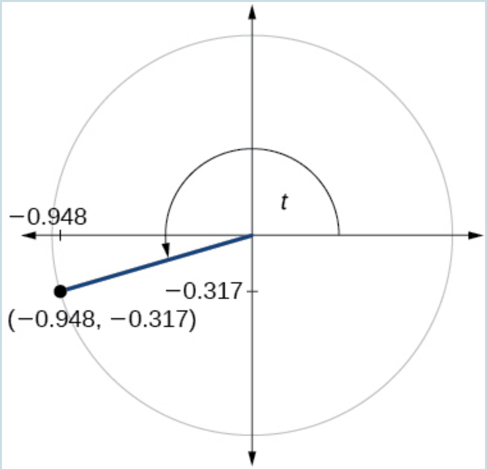
83. 

Show Solution

$\sin t = 0.761, \cos t = -0.649$

84. 

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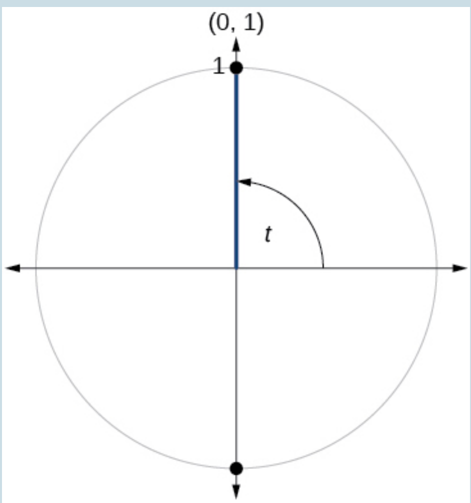
85. 

Show Solution

$\sin t = 1, \cos t = 0$

Technology

For the following exercises, use a graphing calculator to evaluate.

86. $\sin \frac{5\pi}{9}$

87. $\cos \frac{5\pi}{9}$

Show Solution
−0.1736

88. $\sin \frac{\pi}{10}$

89. $\cos \frac{\pi}{10}$

Show Solution
0.9511

90. $\sin \frac{3\pi}{4}$

91. $\cos \frac{3\pi}{4}$

Show Solution
−0.7071

92. $\sin 98$

93. $\cos 98$

Show Solution
−0.1392

94. $\cos 310$

95. $\sin 310$

Show Solution
-0.7660

Extensions

For the following exercises, evaluate.

96. $\sin\left(\frac{11\pi}{3}\right)\ \cos\left(\frac{-5\pi}{6}\right)$

97. $\sin\left(\frac{3\pi}{4}\right)\ \cos\left(\frac{5\pi}{3}\right)$

Show Solution
$\frac{\sqrt{2}}{4}$

98. $\sin\left(-\frac{4\pi}{3}\right)\ \cos\left(\frac{\pi}{2}\right)$

99. $\sin\left(\frac{-9\pi}{4}\right)\ \cos\left(\frac{-\pi}{6}\right)$

Show Solution
$-\frac{\sqrt{6}}{4}$

100. $\sin\left(\frac{\pi}{6}\right)\ \cos\left(\frac{-\pi}{3}\right)$

101. $\sin\left(\frac{7\pi}{4}\right)\cos\left(\frac{-2\pi}{3}\right)$

Show Solution

$\frac{\sqrt{2}}{4}$

102. $\cos\left(\frac{5\pi}{6}\right)\ \cos\left(\frac{2\pi}{3}\right)$

103. $\cos\left(\frac{-\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$

Show Solution

$\frac{\sqrt{2}}{4}$

104. $\sin\left(\frac{-5\pi}{4}\right)\ \sin\left(\frac{11\pi}{6}\right)$

105. $\sin\left(\pi\right)\sin\left(\frac{\pi}{6}\right)$

Show Solution

0

Real-World Applications

For the following exercises, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point $(0, 1)$, that is, on the due north position. Assume the carousel revolves counter clockwise.

106. What are the coordinates of the child after 45 seconds?

107. What are the coordinates of the child after 90 seconds?

Show Solution
$(0, -1)$

108. What are the coordinates of the child after 125 seconds?

109. When will the child have coordinates $(0.707, -0.707)$ if the ride lasts 6 minutes? (There are multiple answers.)

Show Solution
37.5 seconds, 97.5 seconds, 157.5 seconds, 217.5 seconds, 277.5 seconds, 337.5 seconds

110. When will the child have coordinates $(-0.866, -0.5)$ if the ride lasts 6 minutes?

Glossary

cosine function
: the *x*-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity
: a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function
: the *y*-value of the point on a unit circle corresponding to a given angle

CHAPTER 3.5: THE OTHER TRIGONOMETRIC FUNCTIONS

Learning Objectives

In this section you will:

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

We can also define the remaining functions in terms of the unit circle with a point (x, y) corresponding to an angle of t, as shown in (Figure). As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

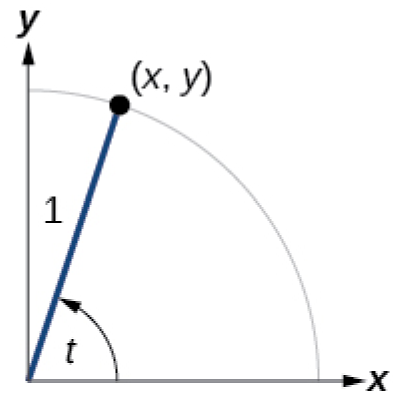


Figure 1.

The first function we will define is the tangent. The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In (Figure), the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as $\tan$. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The secant function is the reciprocal of the cosine function. In (Figure), the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as $\sec$.
- The cotangent function is the reciprocal of the tangent function. In (Figure), the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as $\cot$.
- The cosecant function is the reciprocal of the sine function. In (Figure), the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as $\csc$.

Tangent, Secant, Cosecant, and Cotangent Functions

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\begin{aligned} \tan\ t &= \frac{y}{x}, x \neq 0 \\ \sec\ t &= \frac{1}{x}, x \neq 0 \\ \csc\ t &= \frac{1}{y}, y \neq 0 \\ \cot\ t &= \frac{x}{y}, y \neq 0 \end{aligned}$$

Finding Trigonometric Functions from a Point on the Unit Circle

The point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is on the unit circle, as shown in (Figure). Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t.$

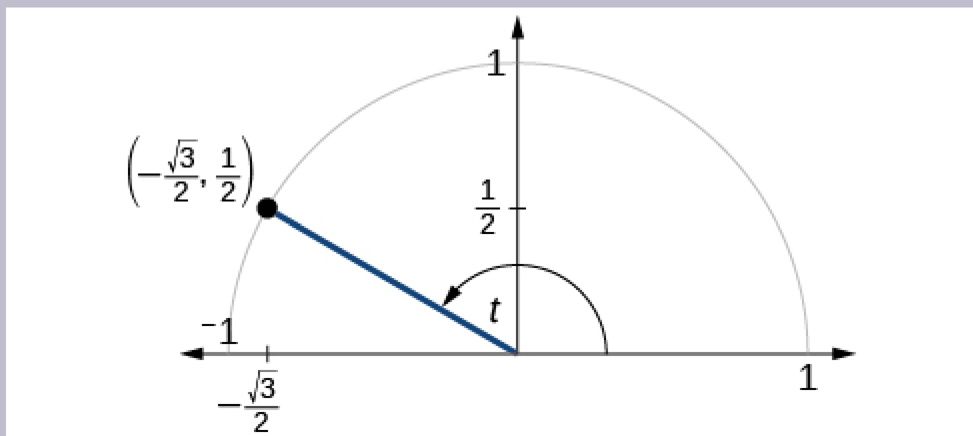


Figure 2.

Show Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle $t,$ we can use those coordinates to find the six functions:

$$\begin{array}{llllll}
\sin\ t & = y & = \frac{1}{2} & & & \\
\cos\ t & = x & = -\frac{\sqrt{3}}{2} & & & \\
\tan\ t & = \frac{y}{x} & = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} & = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) & = -\frac{1}{\sqrt{3}} & = -\frac{\sqrt{3}}{3} \\
\sec\ t & = \frac{1}{x} & = \frac{1}{-\frac{\sqrt{3}}{2}} & = -\frac{2}{\sqrt{3}} & = -\frac{2\sqrt{3}}{3} & \\
\csc\ t & = \frac{1}{y} & = \frac{1}{\frac{1}{2}} & = 2 & & \\
\cot\ t & = \frac{x}{y} & = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} & = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) & = -\sqrt{3} &
\end{array}$$

Try It

The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in (Figure). Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t.$

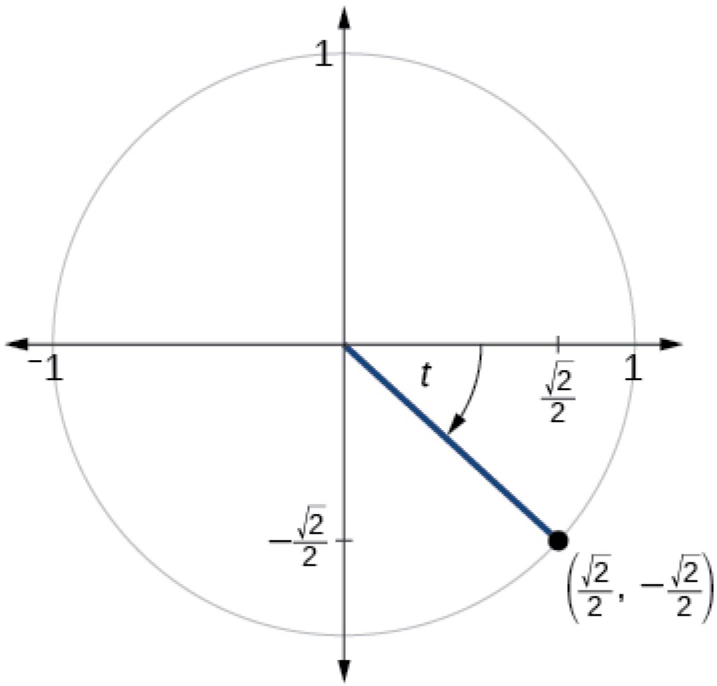


Figure 3.

Show Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$

Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t.$ when $t = \frac{\pi}{6}.$

Show Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\begin{aligned}
\tan\frac{\pi}{6} &= \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} \\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} && = \frac{1}{\sqrt{3}} && = \frac{\sqrt{3}}{3} \\
\sec\frac{\pi}{6} &= \frac{1}{\cos\frac{\pi}{6}} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} && = \frac{2}{\sqrt{3}} && = \frac{2\sqrt{3}}{3} \\
\csc\frac{\pi}{6} &= \frac{1}{\sin\frac{\pi}{6}} && = \frac{1}{\frac{1}{2}} && = 2 \\
\cot\frac{\pi}{6} &= \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} && = \sqrt{3}
\end{aligned}$$

Try It

Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t.$ when $t = \frac{\pi}{3}$.

Show Solution

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{3} = \frac{1}{2}$$
$$\tan\frac{\pi}{3} = \sqrt{3}$$
$$\sec\frac{\pi}{3} = 2$$
$$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}$$
$$\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x equal to the cosine and y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in (Figure).

Angle	0	$\frac{\pi}{6}$, or 30	$\frac{\pi}{4}$, or 45	$\frac{\pi}{3}$, or 60	$\frac{\pi}{2}$, or 90
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the reference angle formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*– and *y*-values in the original quadrant. (Figure) shows which functions are positive in which quadrant.

To help remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

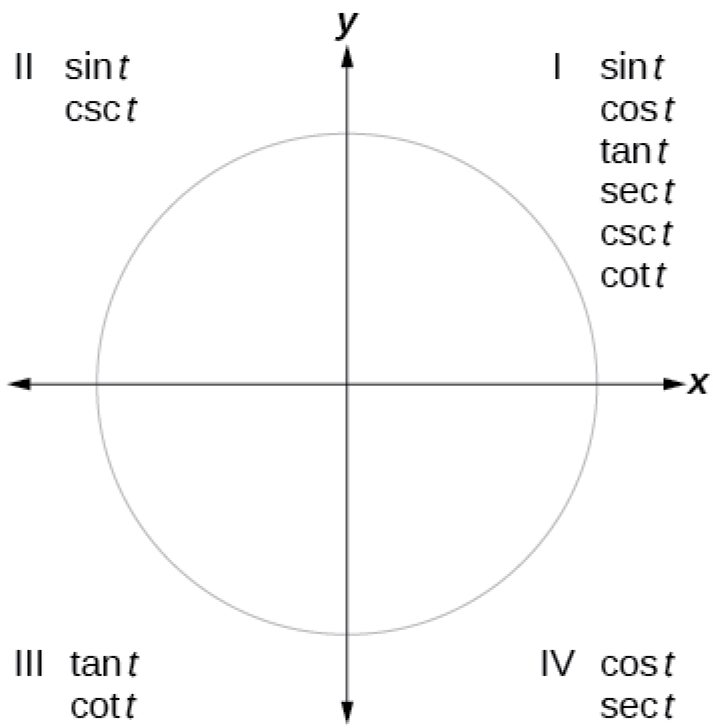


Figure 4. The trigonometric functions are each listed in the quadrants in which they are positive.

How To

Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions.

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis.

This is the reference angle.

2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Show Solution

The angle between this angle's terminal side and the *x*-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\begin{aligned} \cos\left(-\frac{5\pi}{6}\right) &= -\frac{\sqrt{3}}{2}, \sin\left(-\frac{5\pi}{6}\right) &= -\frac{1}{2}, \tan\left(-\frac{5\pi}{6}\right) &= \frac{\sqrt{3}}{3}, \\ \sec\left(-\frac{5\pi}{6}\right) &= -\frac{2\sqrt{3}}{3}, \csc\left(-\frac{5\pi}{6}\right) &= -2, \cot\left(-\frac{5\pi}{6}\right) &= \sqrt{3} \end{aligned}$$

Try It

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Show Solution

$$\sin\left(\frac{-7\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{-7\pi}{4}\right) = \frac{\sqrt{2}}{2}, \tan\left(\frac{-7\pi}{4}\right) = 1,$$
$$\sec\left(\frac{-7\pi}{4}\right) = \sqrt{2}, \csc\left(\frac{-7\pi}{4}\right) = \sqrt{2}, \cot\left(\frac{-7\pi}{4}\right) = 1$$

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f\left(x\right) = x^2$, shown in (Figure). The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $\left(4\right)^2 = \left(-4\right)^2$, $\left(-5\right)^2 = \left(5\right)^2$, and so on. So $f\left(x\right) = x^2$ is an even function, a function such that two inputs that are opposites have the same output. That means $f\left(-x\right) = f\left(x\right)$.

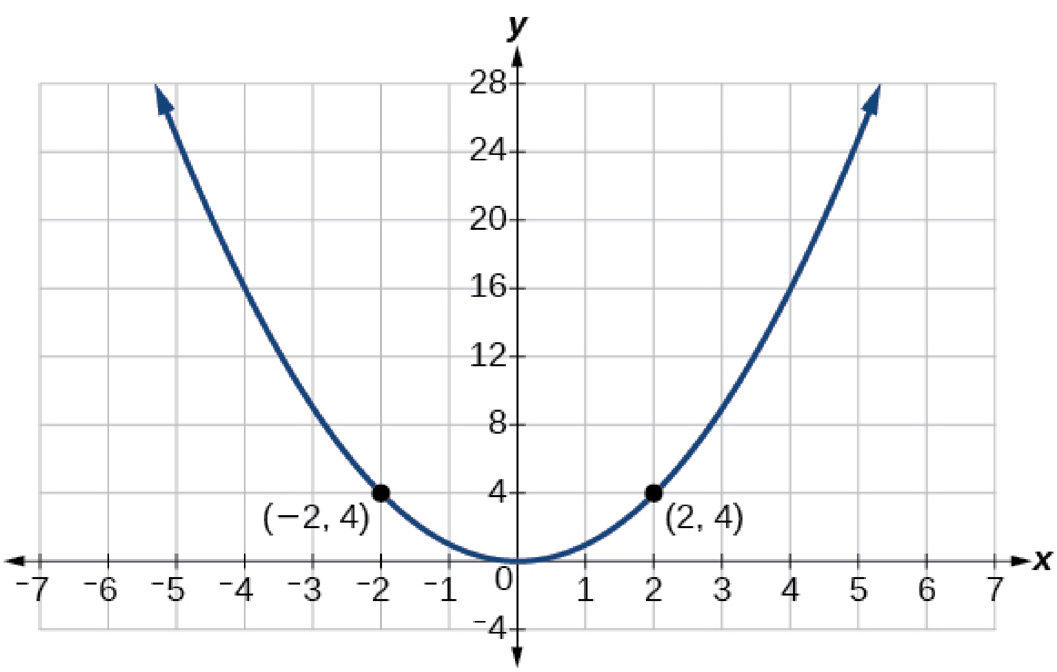


Figure 5. The function $f\left(x\right) = x^2$ is an even function.

Now consider the function $f\left(x\right) = x^3$, shown in (Figure). The graph is not symmetrical about the *y*-axis.

All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f\left(x\right)={x}^{3}$ is an odd function, one such that two inputs that are opposites have outputs that are also opposites. That means $f\left(-x\right)=-f\left(x\right).$

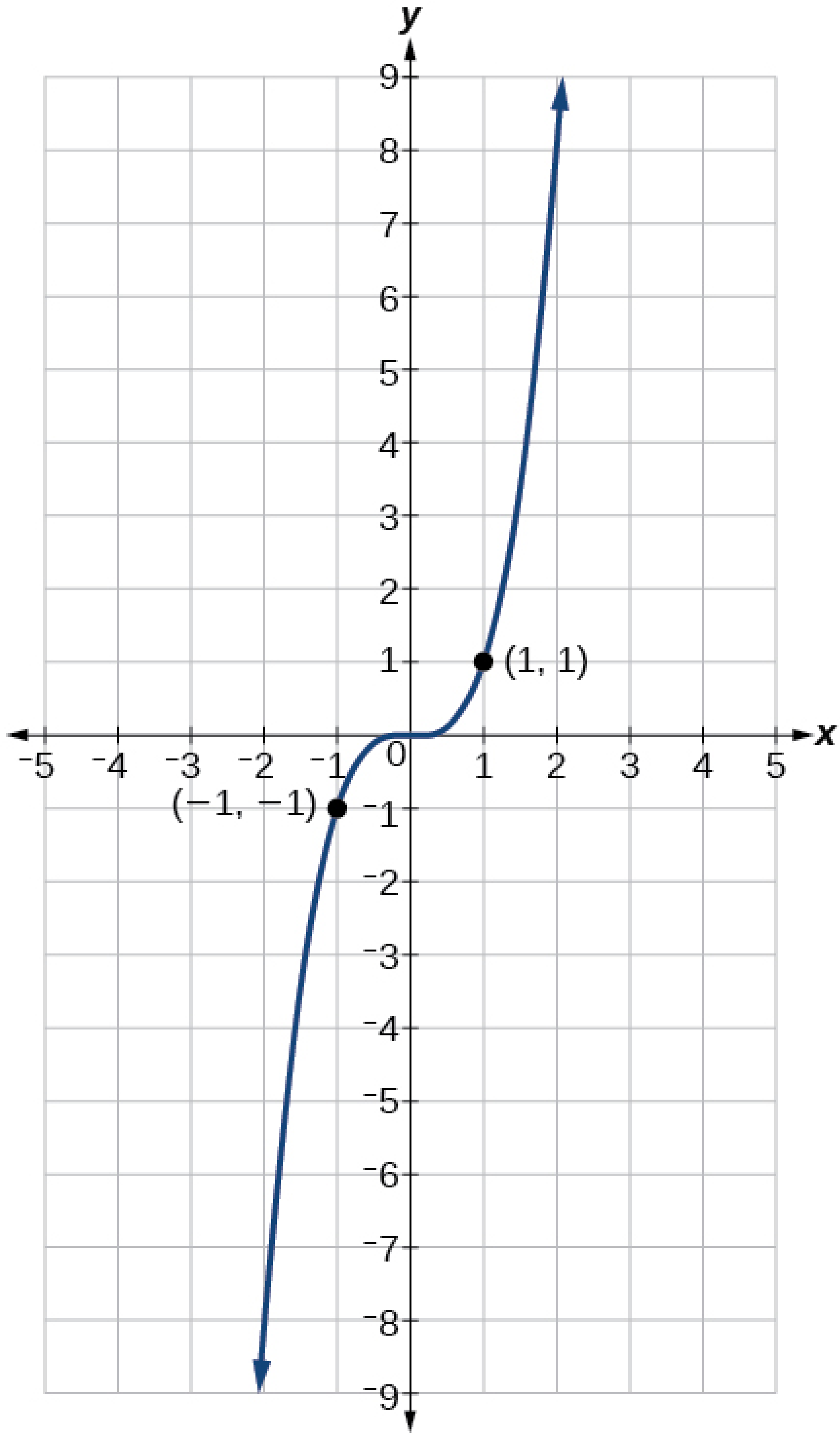


Figure 6. The function $f\left(x\right)={x}^{3}$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a unit circle with a positive and a negative angle, as in (Figure). The sine of the positive angle is y. The sine of the negative angle is $-y$. The sine function, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in (Figure).

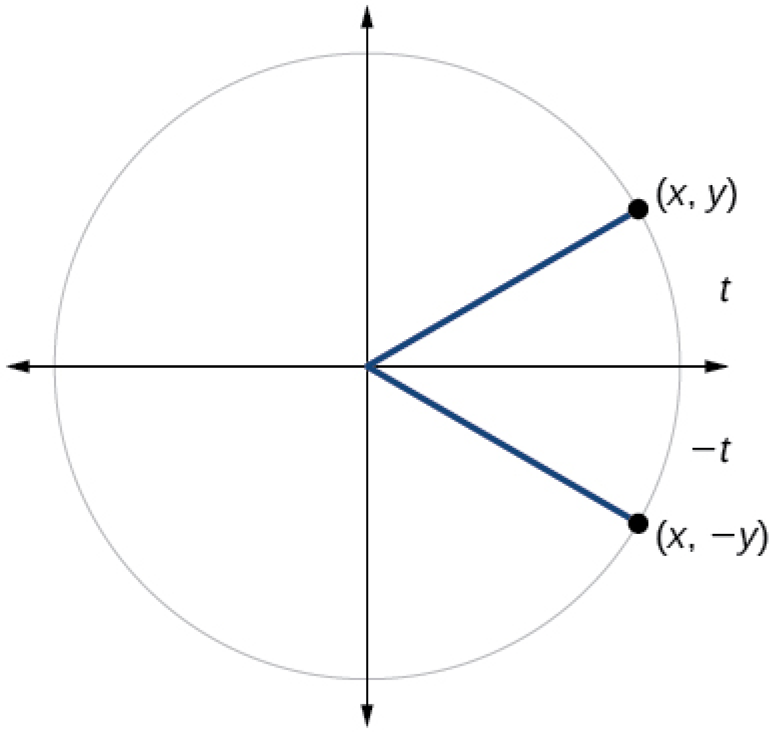


Figure 7.

$$\begin{array}{rclrclrcl}
\sin t & = & y & \cos t & = & x & \tan(t) & = & \frac{y}{x} \\
\sin(-t) & = & -y & \cos(-t) & = & x & \tan(-t) & = & -\frac{y}{x} \\
\sin t & \neq & \sin(-t) & \cos t & = & \cos(-t) & \tan t & \neq & \tan(-t) \\
 & & & & & & & & \\
\sec t & = & \frac{1}{x} & \csc t & = & \frac{1}{y} & \cot t & = & \frac{x}{y} \\
\sec(-t) & = & \frac{1}{x} & \csc(-t) & = & \frac{1}{-y} & \cot(-t) & = & \frac{x}{-y} \\
\sec t & = & \sec(-t) & \csc t & \neq & \csc(-t) & \cot t & \neq & \cot(-t)
\end{array}$$

Even and Odd Trigonometric Functions

An even function is one in which $f(-x) = f(x)$.

An odd function is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\begin{aligned} \cos(-t) &= \cos t \\ \sec(-t) &= \sec t \end{aligned}$$

Sine, tangent, cosecant, and cotangent are odd:

$$\begin{aligned} \sin(-t) &= -\sin t \\ \tan(-t) &= -\tan t \\ \csc(-t) &= -\csc t \\ \cot(-t) &= -\cot t \end{aligned}$$

Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Show Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Try It

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Show Solution

$-\sqrt{3}$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

Fundamental Identities

We can derive some useful identities from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$\tan t = \frac{\sin t}{\cos t}$

$\sec t = \frac{1}{\cos t}$

$\csc t = \frac{1}{\sin t}$

$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45) = \frac{\sqrt{2}}{2}, \cos(45) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45)$.
b. Given $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}, \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, evaluate $\sec\left(\frac{5\pi}{6}\right)$.

Show Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.
$$\begin{aligned}\tan(45) &= \frac{\sin(45)}{\cos(45)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1\end{aligned}$$

---

b.
$$\begin{aligned}\sec\left(\frac{5\pi}{6}\right) &= \frac{1}{\cos\left(\frac{5\pi}{6}\right)} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2\sqrt{3}}{1} \\ &= \frac{-2}{\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3}\end{aligned}$$

Try It

Evaluate $\csc\left(\frac{7\pi}{6}\right)$.

Show Solution

-2

Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Show Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{array}{lll} \frac{\sec\ t}{\tan\ t} & = & \frac{\frac{1}{\cos\ t}}{\frac{\sin\ t}{\cos\ t}} \\ & = & \frac{1}{\cos\ t}\Delta\frac{\cos\ t}{\sin\ t} & \text{Multiply by the reciprocal.} \\ & = & \frac{1}{\sin\ t} = \csc\ t & \text{Simplify and use the identity.} \end{array}$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$, we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Try It

Simplify $(\tan t)\,(\cos t)$.

Show Solution

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternate forms of the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$.

$$\begin{array}{rcl} \frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} & = & \frac{1}{\cos^2 t} \\ 1 + \tan^2 t & = & \sec^2 t \end{array}$$

The other form is obtained by dividing both sides by $\sin^2 t$.

$$\begin{aligned} \frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} &= \frac{1}{\sin^2 t} \\ \cot^2 t + 1 &= \csc^2 t \end{aligned}$$

Alternate Forms of the Pythagorean Identity

$$1 + \tan^2 t = \sec^2 t$$
$$\cot^2 t + 1 = \csc^2 t$$

Using Identities to Relate Trigonometric Functions

If $\cos(t) = \frac{12}{13}$ and t is in quadrant IV, as shown in (Figure), find the values of the other five trigonometric functions.

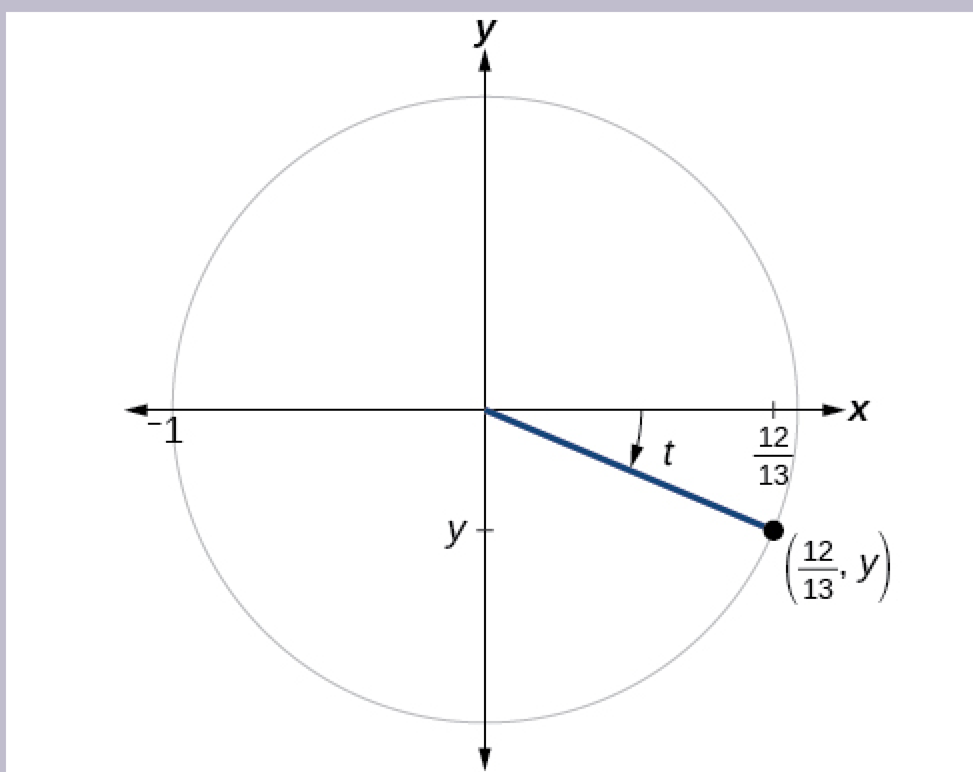


Figure 8.

Show Solution

$\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$\begin{aligned}\left(\frac{12}{13}\right)^2 + \sin^2 t &= 1\\ \sin^2 t &= 1 - \left(\frac{12}{13}\right)^2\\ \sin^2 t &= 1 - \frac{144}{169}\\ \sin^2 t &= \frac{25}{169}\\ \sin\ t &= \sqrt{\frac{25}{169}}\\ \sin\ t &= \frac{\sqrt{25}}{\sqrt{169}}\\ \sin\ t &= \frac{5}{13}\end{aligned}$$

The sign of the sine depends on the y-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the y-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\begin{aligned}\tan\ t &= \frac{\sin\ t}{\cos\ t} &= \frac{-\frac{5}{13}}{\frac{12}{13}} &= -\frac{5}{12}\\ \sec\ t &= \frac{1}{\cos\ t} &= \frac{1}{\frac{12}{13}} &= \frac{13}{12}\\ \csc\ t &= \frac{1}{\sin\ t} &= \frac{1}{-\frac{5}{13}} &= \frac{-13}{5}\\ \cot\ t &= \frac{1}{\tan\ t} &= \frac{1}{-\frac{5}{12}} &= -\frac{12}{5}\end{aligned}$$

Try It

If $\sec\left(t\right) = -\frac{17}{8}$ and $0 < t < \pi$, find the values of the other five functions.

Show Solution

$$\cos t = -\frac{8}{17},\ \sin t = \frac{15}{17},\ \tan t = -\frac{15}{8}$$
$$\csc t = \frac{17}{15},\ \cot t = -\frac{8}{15}$$

As we discussed at the beginning of the chapter, a function that repeats its values in regular intervals is known as a periodic function. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π, will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4) = f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

Period of a Function

The period P of a repeating function f is the number representing the interval such that $f(x+P) = f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on (Figure).

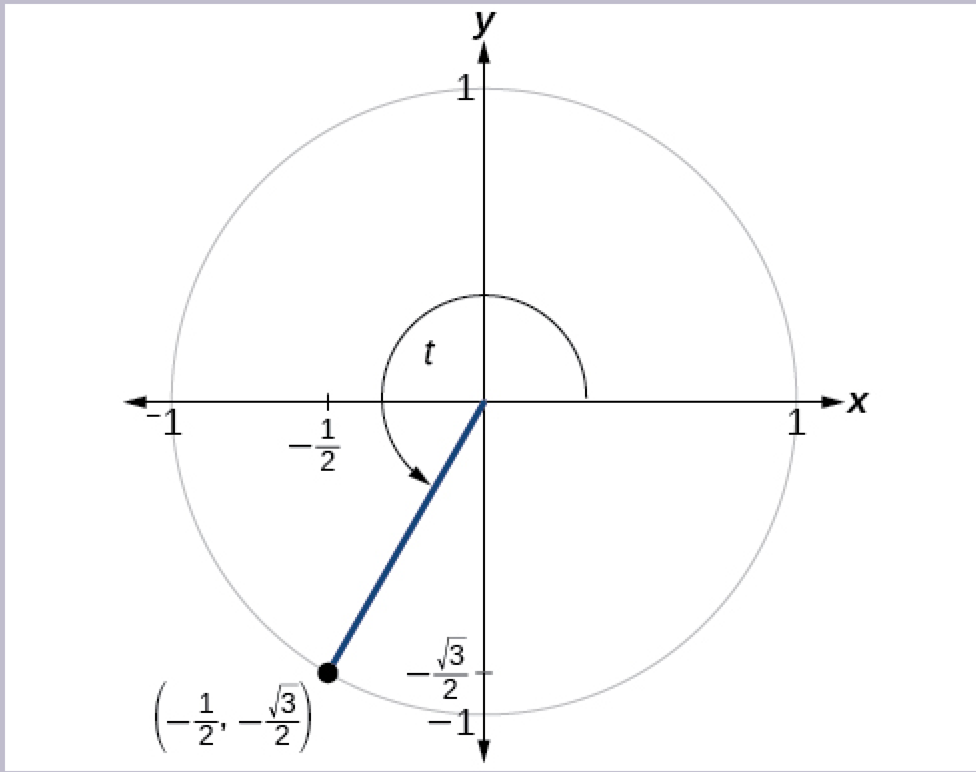


Figure 9.

Show Solution

$$\begin{aligned}
\sin t &= y &&= -\frac{\sqrt{3}}{2} \\
\cos t &= x &&= -\frac{1}{2} \\
\tan t &= \frac{\sin t}{\cos t} &&= \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} &&= \sqrt{3} \\
\sec t &= \frac{1}{\cos t} &&= \frac{1}{-\frac{1}{2}} &&= -2 \\
\csc t &= \frac{1}{\sin t} &&= \frac{1}{-\frac{\sqrt{3}}{2}} &&= -\frac{2\sqrt{3}}{3} \\
\cot t &= \frac{1}{\tan t} &&= \frac{1}{\sqrt{3}} &&= \frac{\sqrt{3}}{3}
\end{aligned}$$

Try It

Find the values of the six trigonometric functions of angle t based on (Figure).

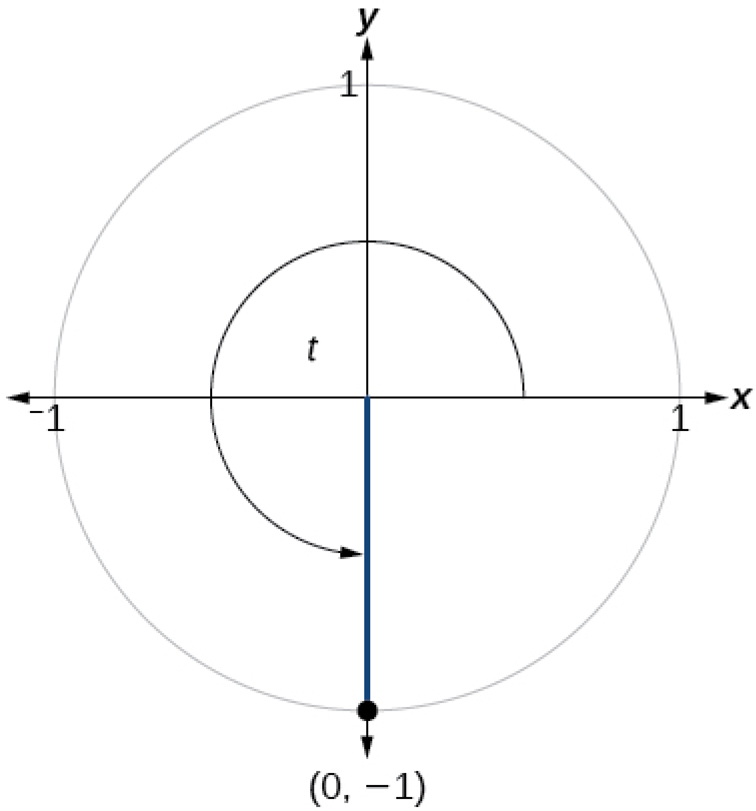


Figure 10.

Show Solution

$\sin t = -1, \cos t = 0, \tan t = \text{Undefined}$
$\sec t = \text{Undefined}, \csc t = -1, \cot t = 0$

Finding the Value of Trigonometric Functions

If $\sin\left(t\right)=-\frac{\sqrt{3}}{2}$ and $\cos\left(t\right)=\frac{1}{2}$, find $\sec\left(t\right),\csc\left(t\right),\tan\left(t\right),\cot\left(t\right).$

Show Solution

$$\begin{array}{llll}\sec\ t & =\frac{1}{\cos\ t} & =\frac{1}{\frac{1}{2}}=2 \\ \csc\ t & =\frac{1}{\sin\ t} & =\frac{1}{-\frac{\sqrt{3}}{2}}-\frac{2\sqrt{3}}{3} \\ \tan\ t & =\frac{\sin\ t}{\cos\ t} & =\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=-\sqrt{3} \\ \cot\ t & =\frac{1}{\tan\ t} & =\frac{1}{-\sqrt{3}}=-\frac{\sqrt{3}}{3}\end{array}$$

Try It

$\sin\left(t\right)=\frac{\sqrt{2}}{2}$ and $\cos\left(t\right)=\frac{\sqrt{2}}{2}$, find $\sec\left(t\right),\csc\left(t\right),\tan\left(t\right)$, and $\cot\left(t\right)$

Show Solution

$\text{sec}t=\sqrt{2},\text{csc}t=\sqrt{2},\text{tan}t=1,\text{cot}t=1$

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30, we could press

$$\text{(for a scientific calculator): } \frac{1}{30\,\frac{\pi}{180}}\ \text{COS}$$
$$\text{or}$$
$$\text{(for a graphing calculator): } \frac{1}{\cos\left(\frac{30\pi}{180}\right)}$$

How To

Given an angle measure in radians, use a scientific calculator to find the cosecant.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: $1/$
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

How To

Given an angle measure in radians, use a graphing utility/calculator to find the cosecant.

- If the graphing utility has degree mode and radian mode, set it to radian mode.
- Enter: $1/$
- Press the SIN key.
- Enter the value of the angle inside parentheses.
- Press the ENTER key.

Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Show Solution

For a scientific calculator, enter information as follows:

$$\begin{aligned} 1/\left(5\ \pi/7\right)\ \text{SIN} & = \\ \csc\left(\frac{5\pi}{7}\right) & \approx \quad 1.279 \end{aligned}$$

Try It

Evaluate the cotangent of $-\frac{\pi}{8}$.

Show Solution

≈ -2.414

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determing Trig Function Values
- More Examples of Determining Trig Functions
- Pythagorean Identities
- Trig Functions on a Calculator

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$
Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See (Figure).
- Trigonometric functions can also be found from an angle. See (Figure).
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See (Figure).
- A function is said to be even if $f\left(-x\right)=f\left(x\right)$ and odd if $f\left(-x\right)=-f\left(x\right)$ for all *x* in the domain of *f*.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See (Figure).
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See (Figure) and (Figure).
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See (Figure).
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f is the smallest interval such that $f\left(x+P\right)=f\left(x\right)$ for any value of $x.$
- The values of trigonometric functions can be found by mathematical analysis. See (Figure) and (Figure).
- To evaluate trigonometric functions of other angles, we can use a calculator or computer

software. See (Figure).

Section Exercises

Verbal

1. On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

Show Solution
Yes, when the reference angle is $\frac{\pi}{4}$ and the terminal side of the angle is in quadrants I and III. Thus, a $x = \frac{\pi}{4}, \frac{5\pi}{4}$, the sine and cosine values are equal.

2. What would you estimate the cosine of π degrees to be? Explain your reasoning.

3. For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

Show Solution
Substitute the sine of the angle in for y in the Pythagorean Theorem $x^2 + y^2 = 1$. Solve for x and take the negative solution.

4. Describe the secant function.

5. Tangent and cotangent have a period of π. What does this tell us about the output of these functions?

Show Solution

The outputs of tangent and cotangent will repeat every π units.

Algebraic

For the following exercises, find the exact value of each expression.

6. $\tan\frac{\pi}{6}$

7. $\sec\frac{\pi}{6}$

Show Solution

$\frac{2\sqrt{3}}{3}$

8. $\csc\frac{\pi}{6}$

9. $\cot\frac{\pi}{6}$

Show Solution

$\sqrt{3}$

10. $\tan\frac{\pi}{4}$

11. $\sec\frac{\pi}{4}$

Show Solution

$\sqrt{2}$

12. $\csc \frac{\pi}{4}$

13. $\cot \frac{\pi}{4}$

Show Solution

1

14. $\tan \frac{\pi}{3}$

15. $\sec \frac{\pi}{3}$

Show Solution

2

16. $\csc \frac{\pi}{3}$

17. $\cot \frac{\pi}{3}$

Show Solution

$\frac{\sqrt{3}}{3}$

For the following exercises, use reference angles to evaluate the expression.

18. $\tan \frac{5\pi}{6}$

19. $\sec \frac{7\pi}{6}$

Show Solution

$-\frac{2\sqrt{3}}{3}$

20. $\csc\frac{11\pi}{6}$

21. $\cot\frac{13\pi}{6}$

Show Solution

$\sqrt{3}$

22. $\tan\frac{7\pi}{4}$

23. $\sec\frac{3\pi}{4}$

Show Solution

$-\sqrt{2}$

24. $\csc\frac{5\pi}{4}$

25. $\cot\frac{11\pi}{4}$

Show Solution

–1

26. $\tan\frac{8\pi}{3}$

27. $\sec\frac{4\pi}{3}$

Show Solution
-2

28. $\csc \frac{2\pi}{3}$

29. $\cot \frac{5\pi}{3}$

Show Solution
$-\frac{\sqrt{3}}{3}$

30. $\tan 225°$

31. $\sec 300°$

Show Solution
2

32. $\csc 150°$

33. $\cot 240°$

Show Solution
$\frac{\sqrt{3}}{3}$

34. $\tan 330°$

35. $\sec 120°$

Show Solution

–2

36. $\csc 210°$

37. $\cot 315°$

Show Solution

–1

38. If $\sin t = \frac{3}{4}$, and t is in quadrant II, find $\cos t$, $\sec t$, $\csc t$, $\tan t$, and $\cot t$.

39. If $\cos t = -\frac{1}{3}$, and t is in quadrant III, find $\sin t$, $\sec t$, $\csc t$, $\tan t$, and $\cot t$.

Show Solution

$\sin t = -\frac{2\sqrt{2}}{3}, \sec t = -3, \csc t = -\frac{3\sqrt{2}}{4}, \tan t = 2\sqrt{2}, \cot t = \frac{\sqrt{2}}{4}$

40. If $\tan t = \frac{12}{5}$, and $0 \leq t < \frac{\pi}{2}$, find $\sin t$, $\cos t$, $\sec t$, $\csc t$, $\text{and} \cot t$.

41. If $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$, find $\sec t$, $\csc t$, $\tan t$, and $\cot t$.

Show Solution

$\text{sec}t = 2, \text{csc}t = \frac{2\sqrt{3}}{3}, \text{tan}t = \sqrt{3}, \text{cot}t = \frac{\sqrt{3}}{3}$

42. If $\sin 40° \approx 0.643$ and $\cos 40° \approx 0.766$, find $\sec 40°$, $\csc 40°$, $\tan 40°$, and $\cot 40°$.

43. If $\sin t = \frac{\sqrt{2}}{2}$, what is the $\sin(-t)$?

Show Solution

$-\frac{\sqrt{2}}{2}$

44. If $\cos t = \frac{1}{2}$, what is the $\cos(-t)$?

45. If $\sec t = 3.1$, what is the $\sec(-t)$?

Show Solution

3.1

46. If $\csc t = 0.34$, what is the $\csc(-t)$?

47. If $\tan t = -1.4$, what is the $\tan(-t)$?

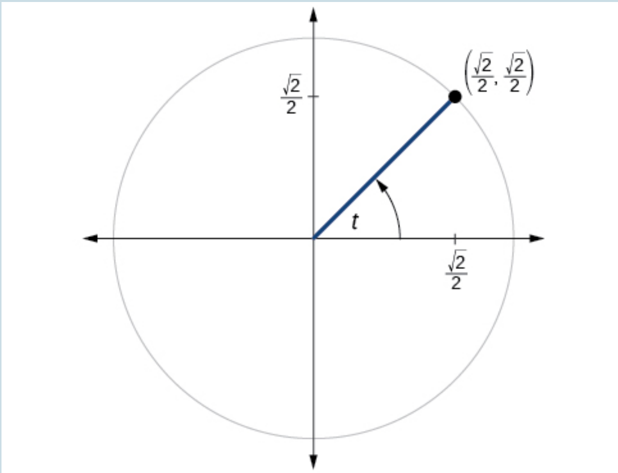
Show Solution

1.4

48. If $\cot t = 9.23$, what is the $\cot(-t)$?

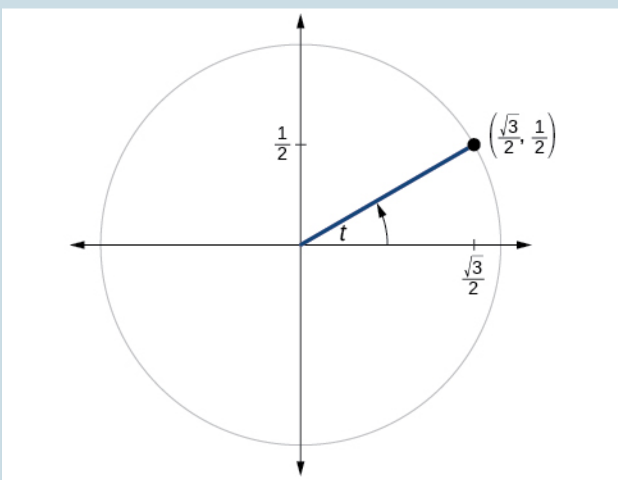
Graphical

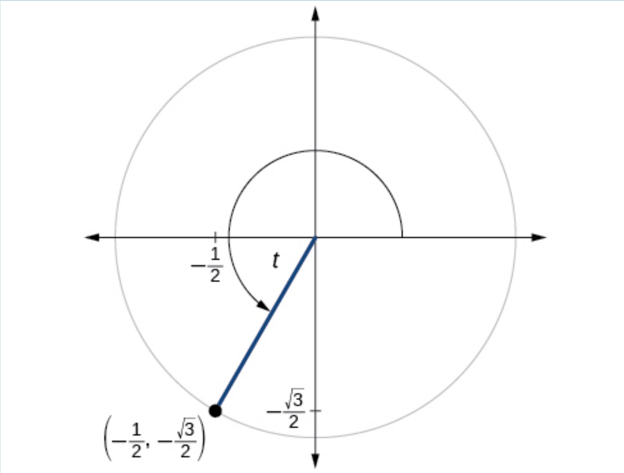
For the following exercises, use the angle in the unit circle to find the value of the each of the six trigonometric functions.

49. 

Show Solution

$\sin t = \frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = 1, \cot t = 1, \sec t = \sqrt{2}, \csc t = \sqrt{2}$

50. 

51. 

Show Solution

$\sin t = -\frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}, \tan t = \sqrt{3}, \cot t = \frac{\sqrt{3}}{3}, \sec t = -2, \csc t = -\frac{2\sqrt{3}}{3}$

Technology

For the following exercises, use a graphing calculator to evaluate to three decimal places.

52. $\csc \frac{5\pi}{9}$

53. $\cot \frac{4\pi}{7}$

Show Solution

–0.228

54. $\sec \frac{\pi}{10}$

55. $\tan \frac{5\pi}{8}$

Show Solution

–2.414

56. $\sec \frac{3\pi}{4}$

57. $\csc \frac{\pi}{4}$

Show Solution
1.414

58. $\tan 98°$

59. $\cot 33°$

Show Solution
1.540

60. $\cot 140^{/circ}$

61. $\sec 310^{/circ}$

Show Solution
1.556

Extensions

For the following exercises, use identities to evaluate the expression.

62. If $\tan(t) \approx 2.7$, and $\sin(t) \approx 0.94$, find $\cos(t)$.

63. If $\tan(t) \approx 1.3$, and $\cos(t) \approx 0.61$, find $\sin(t)$.

Show Solution
$\sin(t) \approx 0.79$

64. If $\csc(t) \approx 3.2$, and $\cos(t) \approx 0.95$, find $\tan(t)$.

65. If $\cot(t) \approx 0.58$, and $\cos(t) \approx 0.5$, find $\csc(t)$.

Show Solution
$\csc t \approx 1.16$

67. Determine whether the function $f(x) = 2\sin x \cos x$ is even, odd, or neither.

68. Determine whether the function $f(x) = 3\sin^2 x \cos x + \sec x$ is even, odd, or neither.

Show Solution
even

69. Determine whether the function $f(x) = \sin x - 2\cos^2 x$ is even, odd, or neither.

70. Determine whether the function $f(x) = \csc^2 x + \sec x$ is even, odd, or neither.

Show Solution
even

For the following exercises, use identities to simplify the expression.

71. $\csc t \tan t$

72. $\frac{\sec t}{\csc t}$

Show Solution
$\frac{\sin t}{\cos t} = \tan t$

Real-World Applications

73. The amount of sunlight in a certain city can be modeled by the function $h = 15\cos\left(\frac{1}{600}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on February 10, the 42nd day of the year. State the period of the function.

74. The amount of sunlight in a certain city can be modeled by the function $h = 16\cos\left(\frac{1}{500}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on September 24, the 267th day of the year. State the period of the function.

Show Solution
13.77 hours, period: 1000π

75. The equation $P = 20\sin(2\pi t) + 100$ models the blood pressure, P, where t represents time in seconds. (a) Find the blood pressure after 15 seconds. (b) What are the maximum and minimum blood pressures?

76. The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 6$, where x represents the crank angle. Find the height of the piston when the crank angle is $55°$.

Show Solution

7.73 inches

77. The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 5$, where x represents the crank angle. Find the height of the piston when the crank angle is $55°$.

Chapter Review Exercises

Angles

For the following exercises, convert the angle measures to degrees.

1. $\frac{\pi}{4}$

Show Solution
$45°$

2. $-\frac{5\pi}{3}$

For the following exercises, convert the angle measures to radians.

3. $-210°$

Show Solution
$-\frac{7\pi}{6}$

4. $180°$

5. Find the length of an arc in a circle of radius 7 meters subtended by the central angle of $85^{/circ}$.

Show Solution
10.385 meters

6. Find the area of the sector of a circle with diameter 32 feet and an angle of $\frac{3\pi}{5}$ radians.

For the following exercises, find the angle between $0°$ and $360°$ that is coterminal with the given angle.

7. $420°$

Show Solution
$60°$

8. $-80^{/circ}$

For the following exercises, find the angle between 0 and 2π in radians that is coterminal with the given angle.

9. $-\frac{20\pi}{11}$

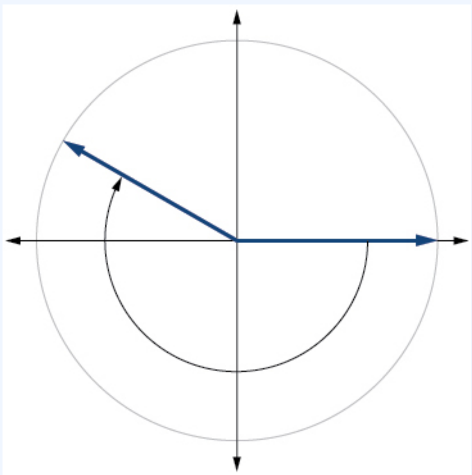
Show Solution
$\frac{2\pi}{11}$

10. $\frac{14\pi}{5}$

For the following exercises, draw the angle provided in standard position on the Cartesian plane.

11. $-210°$

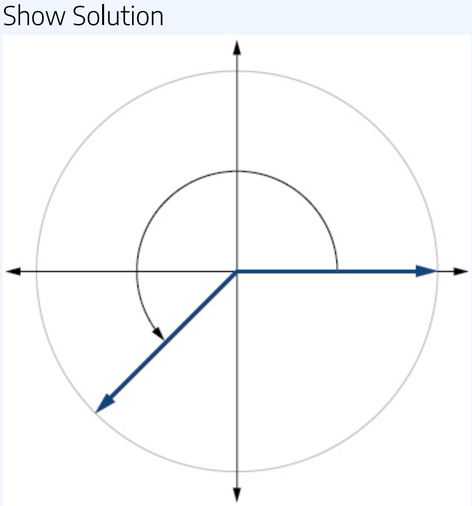
Show Solution



12. $75°$

13. $\frac{5\pi}{4}$

Show Solution



14. $-\frac{\pi}{3}$

15. Find the linear speed of a point on the equator of the earth if the earth has a radius of 3,960 miles and the earth rotates on its axis every 24 hours. Express answer in miles per hour. Round to the nearest hundredth.

Show Solution
1036.73 miles per hour

16. A car wheel with a diameter of 18 inches spins at the rate of 10 revolutions per second. What is the car's speed in miles per hour? Round to the nearest hundredth.

Right Triangle Trigonometry

For the following exercises, use side lengths to evaluate.

17. $\cos\frac{\pi}{4}$

Show Solution
$\frac{\sqrt{2}}{2}$

18. $\cot\frac{\pi}{3}$

19. $\tan\frac{\pi}{6}$

Show Solution
$\frac{\sqrt{3}}{3}$

18. $\cos\left(\frac{\pi}{2}\right) = \sin\left(\_\_\_^{/circ}\right)$

19. $\csc\left(18^{/circ}\right) = \sec\left(\_\_\_^{/circ}\right)$

Show Solution
$72°$

For the following exercises, use the given information to find the lengths of the other two sides of the right triangle.

20. $\cos B = \frac{3}{5}, a = 6$

21. $\tan A = \frac{5}{9}, b = 6$

Show Solution
$a = \frac{10}{3}, c = \frac{2\sqrt{106}}{3}$

For the following exercises, use (Figure) to evaluate each trigonometric function.

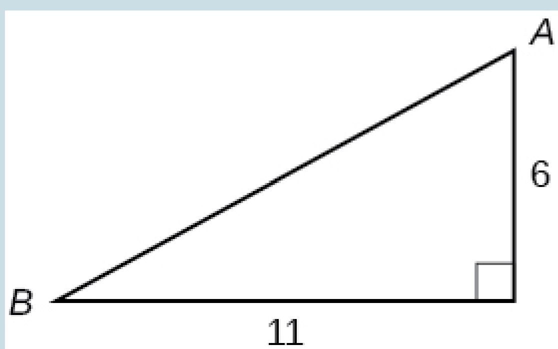


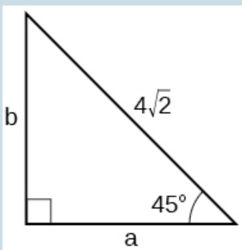
Figure 11.

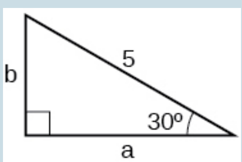
22. $\sin\ A$

23. $\tan B$

Show Solution
$\frac{6}{11}$

For the following exercises, solve for the unknown sides of the given triangle.

24. 

25. 

Show Solution
$a = \frac{5\sqrt{3}}{2}, b = \frac{5}{2}$

26. A 15-ft ladder leans against a building so that the angle between the ground and the ladder is $70°$. How high does the ladder reach up the side of the building? Find the answer to four decimal places.

27. The angle of elevation to the top of a building in Baltimore is found to be 4 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building. Find the answer to four decimal places.

Show Solution
369.2136 ft

Unit Circle

28. Find the exact value of $\sin\frac{\pi}{3}$.

29. Find the exact value of $\cos\frac{\pi}{4}$.

Show Solution
$\frac{\sqrt{2}}{2}$

30. Find the exact value of $\cos\pi$.

31. State the reference angle for $300°$.

Show Solution
$60°$

32. State the reference angle for $\frac{3\pi}{4}$.

33. Compute cosine of $330°$.

Show Solution
$\frac{\sqrt{3}}{2}$

34. Compute sine of $\frac{5\pi}{4}$.

35. State the domain of the sine and cosine functions.

Show Solution

all real numbers

36. State the range of the sine and cosine functions.

The Other Trigonometric Functions

For the following exercises, find the exact value of the given expression.

37. $\cos\frac{\pi}{6}$

Show Solution
$\frac{\sqrt{3}}{2}$

38. $\tan\frac{\pi}{4}$

39. $\csc\frac{\pi}{3}$

Show Solution
$\frac{2\sqrt{3}}{3}$

40. $\sec\frac{\pi}{4}$

For the following exercises, use reference angles to evaluate the given expression.

41. $\sec\frac{11\pi}{3}$

Show Solution

2

42. $\sec 315°$

43. If $\sec(t) = -2.5$, what is the $\sec(-t)$?

Show Solution

–2.5

44. If $\tan(t) = -0.6$, what is the $\tan(-t)$?

45. If $\tan(t) = \frac{1}{3}$, find $\tan(t - \pi)$.

Show Solution

$\frac{1}{3}$

46. If $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sin(t + 2\pi)$.

47. Which trigonometric functions are even?

Show Solution

cosine, secant

48. Which trigonometric functions are odd?

Glossary

cosecant
: the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cotangent
: the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities
: statements that are true for all values of the input on which they are defined

period
: the smallest interval P of a repeating function f such that $f\left(x+P\right)=f\left(x\right)$

secant
: the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

tangent
: the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

CHAPTER 4: FURTHER APPLICATIONS OF TRIGONOMETRY

CHAPTER 4.1: INTRODUCTION TO FURTHER APPLICATIONS OF TRIGONOMETRY

Figure 1. General Sherman, the world's largest living tree. (credit: Mike Baird, Flickr)

The world's largest tree by volume, named General Sherman, stands 274.9 feet tall and resides in Northern California.[1] Just how do scientists know its true height? A common way to measure the height involves determining the angle of elevation, which is formed by the tree and the ground at a point some distance away from the base of the tree. This method is much more practical than climbing the tree and dropping a very long tape measure.

1. Source: National Park Service. "The General Sherman Tree." http://www.nps.gov/seki/naturescience/sherman.htm. Accessed April 25, 2014.

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

CHAPTER 4.2: NON-RIGHT TRIANGLES: LAW OF SINES

Learning Objectives

In this section, you will:

- Use the Law of Sines to solve oblique triangles.
- Find the area of an oblique triangle using the sine function.
- Solve applied problems using the Law of Sines.

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the angle of elevation measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in (Figure) that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving non-right triangles.

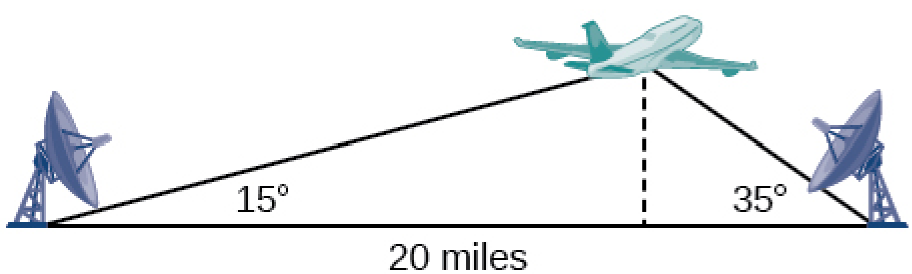


Figure 1.

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an altitude, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an oblique triangle. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

1. **ASA (angle-side-angle)** We know the measurements of two angles and the included side. See (Figure).

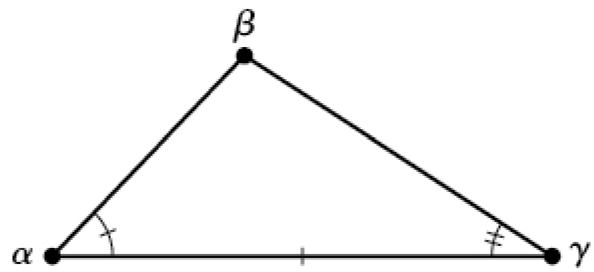


Figure 2.

2. **AAS (angle-angle-side)** We know the measurements of two angles and a side that is not between the known angles. See (Figure).

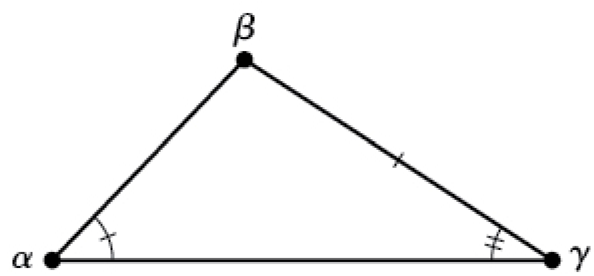


Figure 3.

3. **SSA (side-side-angle)** We know the measurements of two sides and an angle that is not between the known sides. See (Figure).

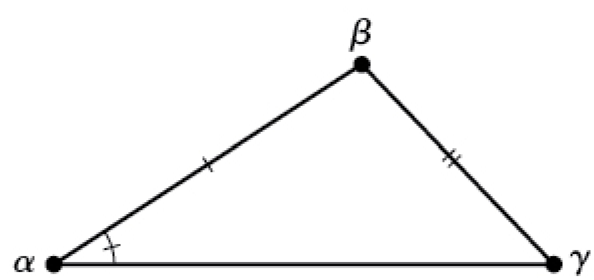


Figure 4.

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in (Figure).

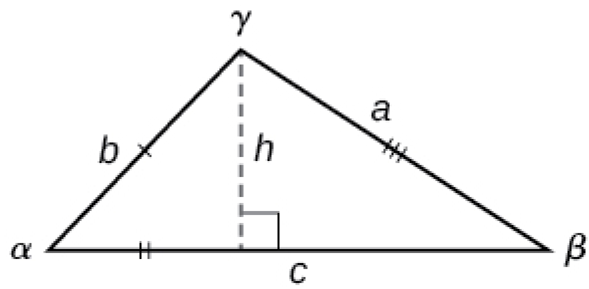


Figure 5.

Using the right triangle relationships, we know that $\sin\alpha = \frac{h}{b}$ and $\sin\beta = \frac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin\alpha \text{ and } h = a\sin\beta$$

We then set the expressions equal to each other.

$$b\sin\alpha = a\sin\beta$$
$$\left(\frac{1}{ab}\right)(b\sin\alpha) = (a\sin\beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab}.$$
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \text{ and } \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Collectively, these relationships are called the **Law of Sines**.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\lambda}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See (Figure).

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

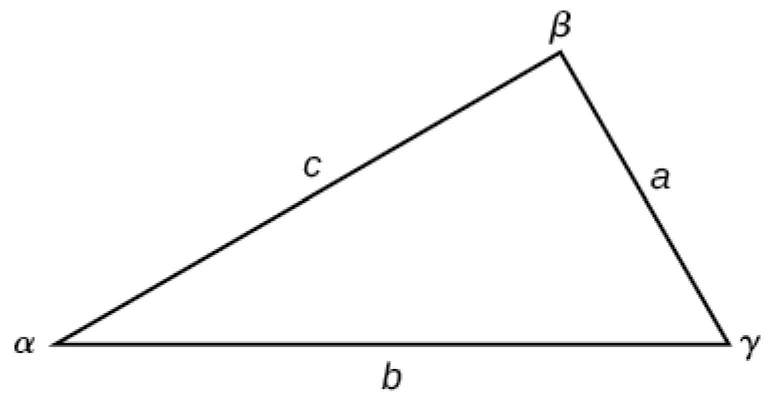


Figure 6.

Law of Sines

Given a triangle with angles and opposite sides labeled as in (Figure), the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The Law of Sines is based on proportions and is presented symbolically two ways.

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$
$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

To solve an oblique triangle, use any pair of applicable ratios.

Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in (Figure) to the nearest tenth.

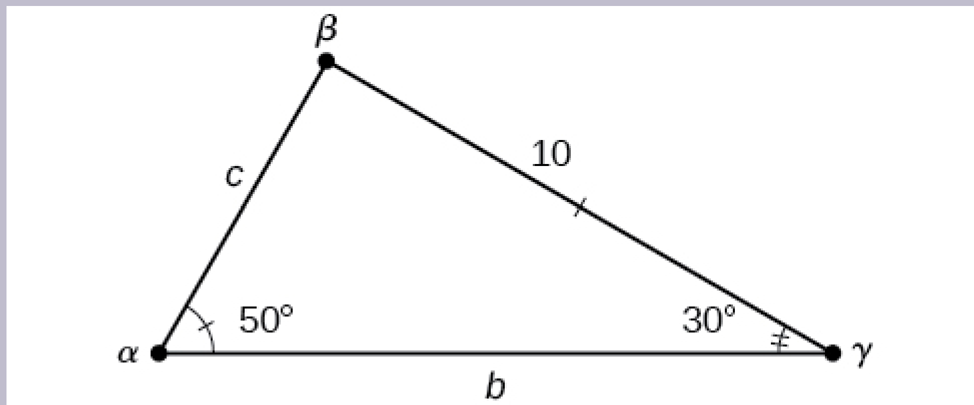


Figure 7.

Show Solution

The three angles must add up to 180 degrees. From this, we can determine that

$$\beta = 180° - 50° - 30°$$
$$= 100°$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50°$ and its corresponding side $a = 10.$ We can use the following proportion from the Law of Sines to find the length of $c.$

$$\frac{\sin(50°)}{10} = \frac{\sin(30°)}{c}$$
$$c\frac{\sin(50°)}{10} = \sin(30°) \qquad \text{Multiply both sides by } c.$$
$$c = \sin(30°)\frac{10}{\sin(50°)} \qquad \text{Multiply by the reciprocal to isolate } c.$$
$$c \approx 6.5$$

Similarly, to solve for $b,$ we set up another proportion.

$$\frac{\sin(50°)}{10} = \frac{\sin(100°)}{b}$$
$$b\sin(50°) = 10\sin(100°) \qquad \text{Multiply both sides by } b.$$
$$b = \frac{10\sin(100°)}{\sin(50°)} \qquad \text{Multiply by the reciprocal to isolate } b.$$
$$b \approx 12.9$$

Therefore, the complete set of angles and sides is

$$\begin{array}{ll} \alpha = 50° & a = 10 \\ \beta = 100° & b \approx 12.9 \\ \gamma = 30° & c \approx 6.5 \end{array}$$

Try It

Solve the triangle shown in (Figure) to the nearest tenth.

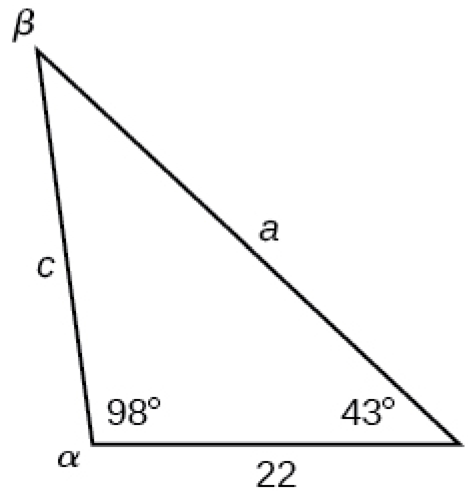


Figure 8.

Show Solution

$$\begin{array}{ll} \alpha = 98^\circ & a = 34.6 \\ \beta = 39^\circ & b = 22 \\ \gamma = 43^\circ & c = 23.8 \end{array}$$

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ambiguous case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

Possible Outcomes for SSA Triangles

Oblique triangles in the category SSA may have four different outcomes. (Figure) illustrates the solutions with the known sides a and b and known angle α.

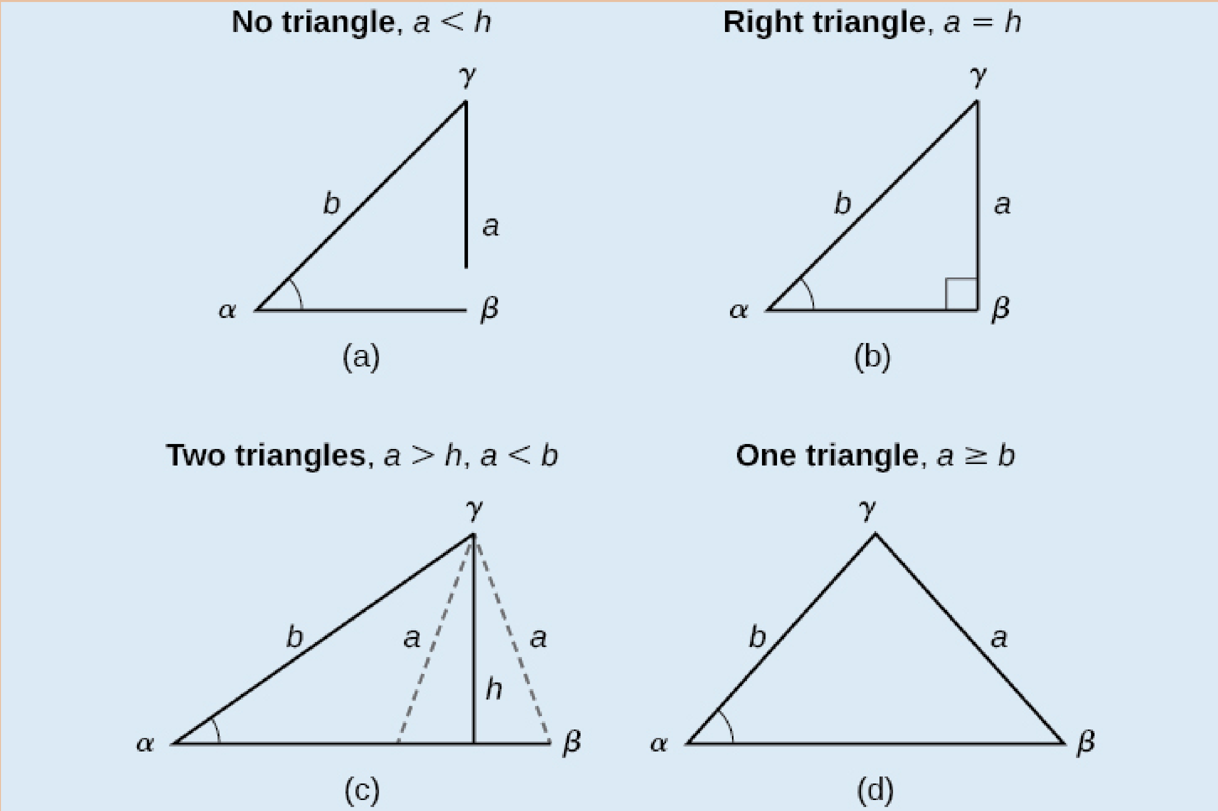


Figure 9.

Solving an Oblique SSA Triangle

Solve the triangle in (Figure) for the missing side and find the missing angle measures to the nearest tenth.

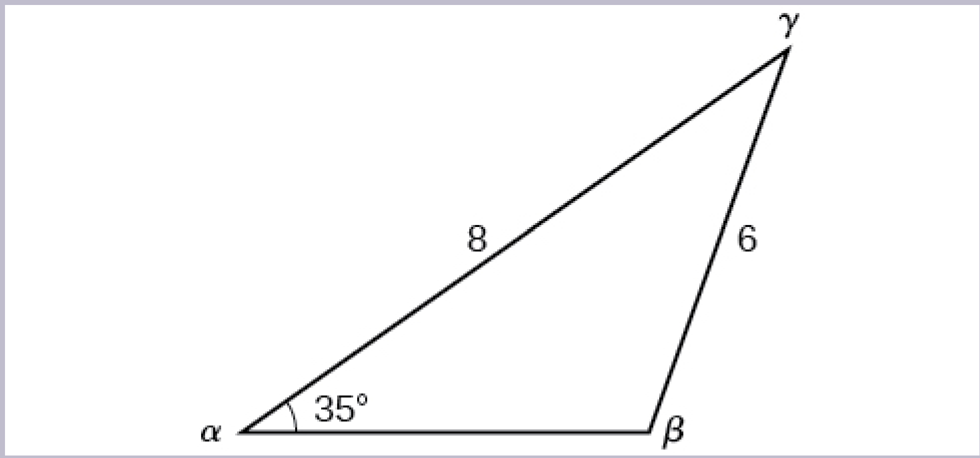


Figure 10.

Show Solution

Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\begin{aligned} \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\ \frac{\sin(35^\circ)}{6} &= \frac{\sin\beta}{8} \\ \frac{8\sin(35^\circ)}{6} &= \sin\beta \\ 0.7648 &\approx \sin\beta \\ \sin^{-1}(0.7648) &\approx 49.9^\circ \\ \beta &\approx 49.9^\circ \end{aligned}$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have (Figure). It appears that there may be a second triangle that will fit the given criteria.

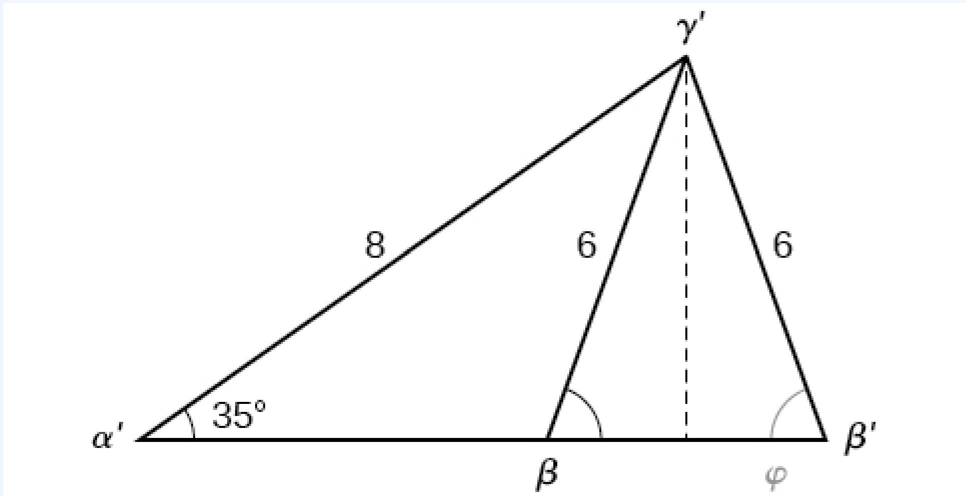


Figure 11.

The angle supplementary to β is approximately equal to $49.9°$, which means that $\beta = 180° - 49.9° = 130.1°$. (Remember that the sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\gamma = 180° - 35° - 130.1° \approx 14.9°$$

We can then use these measurements to solve the other triangle. Since γ' is supplementary to the sum of α' and β', we have

$$\gamma' = 180° - 35° - 49.9° \approx 95.1°$$

Now we need to find c and c'.

We have

$$\frac{c}{\sin(14.9°)} = \frac{6}{\sin(35°)}$$
$$c = \frac{6\sin(14.9°)}{\sin(35°)} \approx 2.7$$

Finally,

$$\frac{c'}{\sin(95.1°)} = \frac{6}{\sin(35°)}$$
$$c' = \frac{6\sin(95.1°)}{\sin(35°)} \approx 10.4$$

To summarize, there are two triangles with an angle of $35°$, an adjacent side of 8, and an opposite side of 6, as shown in (Figure).

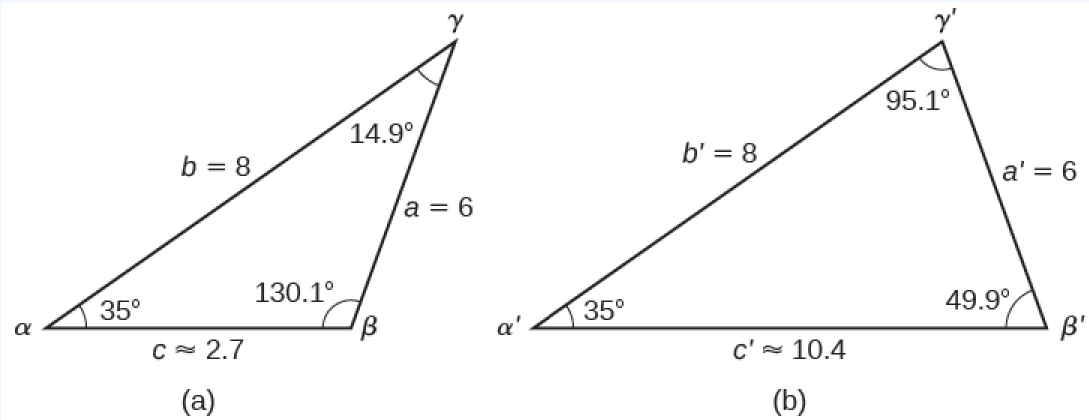


Figure 12.

However, we were looking for the values for the triangle with an obtuse angle $\beta.$ We can see them in the first triangle (a) in (Figure).

Try It

Given $\alpha = 80°, a = 120,$ and $b = 121,$ find the missing side and angles. If there is more than one possible solution, show both.

Show Solution

Solution 1

$$\begin{array}{ll} \alpha = 80° & a = 120 \\ \beta \approx 83.2° & b = 121 \\ \gamma \approx 16.8° & c \approx 35.2 \end{array}$$

Solution 2

$$\begin{array}{ll} \alpha' = 80^\circ & a' = 120 \\ \beta' \approx 96.8^\circ & b' = 121 \\ \gamma' \approx 3.2^\circ & c' \approx 6.8 \end{array}$$

Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in (Figure), solve for the unknown side and angles. Round your answers to the nearest tenth.

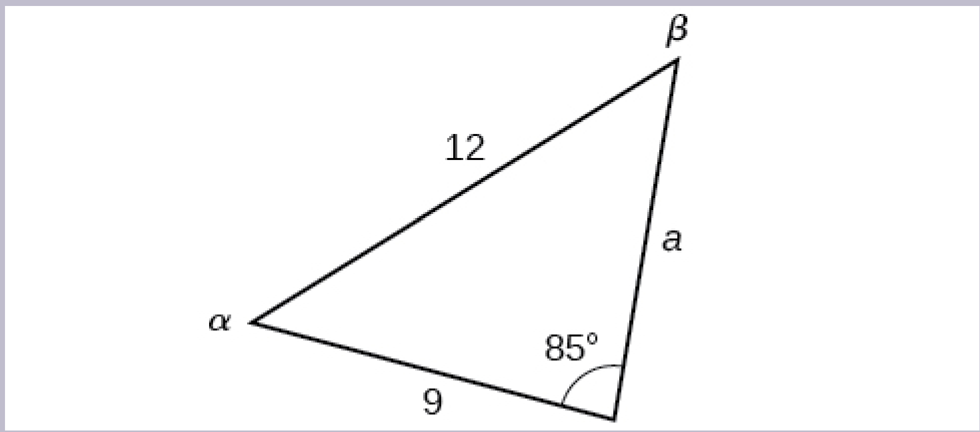


Figure 13.

Show Solution

In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle $\gamma = 85^\circ$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\begin{array}{ll} \frac{\sin(85^\circ)}{12} = \frac{\sin\beta}{9} & \text{Isolate the unknown.} \\ \frac{9\sin(85^\circ)}{12} = \sin\beta & \end{array}$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85°)}{12}\right)$$
$$\beta \approx \sin^{-1}(0.7471)$$
$$\beta \approx 48.3°$$

In this case, if we subtract β from 180^{\circ}, we find that there may be a second possible solution. Thus, $\beta = 180° - 48.3° \approx 131.7°$. To check the solution, subtract both angles, 131.7^{\circ} and 85^{\circ}, from 180^{\circ}. This gives

$$\alpha = 180° - 85° - 131.7° \approx -36.7°,$$

which is impossible, and so $\beta \approx 48.3°$.

To find the remaining missing values, we calculate $\alpha = 180° - 85° - 48.3° \approx 46.7°$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85°)}{12} = \frac{\sin(46.7°)}{a}$$
$$a\frac{\sin(85°)}{12} = \sin(46.7°)$$
$$a = \frac{12\sin(46.7°)}{\sin(85°)} \approx 8.8$$

The complete set of solutions for the given triangle is

$$\alpha \approx 46.7° \ a \approx 8.8$$
$$\beta \approx 48.3° \ b = 9$$
$$\gamma = 85° \ c = 12$$

Try It

Given $\alpha = 80°, a = 100,\ b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Show Solution

$\beta \approx 5.7°, \gamma \approx 94.3°, c \approx 101.3$

Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Show Solution

Using the given information, we can solve for the angle opposite the side of length 10. See (Figure).

$$\frac{\sin\alpha}{10} = \frac{\sin(50°)}{4}$$
$$\sin\alpha = \frac{10\sin(50°)}{4}$$
$$\sin\alpha \approx 1.915$$

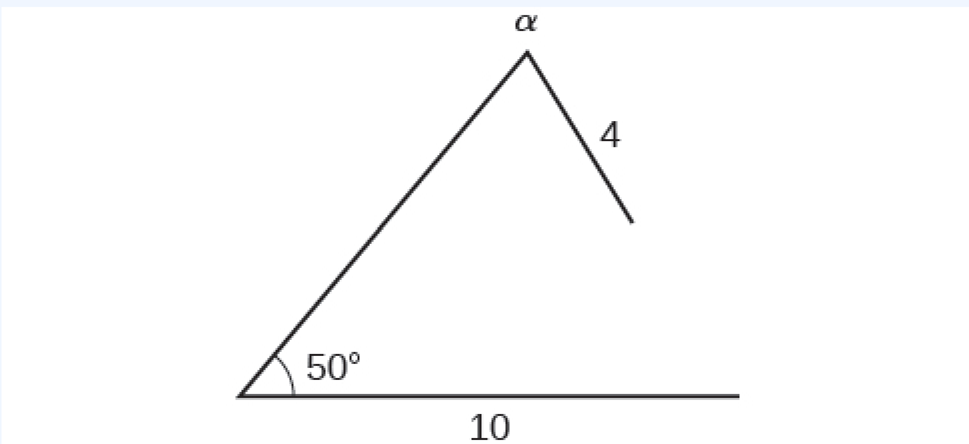


Figure 14.

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$ in a graphing calculator generates an ERROR DOMAIN. Therefore, no triangles can be drawn with the provided dimensions.

Try It

Determine the number of triangles possible given $a = 31,\ b = 26,\ \beta = 48°$.

Show Solution
two

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as $\text{Area} = \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in (Figure), one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin\alpha = \frac{h}{c}$ or $c\sin\alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.

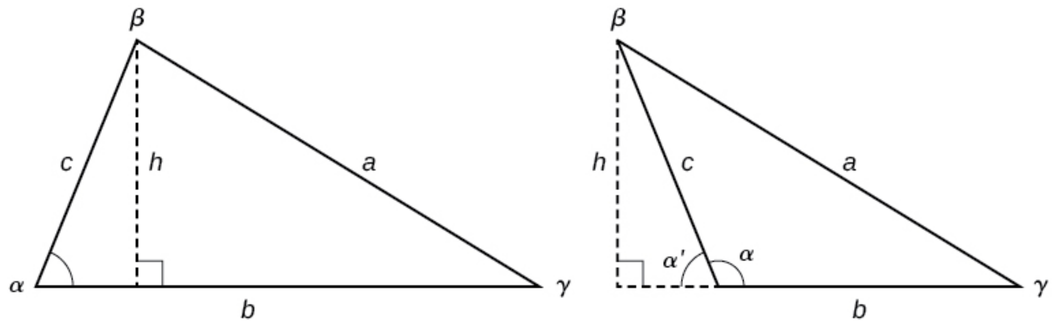


Figure 15.

Thus,

$\text{Area} = \frac{1}{2}\,(\text{base})\,(\text{height}) = \frac{1}{2}b\,(c\sin\alpha)$

Similarly,

$\text{Area} = \frac{1}{2}a\,(b\sin\gamma) = \frac{1}{2}a\,(c\sin\beta)$

Area of an Oblique Triangle

The formula for the area of an oblique triangle is given by

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}bc\sin\alpha \\ &= \tfrac{1}{2}ac\sin\beta \\ &= \tfrac{1}{2}ab\sin\gamma\end{aligned}$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90, b = 52,$ and angle $\gamma = 102°$. Round the area to the nearest integer.

Show Solution

Using the formula, we have

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}ab\sin\gamma\\ \text{Area} &= \tfrac{1}{2}(90)(52)\sin(102^\circ)\\ \text{Area} &\approx 2289\ \text{square units}\end{aligned}$$

Try It

Find the area of the triangle given $\beta = 42^\circ,\ a = 7.2\,\text{ft},\ c = 3.4\,\text{ft}$. Round the area to the nearest tenth.

Show Solution

about $8.2\ \text{square feet}$

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in (Figure). Round the altitude to the nearest tenth of a mile.

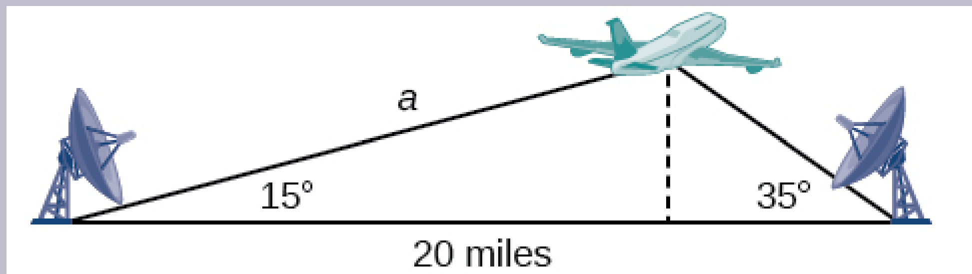


Figure 16.

Show Solution

To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side a, and then use right triangle relationships to find the height of the aircraft, h.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180°15°35° = 130°$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\frac{\sin(130°)}{20} = \frac{\sin(35°)}{a}$$
$$a\sin(130°) = 20\sin(35°)$$
$$a = \frac{20\sin(35°)}{\sin(130°)}$$
$$a \approx 14.98$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know a, we can use right triangle relationships to solve for h.

$$\sin(15°) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin(15°) = \frac{h}{a}$$
$$\sin(15°) = \frac{h}{14.98}$$
$$h = 14.98\sin(15°)$$
$$h \approx 3.88$$

The aircraft is at an altitude of approximately 3.9 miles.

The diagram shown in (Figure) represents the height of a blimp flying over a football stadium.

Find the height of the blimp if the angle of elevation at the southern end zone, point A, is $70°$, the angle of elevation from the northern end zone, point $B,$ is $62°$, and the distance between the viewing points of the two end zones is 145 yards.

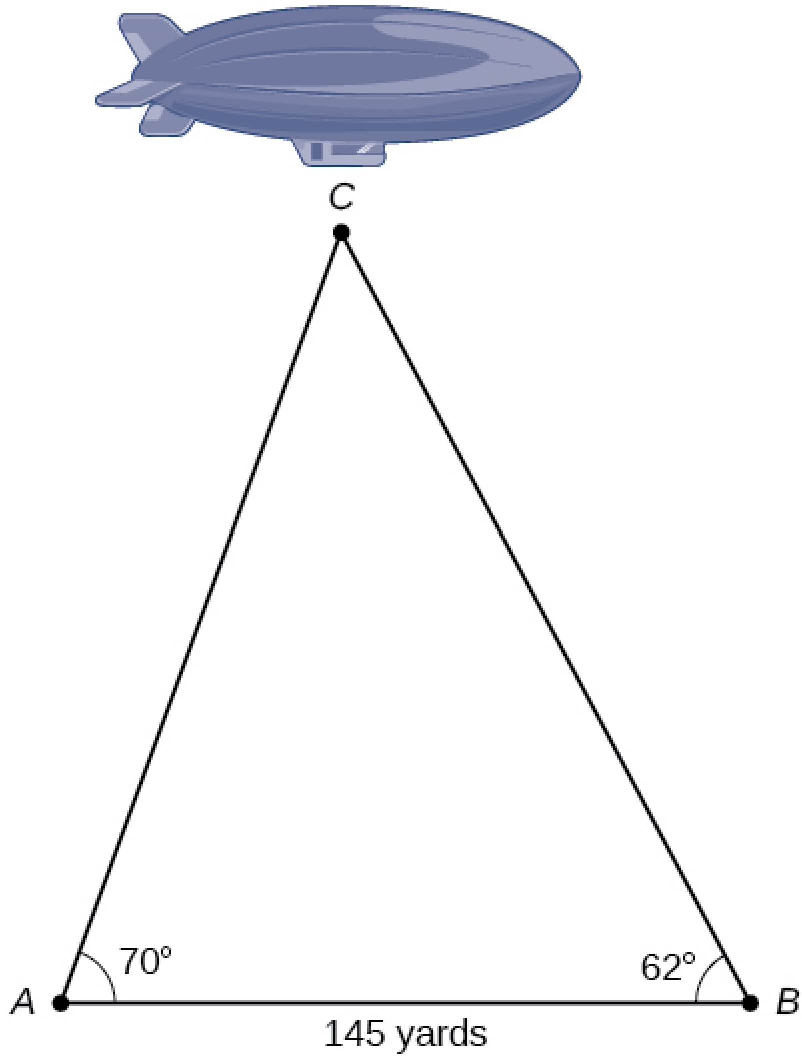


Figure 17.

Show Solution
161.9 yd.

Access these online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics
- Law of Sines: The Ambiguous Case

Key Equations

Law of Sines	$\frac{\sin \alpha}{a}=\frac{\sin \beta}{b}=\frac{\sin \gamma}{c}$ $\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}=\frac{c}{\sin \gamma}$
Area for oblique triangles	$\text{Area}=\frac{1}{2}bc\sin \alpha$ $=\frac{1}{2}ac\sin \beta$ $=\frac{1}{2}ab\sin \gamma$

Key Concepts

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See (Figure).
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See (Figure) and (Figure).
- The Law of Sines can be used to solve triangles with given criteria. See (Figure).
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See (Figure).
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See (Figure).

Section Exercises

Verbal

1. Describe the altitude of a triangle.

Show Solution
The altitude extends from any vertex to the opposite side or to the line containing the opposite side at a 90° angle.

2. Compare right triangles and oblique triangles.

3. When can you use the Law of Sines to find a missing angle?

Show Solution
When the known values are the side opposite the missing angle and another side and its opposite angle.

4. In the Law of Sines, what is the relationship between the angle in the numerator and the side in the denominator?

5. What type of triangle results in an ambiguous case?

Show Solution
A triangle with two given sides and a non-included angle.

Algebraic

For the following exercises, assume α is opposite side $a,$ β is opposite side $b,$ and γ is opposite side $c.$ Solve each triangle, if possible. Round each answer to the nearest tenth.

6. $\alpha = 43°, \gamma = 69°, a = 20$

7. $\alpha = 35°, \gamma = 73°, c = 20$

Show Solution
$\beta = 72°, a \approx 12.0, b \approx 19.9$

8. $\alpha = 60°,\ \beta = 60°,\ \gamma = 60°$

9. $a = 4,\ \ \alpha = \ 60°,\ \beta = 100°$

Show Solution
$\gamma = 20°, b \approx 4.5, c \approx 1.6$

10. $b = 10,\ \beta = 95°, \gamma = \ 30°$

For the following exercises, use the Law of Sines to solve for the missing side for each oblique triangle. Round each answer to the nearest hundredth. Assume that angle A is opposite side a, angle B is opposite side b, and angle C is opposite side c.

11. Find side b when $A = 37,\ \ B = 49,\ c = 5.$

Show Solution
$b \approx 3.78$

12. Find side a when $A = 132°, C = 23°, b = 10.$

13. Find side c when $B = 37°, C = 21°,\ b = 23.$

Show Solution
$c \approx 13.70$

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Determine whether there is no triangle, one triangle, or two triangles. Then solve each triangle, if possible. Round each answer to the nearest tenth.

14. $\alpha = 119°, a = 14, b = 26$

15. $\gamma = 113°, b = 10, c = 32$

Show Solution
one triangle, $\alpha \approx 50.3°, \beta \approx 16.7°, a \approx 26.7$

16. $b = 3.5,\ \ c = 5.3,\ \ \gamma =\ 80°$

17. $a = 12,\ \ c = 17,\ \ \alpha =\ 35°$

Show Solution
two triangles, $\gamma \approx 54.3°, \beta \approx 90.7°, b \approx 20.9$ or
$\gamma' \approx 125.7°, \beta' \approx 19.3°, b' \approx 6.9$

18. $a = 20.5,\ \ b = 35.0,\ \ \beta = 25°$

19. $a = 7,\ c = 9,\ \ \alpha =\ 43°$

Show Solution

two triangles, $\beta \approx 75.7°, \gamma \approx 61.3°, b \approx 9.9$ or
$\beta' \approx 18.3°, \gamma' \approx 118.7°, b' \approx 3.2$

20. $a = 7, b = 3, \beta = 24°$

21. $b = 13, c = 5, \gamma = 10°$

Show Solution
two triangles, $\alpha \approx 143.2°, \beta \approx 26.8°, a \approx 17.3$ or
$\alpha' \approx 16.8°, \beta' \approx 153.2°, a' \approx 8.3$

22. $a = 2.3, c = 1.8, \gamma = 28°$

23. $\beta = 119°, b = 8.2, a = 11.3$

Show Solution
no triangle possible

For the following exercises, use the Law of Sines to solve, if possible, the missing side or angle for each triangle or triangles in the ambiguous case. Round each answer to the nearest tenth.

24. Find angle A when $a = 24, b = 5, B = 22°.$

25. Find angle A when $a = 13, b = 6, B = 20°.$

Show Solution
$A \approx 47.8°$ or $A' \approx 132.2°$

26. Find angle B when $A = 12°, a = 2, b = 9.$

For the following exercises, find the area of the triangle with the given measurements. Round each answer to the nearest tenth.

27. $a = 5, c = 6, \beta = \ 35°$

Show Solution
8.6

28. $b = 11, c = 8, \alpha = 28°$

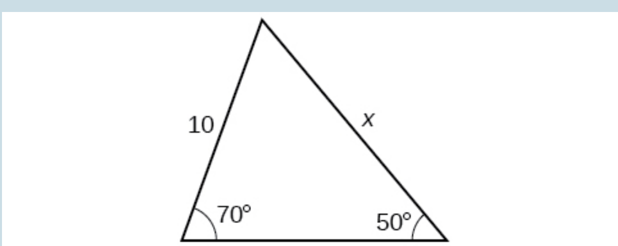
29. $a = 32, b = 24, \gamma = 75°$

Show Solution
370.9

30. $a = 7.2, b = 4.5, \gamma = 43°$

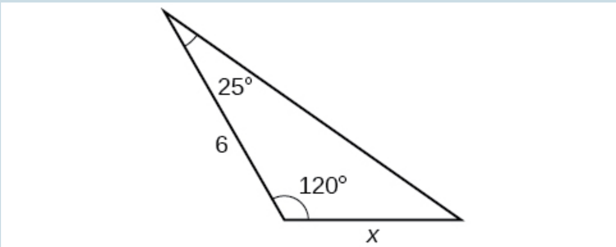
Graphical

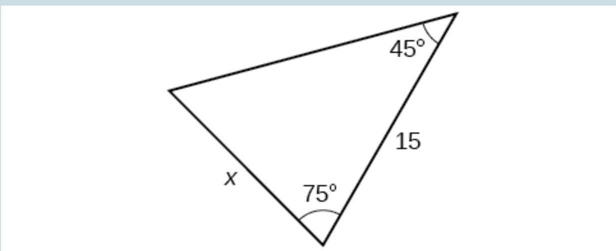
For the following exercises, find the length of side x. Round to the nearest tenth.

31. 

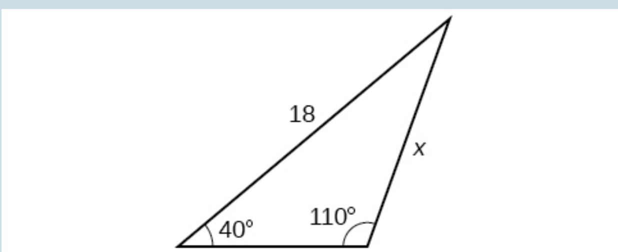
Show Solution

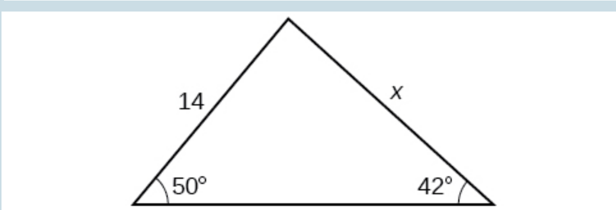
12.3

32. 

33. 

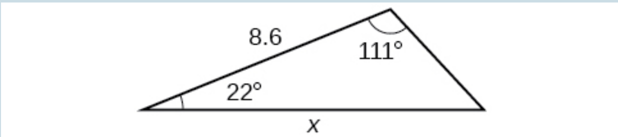
Show Solution
12.2

34. 

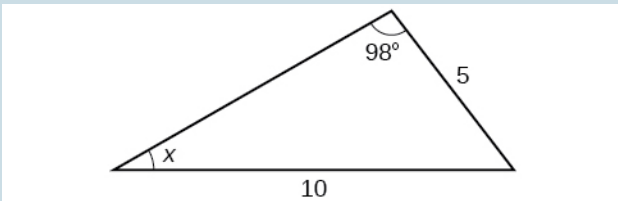
35. 

Show Solution

16.0

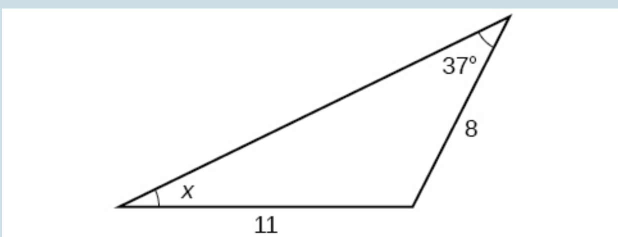
36. 

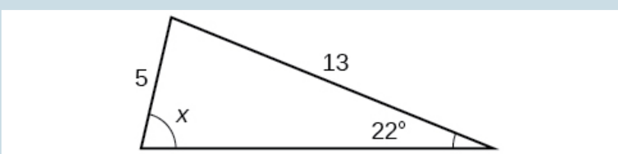
For the following exercises, find the measure of angle x, if possible. Round to the nearest tenth.

37. 

Show Solution

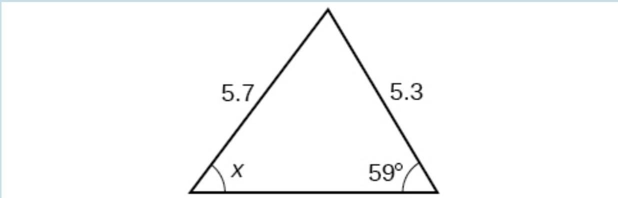
$29.7°$

38. 

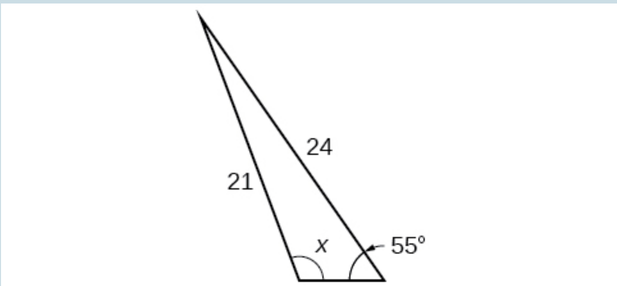
39. 

Show Solution

$x = 76.9°$or $x = 103.1°$

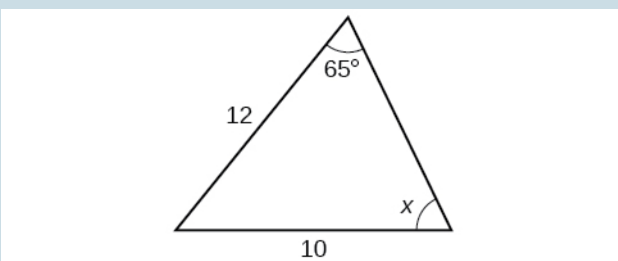
40. 

Notice that x is an obtuse angle.

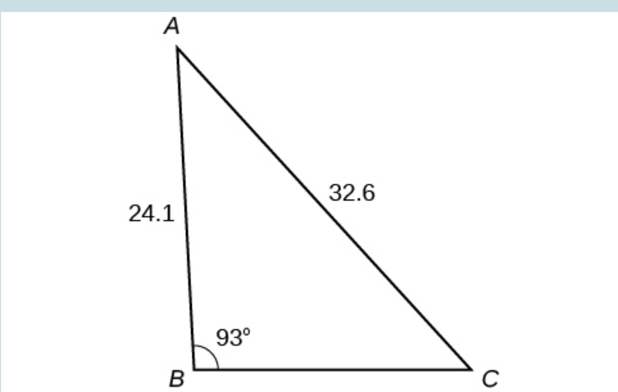
41. 

Show Solution

110.6

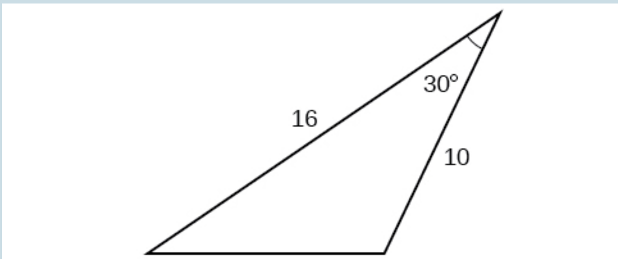
42. 

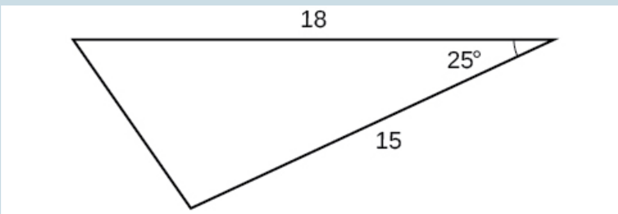
For the following exercises, find the area of each triangle. Round each answer to the nearest tenth.

43. 

Show Solution

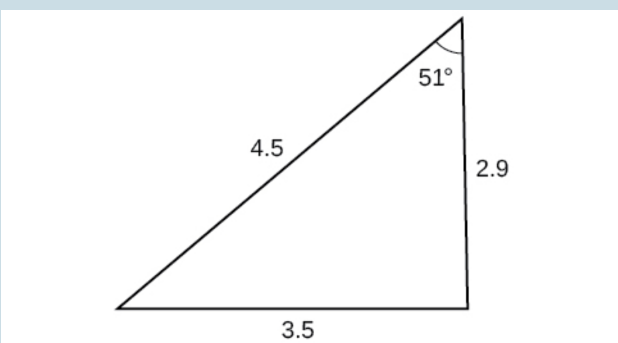
$A \approx 39.4, \ C \approx 47.6, \ BC \approx 20.7$

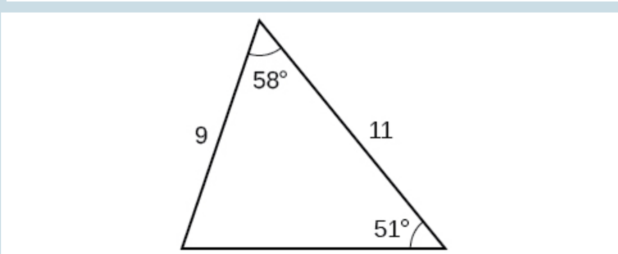
44. 

45. 

Show Solution

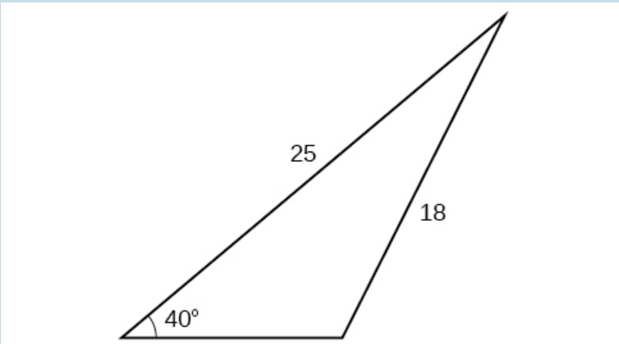
57.1

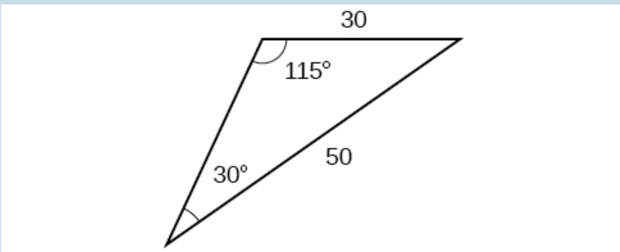
46. 

47. 

Show Solution

42.0

48. 

49. 

Show Solution

430.2

Extensions

50. Find the radius of the circle in (Figure). Round to the nearest tenth.

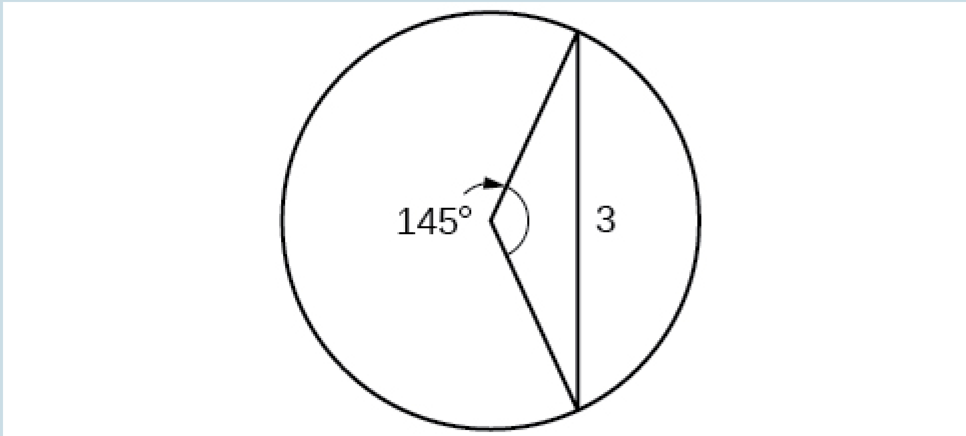


Figure 18.

51. Find the diameter of the circle in (Figure). Round to the nearest tenth.

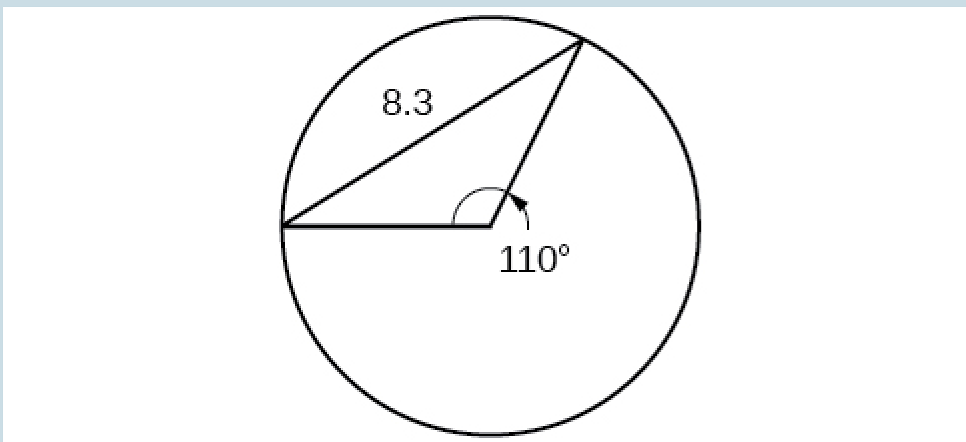


Figure 19.

Show Solution

10.1

52. Find $m\angle ADC$ in (Figure). Round to the nearest tenth.

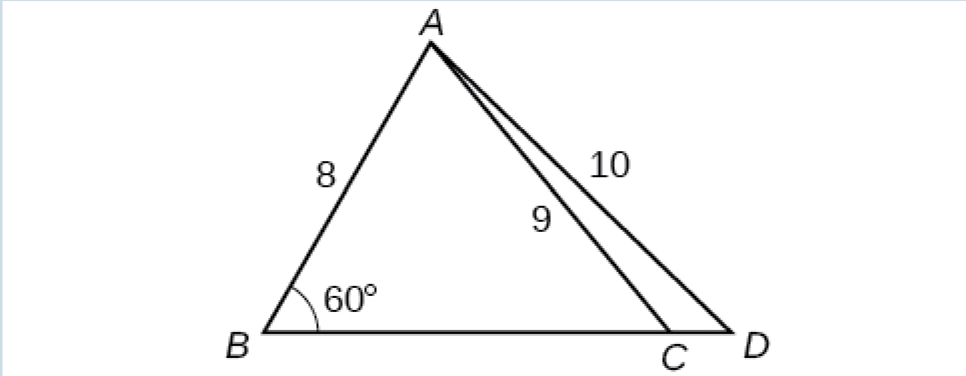


Figure 20.

53. Find AD in (Figure). Round to the nearest tenth.

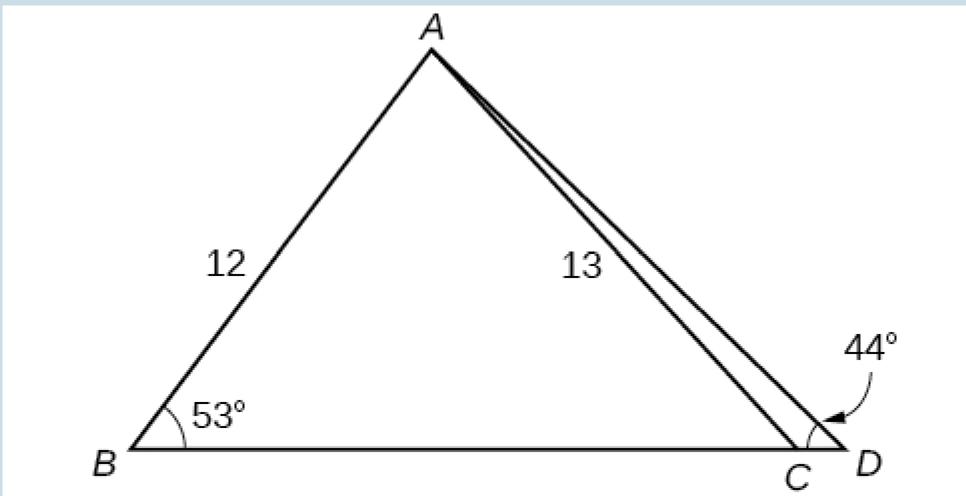


Figure 21.

Show Solution

$AD \approx\ 13.8$

54. Solve both triangles in (Figure). Round each answer to the nearest tenth.

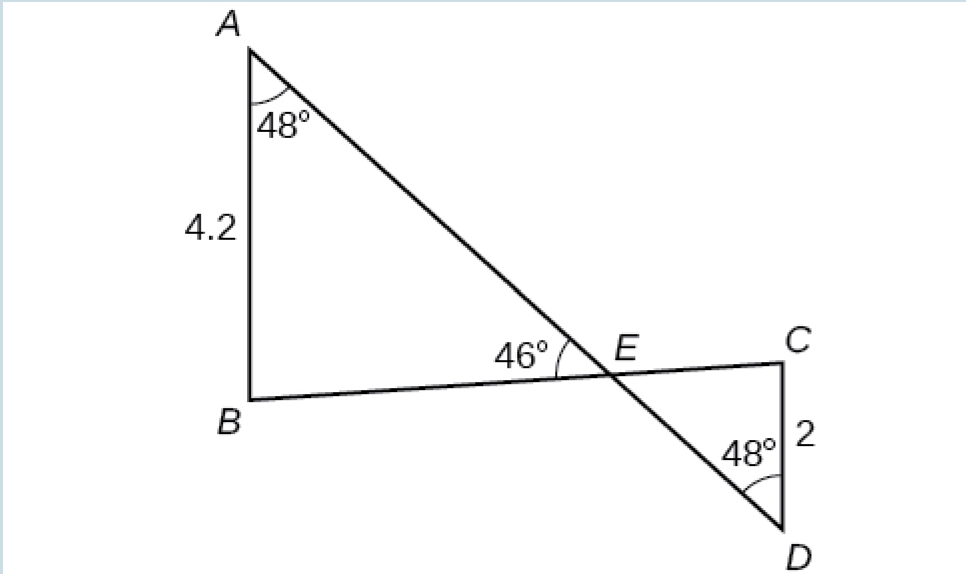


Figure 22.

55. Find AB in the parallelogram shown in (Figure).

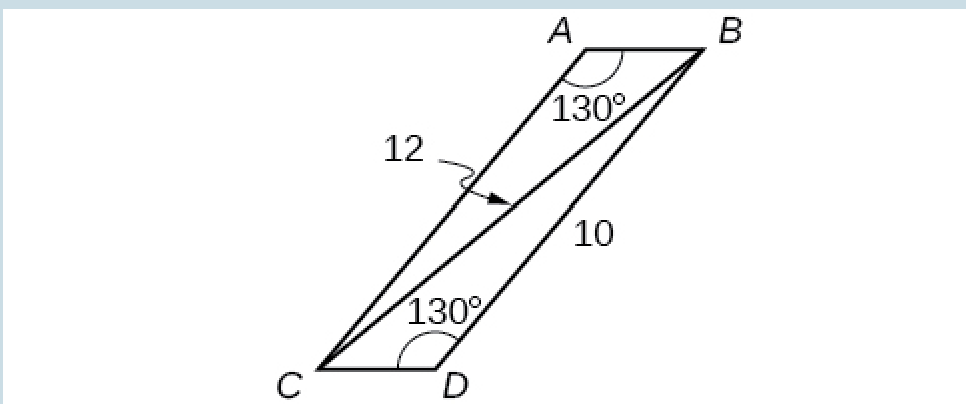


Figure 23.

Show Solution

$AB \approx 2.8$

56. Solve the triangle in (Figure). (Hint: Draw a perpendicular from H to JK). Round each answer to the nearest tenth.

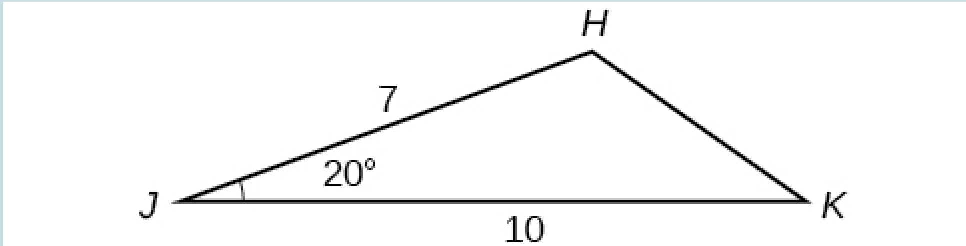


Figure 24.

57. Solve the triangle in (Figure). (Hint: Draw a perpendicular from N to LM). Round each answer to the nearest tenth.

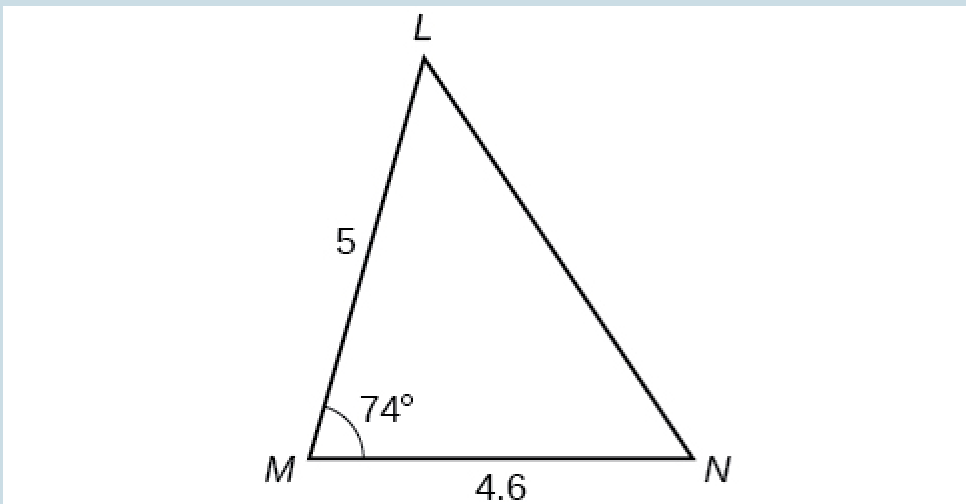


Figure 25.

Show Solution

$L \approx 49.7°,\ N \approx 56.3°,\ LN \approx 5.8$

58. In (Figure), $ABCD$ is not a parallelogram. $\angle m$ is obtuse. Solve both triangles. Round each answer to the nearest tenth.

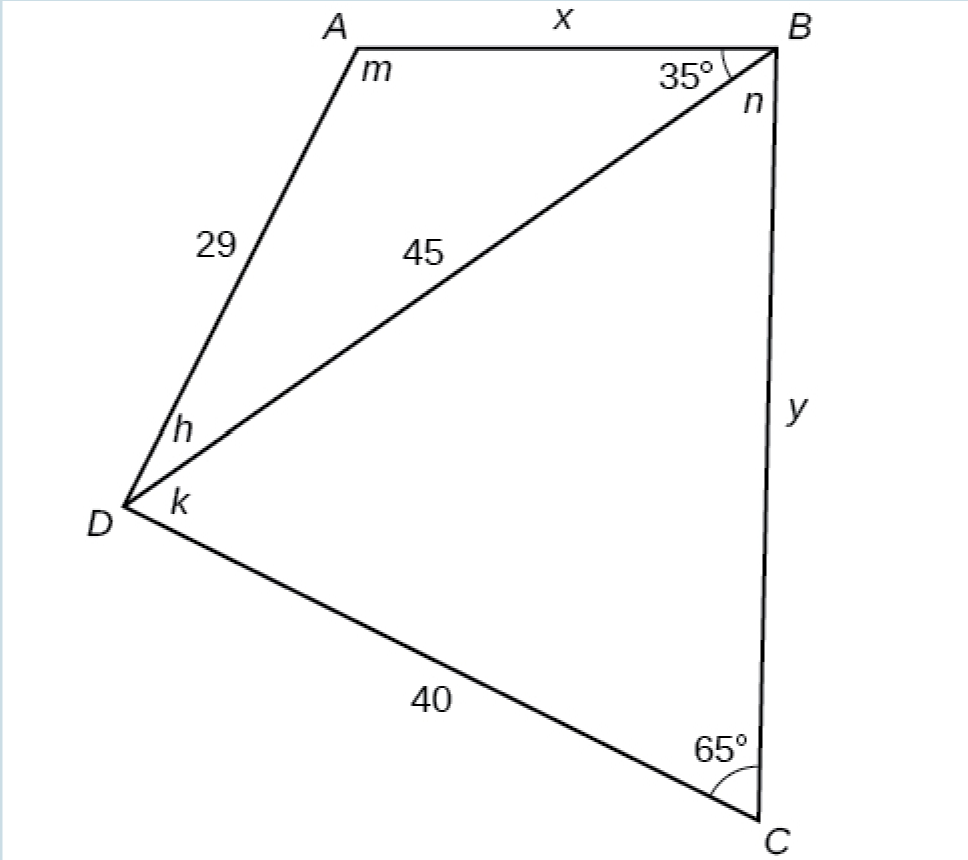


Figure 26.

Real-World Applications

59. A pole leans away from the sun at an angle of $7°$ to the vertical, as shown in (Figure). When the elevation of the sun is $55°$, the pole casts a shadow 42 feet long on the level ground. How long is the pole? Round the answer to the nearest tenth.

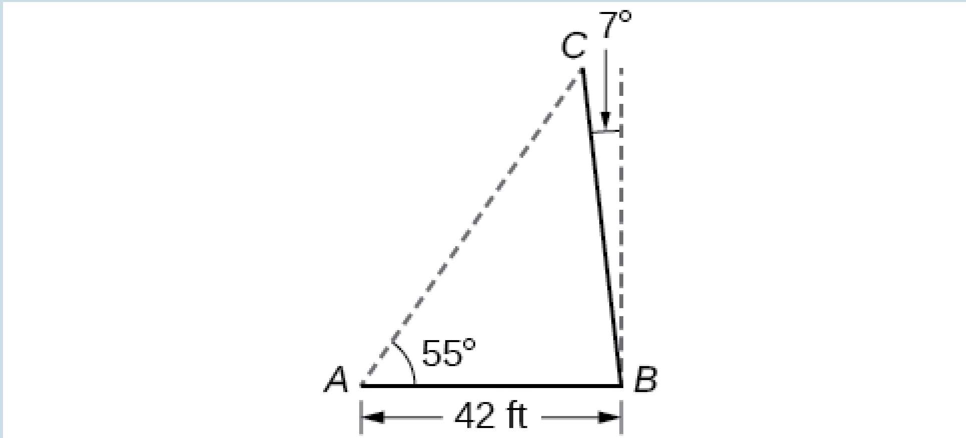


Figure 27.

Show Solution
51.4 feet

60. To determine how far a boat is from shore, two radar stations 500 feet apart find the angles out to the boat, as shown in (Figure). Determine the distance of the boat from station A and the distance of the boat from shore. Round your answers to the nearest whole foot.

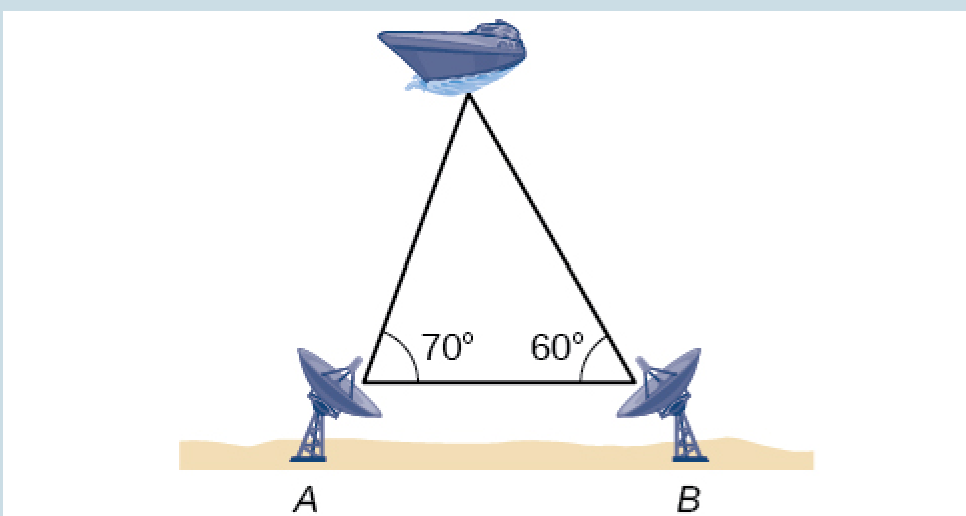


Figure 28.

61. (Figure) shows a satellite orbiting Earth. The satellite passes directly over two tracking stations

A and B, which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be $86.2°$ and $83.9°$, respectively. How far is the satellite from station A and how high is the satellite above the ground? Round answers to the nearest whole mile.

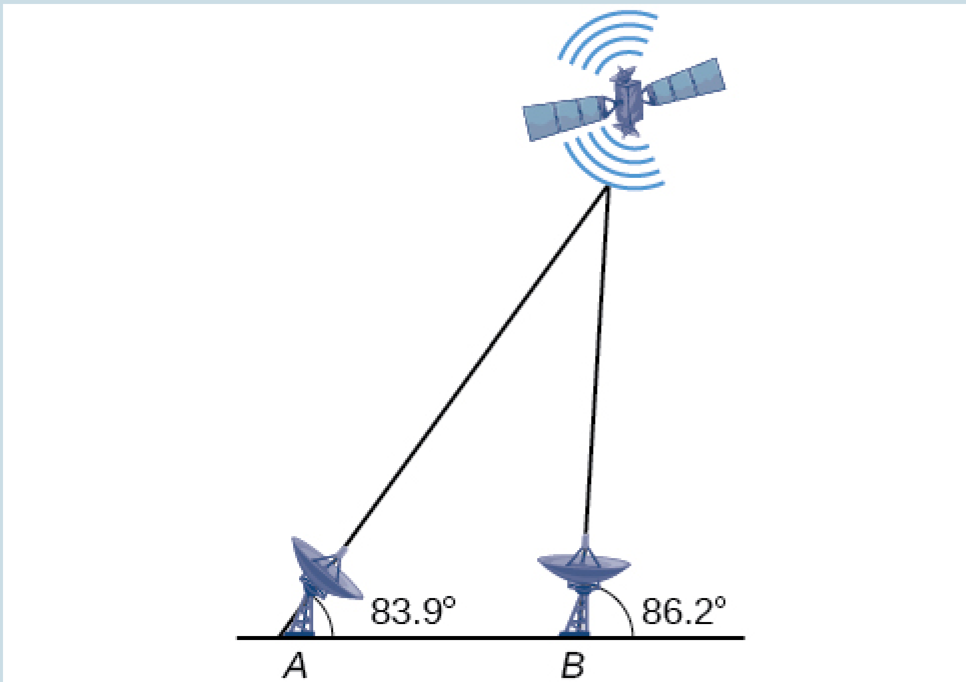


Figure 29.

Show Solution
The distance from the satellite to station A is approximately 1716 miles. The satellite is approximately 1706 miles above the ground.

62. A communications tower is located at the top of a steep hill, as shown in (Figure). The angle of inclination of the hill is $67°$. A guy wire is to be attached to the top of the tower and to the ground, 165 meters downhill from the base of the tower. The angle formed by the guy wire and the hill is $16°$. Find the length of the cable required for the guy wire to the nearest whole meter.

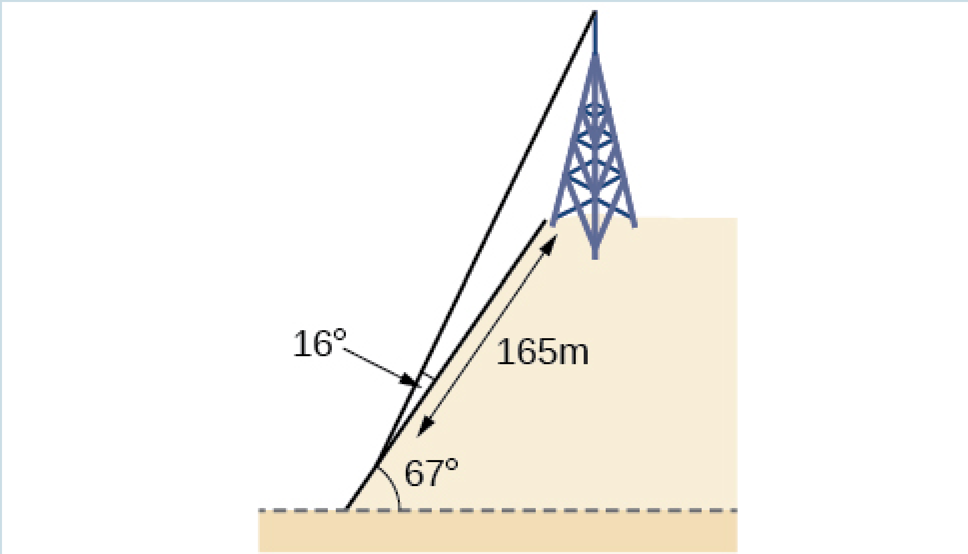


Figure 30.

63. The roof of a house is at a $20°$ angle. An 8-foot solar panel is to be mounted on the roof and should be angled $38°$ relative to the horizontal for optimal results. (See (Figure)). How long does the vertical support holding up the back of the panel need to be? Round to the nearest tenth.

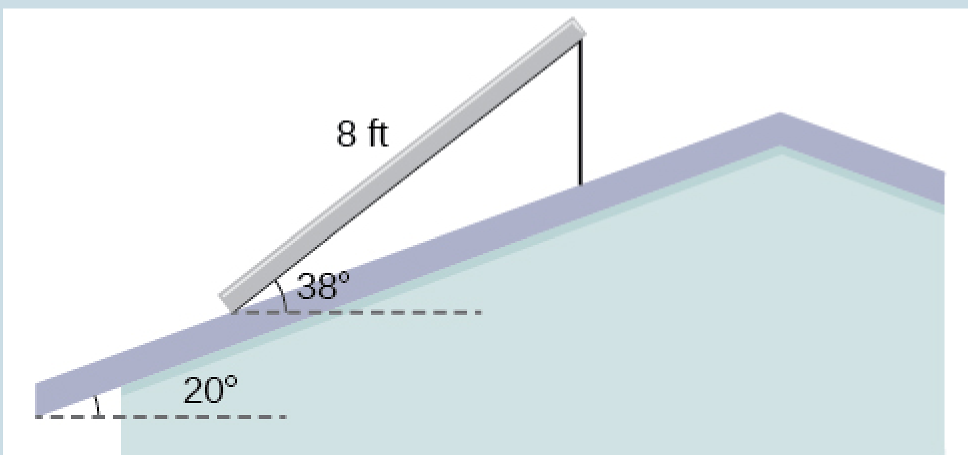


Figure 31.

Show Solution

2.6 ft

64. Similar to an angle of elevation, an *angle of depression* is the acute angle formed by a horizontal line and an observer's line of sight to an object below the horizontal. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 6.6 km apart, to be $37°$ and $44°$, as shown in (Figure). Find the distance of the plane from point A to the nearest tenth of a kilometer.

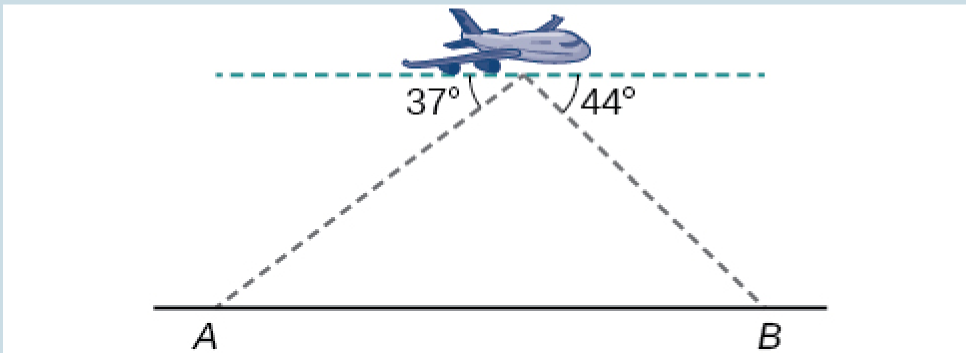


Figure 32.

65. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 4.3 km apart, to be $32°$ and $56°$, as shown in (Figure). Find the distance of the plane from point A to the nearest tenth of a kilometer.

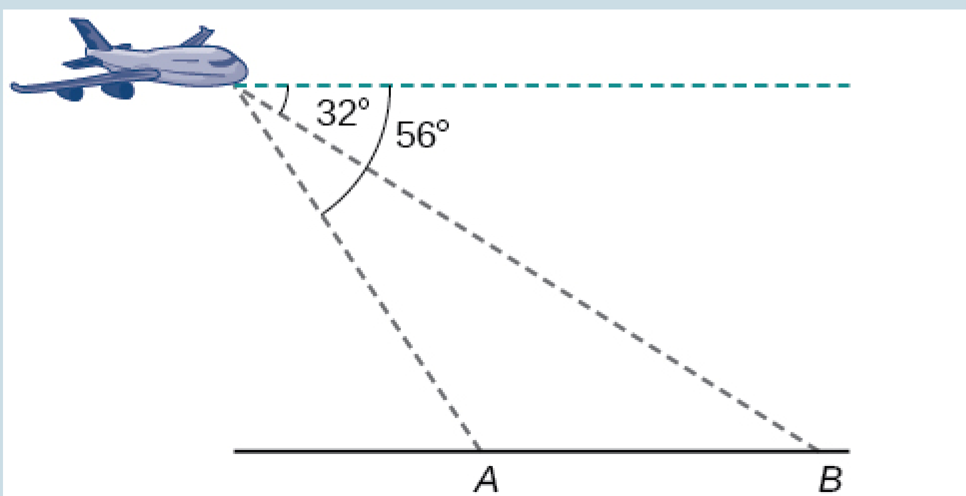


Figure 33.

Show Solution
5.6 km

66. In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be $39°$. They then move 300 feet closer to the building and find the angle of elevation to be $50°$. Assuming that the street is level, estimate the height of the building to the nearest foot.

67. In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be $35°$. They then move 250 feet closer to the building and find the angle of elevation to be $53°$. Assuming that the street is level, estimate the height of the building to the nearest foot.

Show Solution
371 ft

68. Points A and B are on opposite sides of a lake. Point C is 97 meters from $A.$ The measure of angle BAC is determined to be $101°$, and the measure of angle ACB is determined to be $53°$. What is the distance from A to $B,$ rounded to the nearest whole meter?

69. A man and a woman standing $3\frac{1}{2}$ miles apart spot a hot air balloon at the same time. If the angle of elevation from the man to the balloon is $27°$, and the angle of elevation from the woman to the balloon is $41°$, find the altitude of the balloon to the nearest foot.

Show Solution
5936 ft

70. Two search teams spot a stranded climber on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first search team to the stranded climber is 15° . The angle of elevation from the second search team to the climber is 22° . What is the altitude of the climber? Round to the nearest tenth of a mile.

71. A street light is mounted on a pole. A 6-foot-tall man is standing on the street a short distance from the pole, casting a shadow. The angle of elevation from the tip of the man's shadow to the top of his head of 28° . A 6-foot-tall woman is standing on the same street on the opposite side of the pole from the man. The angle of elevation from the tip of her shadow to the top of her head is 28° . If the man and woman are 20 feet apart, how far is the street light from the tip of the shadow of each person? Round the distance to the nearest tenth of a foot.

Show Solution
24.1 ft

72. Three cities, $A, B,$ and $C,$ are located so that city A is due east of city $B.$ If city C is located 35^{\circ} west of north from city B and is 100 miles from city A and 70 miles from city $B,$ how far is city A from city $B?$ Round the distance to the nearest tenth of a mile.

73. Two streets meet at an 80° angle. At the corner, a park is being built in the shape of a triangle. Find the area of the park if, along one road, the park measures 180 feet, and along the other road, the park measures 215 feet.

Show Solution
19,056 ft^{2}

74. Brian's house is on a corner lot. Find the area of the front yard if the edges measure 40 and 56 feet, as shown in (Figure).

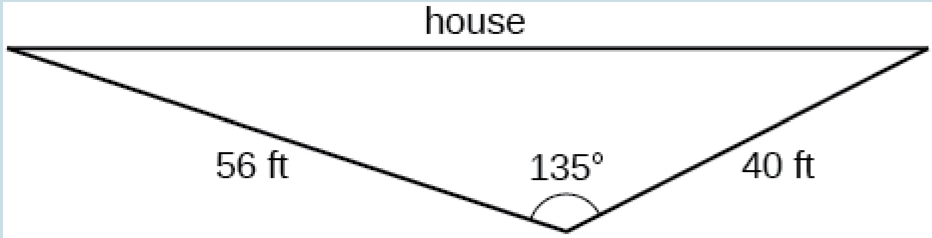


Figure 34.

75. The Bermuda triangle is a region of the Atlantic Ocean that connects Bermuda, Florida, and Puerto Rico. Find the area of the Bermuda triangle if the distance from Florida to Bermuda is 1030 miles, the distance from Puerto Rico to Bermuda is 980 miles, and the angle created by the two distances is 62°.

Show Solution
445,624 square miles

A yield sign measures 30 inches on all three sides. What is the area of the sign?

Naomi bought a modern dining table whose top is in the shape of a triangle. Find the area of the table top if two of the sides measure 4 feet and 4.5 feet, and the smaller angles measure $32°$ and 42°, as shown in (Figure).

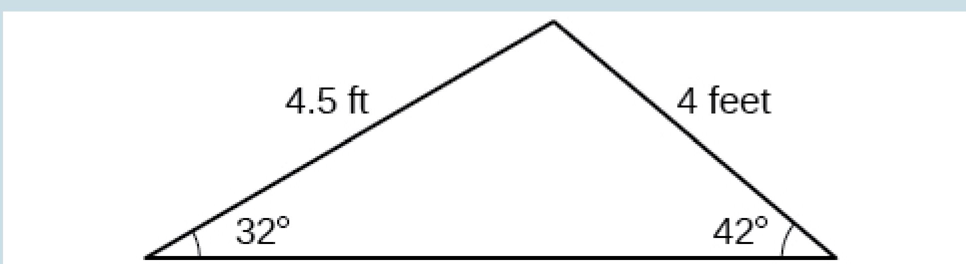


Figure 35.

Show Solution

8.65 ft^2

Glossary

altitude
: a perpendicular line from one vertex of a triangle to the opposite side, or in the case of an obtuse triangle, to the line containing the opposite side, forming two right triangles

ambiguous case
: a scenario in which more than one triangle is a valid solution for a given oblique SSA triangle

Law of Sines
: states that the ratio of the measurement of one angle of a triangle to the length of its opposite side is equal to the remaining two ratios of angle measure to opposite side; any pair of proportions may be used to solve for a missing angle or side

oblique triangle
: any triangle that is not a right triangle

CHAPTER 4.3: NON-RIGHT TRIANGLES: LAW OF COSINES

Learning Objectives

In this section, you will:

- Use the Law of Cosines to solve oblique triangles.
- Solve applied problems using the Law of Cosines.
- Use Heron's formula to find the area of a triangle.

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in (Figure). How far from port is the boat?

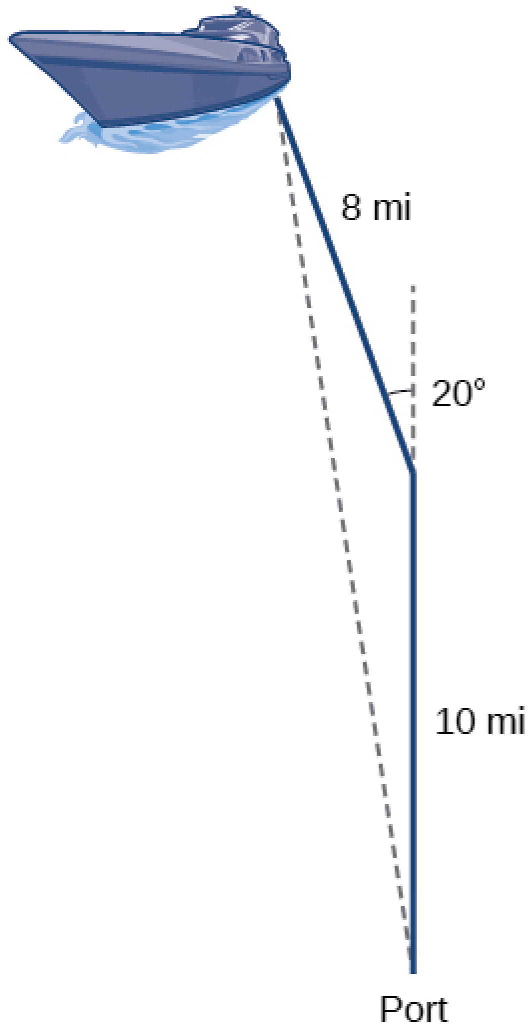


Figure 1.

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the **Law of Cosines**, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the *x*-axis, and vertex C located at some point (x, y) in the plane, as illustrated in (Figure). Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

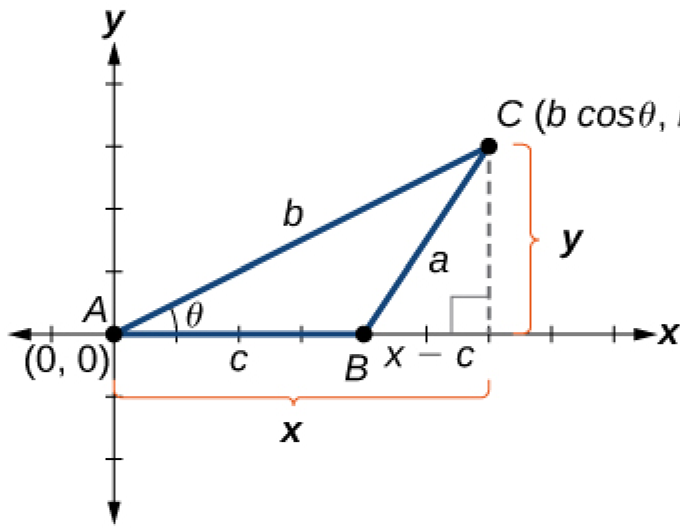


Figure 2.

We can drop a perpendicular from C to the *x*-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that
$\cos\theta = \frac{x(\text{adjacent})}{b(\text{hypotenuse})}$ and $\sin\theta = \frac{y(\text{opposite})}{b(\text{hypotenuse})}$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta,\ b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$
\begin{array}{ll}
a^2 = (x-c)^2 + y^2 & \\
\quad = (b\cos\theta - c)^2 + (b\sin\theta)^2 & \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y. \\
\quad = \left(b^2\cos^2\theta - 2bc\cos\theta + c^2\right) + b^2\sin^2\theta & \text{Expand the perfect square.} \\
\quad = b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta & \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1. \\
\quad = b^2\left(\cos^2\theta + \sin^2\theta\right) + c^2 - 2bc\cos\theta & \text{Factor out } b^2. \\
a^2 = b^2 + c^2 - 2bc\cos\theta &
\end{array}
$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

Law of Cosines

The Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle. For triangles labeled as in (Figure), with angles α, β, and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\ \cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\ \cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\ \cos\gamma$$

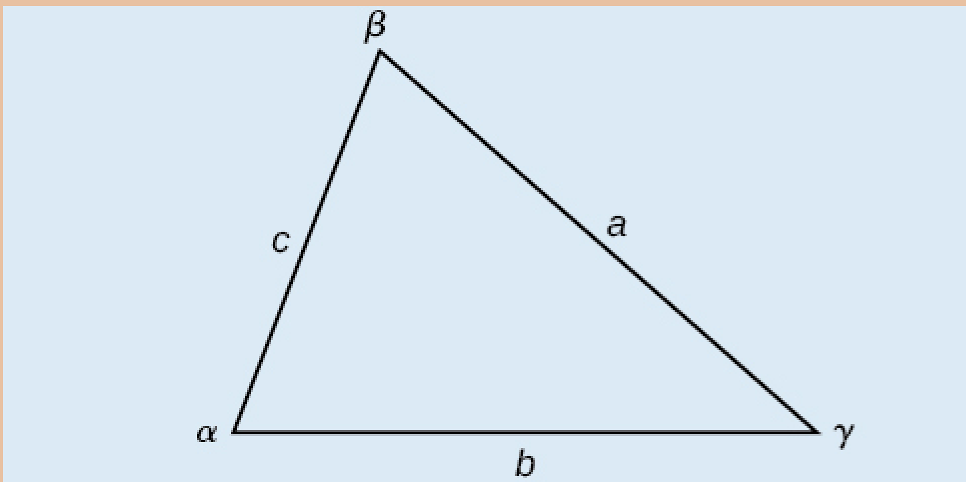


Figure 3.

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\ \alpha = \frac{b^2+c^2-a^2}{2bc}$$
$$\cos\ \beta = \frac{a^2+c^2-b^2}{2ac}$$
$$\cos\ \gamma = \frac{a^2+b^2-c^2}{2ab}$$

Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle.

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.

2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in (Figure).

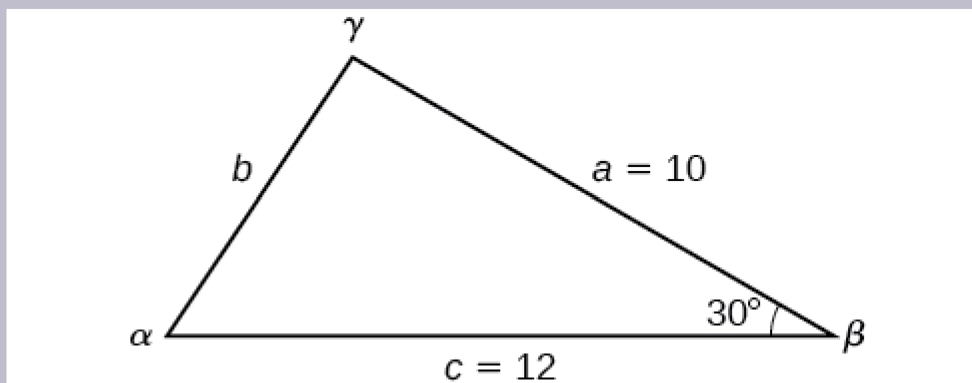


Figure 4.

Show Solution

First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side $b,$ as we know the measurement of the opposite angle $\beta.$

$$\begin{array}{ll} b^2 = a^2 + c^2 - 2ac\cos\beta & \\ b^2 = 10^2 + 12^2 - 2\left(10\right)\left(12\right)\cos\left(30^\circ\right) & \text{Substitute the measurements for the known quantities.} \\ b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) & \text{Evaluate the cosine and begin to simplify.} \\ b^2 = 244 - 120\sqrt{3} & \\ b = \sqrt{244 - 120\sqrt{3}} & \text{Use the square root property.} \\ b \approx 6.013 & \end{array}$$

Because we are solving for a length, we use only the positive square root. Now that we know the length $b,$ we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle $\alpha,$ we have

$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$
$\frac{\sin\alpha}{10} = \frac{\sin(30°)}{6.013}$
$\sin\alpha = \frac{10\sin(30°)}{6.013}$ Multiply both sides of the equation by 10.
$\alpha = \sin^{-1}\left(\frac{10\sin(30°)}{6.013}\right)$ Find the inverse sine of $\frac{10\sin(30°)}{6.013}$.
$\alpha \approx 56.3°$

The other possibility for α would be $\alpha = 180° - 56.3° \approx 123.7°$. In the original diagram, α is adjacent to the longest side, so α is an acute angle and, therefore, $123.7°$ does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between $0°$ and $180°$. Proceeding with $\alpha \approx 56.3°$, we can then find the third angle of the triangle.

$$\gamma = 180° - 30° - 56.3° \approx 93.7°$$

The complete set of angles and sides is

$\alpha \approx 56.3°$	$a = 10$
$\beta = 30°$	$b \approx 6.013$
$\gamma \approx 93.7°$	$c = 12$

Try It

Find the missing side and angles of the given triangle: $\alpha = 30°,\ b = 12,\ c = 24$.

Show Solution

$a \approx 14.9,\ \beta \approx 23.8°,\ \gamma \approx 126.2°.$

Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a = 20$, side $b = 25$, and side $c = 18$.

Show Solution

For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$\begin{array}{ll} a^2 = b^2 + c^2 - 2bc\cos\alpha & \\ 20^2 = 25^2 + 18^2 - 2(25)(18)\cos\alpha & \text{Substitute the appropriate measurements.} \\ 400 = 625 + 324 - 900\cos\alpha & \text{Simplify in each step.} \\ 400 = 949 - 900\cos\alpha & \\ -549 = -900\cos\alpha & \text{Isolate } \cos\alpha. \\ \frac{-549}{-900} = \cos\alpha & \\ 0.61 \approx \cos\alpha & \\ \cos^{-1}(0.61) \approx \alpha & \text{Find the inverse cosine.} \\ \alpha \approx 52.4^\circ & \end{array}$$

See (Figure).

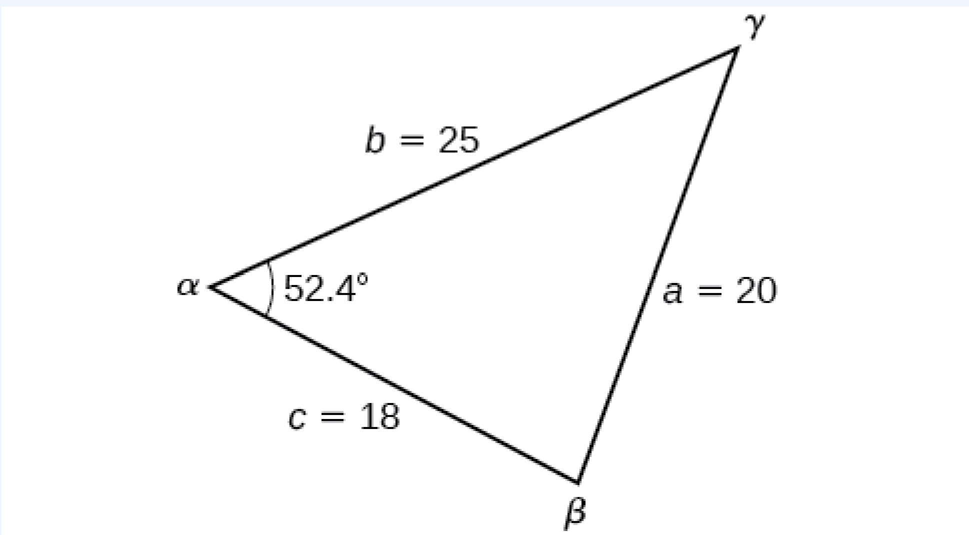


Figure 5.

Analysis

Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

Try It

Given $a = 5, b = 7$, and $c = 10$, find the missing angles.

Show Solution

$\alpha \approx 27.7°,\ \beta \approx 40.5°,\ \gamma \approx 111.8°$

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Show Solution

For simplicity, we start by drawing a diagram similar to (Figure) and labeling our given information.

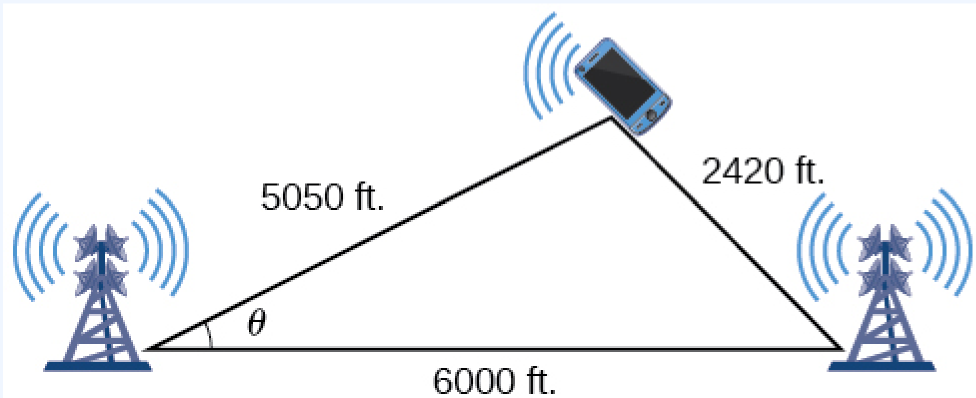


Figure 6.

Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420, b = 5050,$ and $c = 6000.$ Thus, θ corresponds to the opposite side $a = 2420.$

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$
$$(2420)^2 = (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta$$
$$(2420)^2 - (5050)^2 - (6000)^2 = -2(5050)(6000)\cos\theta$$
$$\frac{(2420)^2-(5050)^2-(6000)^2}{-2(5050)(6000)} = \cos\theta$$
$$\cos\theta \approx 0.9183$$
$$\theta \approx \cos^{-1}(0.9183)$$
$$\theta \approx 23.3°$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in (Figure). This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

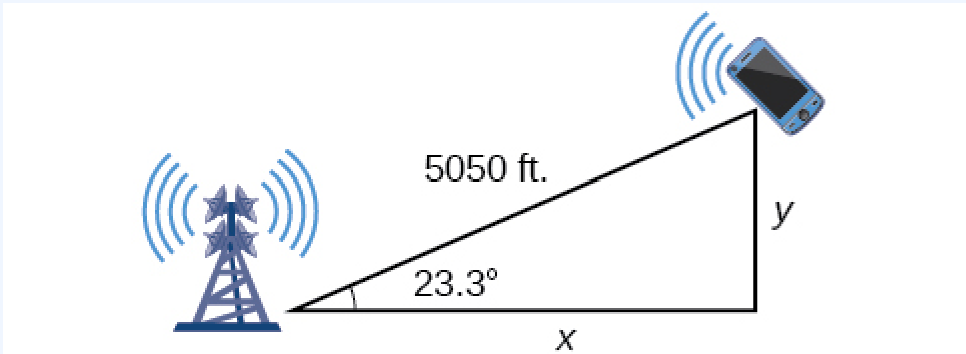


Figure 7.

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$\cos(23.3°) = \frac{x}{5050}$$
$$x = 5050\cos(23.3°)$$
$$x \approx 4638.15 \text{ feet}$$
$$\sin(23.3°) = \frac{y}{5050}$$
$$y = 5050\sin(23.3°)$$
$$y \approx 1997.5 \text{ feet}$$

The cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in (Figure).

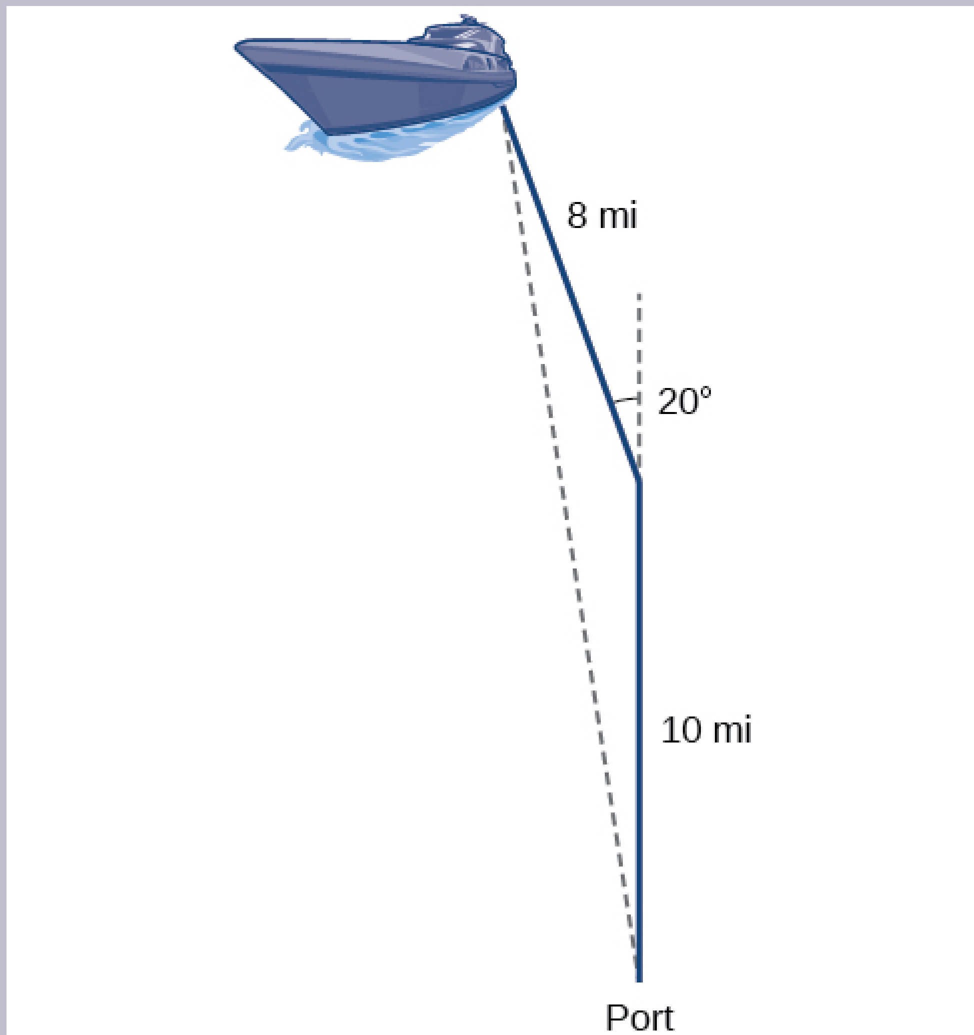


Figure 8.

Show Solution

The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180° - 20° = 160°$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$\begin{aligned} x^2 &= 8^2 + 10^2 - 2\left(8\right)\left(10\right)\cos\left(160°\right) \\ x^2 &= 314.35 \\ x &= \sqrt{314.35} \\ x &\approx 17.7 \text{ miles} \end{aligned}$$

The boat is about 17.7 miles from port.

Using Heron's Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use Heron's formula instead of finding the height. Heron of Alexandria was a geometer who lived during the first century A.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

Heron's Formula

Heron's formula finds the area of oblique triangles in which sides a, b, and c are known.

$$\text{Area} = \sqrt{s\left(s-a\right)\left(s-b\right)\left(s-c\right)}$$

where $s = \frac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Using Heron's Formula to Find the Area of a Given Triangle

Find the area of the triangle in (Figure) using Heron's formula.

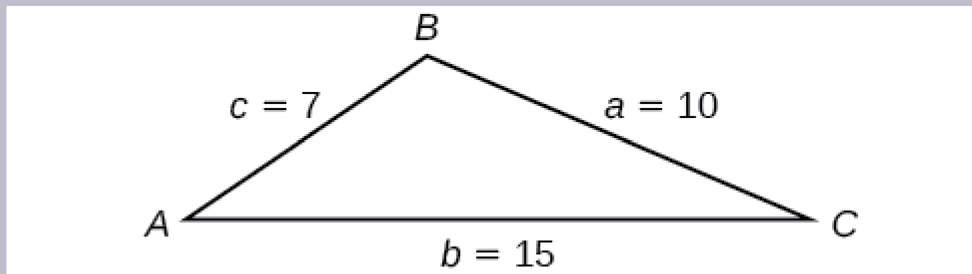


Figure 9.

Show Solution

First, we calculate s.

$$s = \frac{(a+b+c)}{2}$$
$$s = \frac{(10+15+7)}{2} = 16$$

Then we apply the formula.

$$\text{Area} = \sqrt{s\left(s-a\right)\left(s-b\right)\left(s-c\right)}$$
$$\text{Area} = \sqrt{16\left(16-10\right)\left(16-15\right)\left(16-7\right)}$$
$$\text{Area} \approx 29.4$$

The area is approximately 29.4 square units.

Try It

Use Heron's formula to find the area of a triangle with sides of lengths $a = 29.7\,\text{ft}, b = 42.3\,\text{ft},$ and $c = 38.4\,\text{ft}.$

Show Solution

Area = 552 square feet

Applying Heron's Formula to a Real-World Problem

A Chicago city developer wants to construct a building consisting of artist's lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? See (Figure) for a view of the city property.

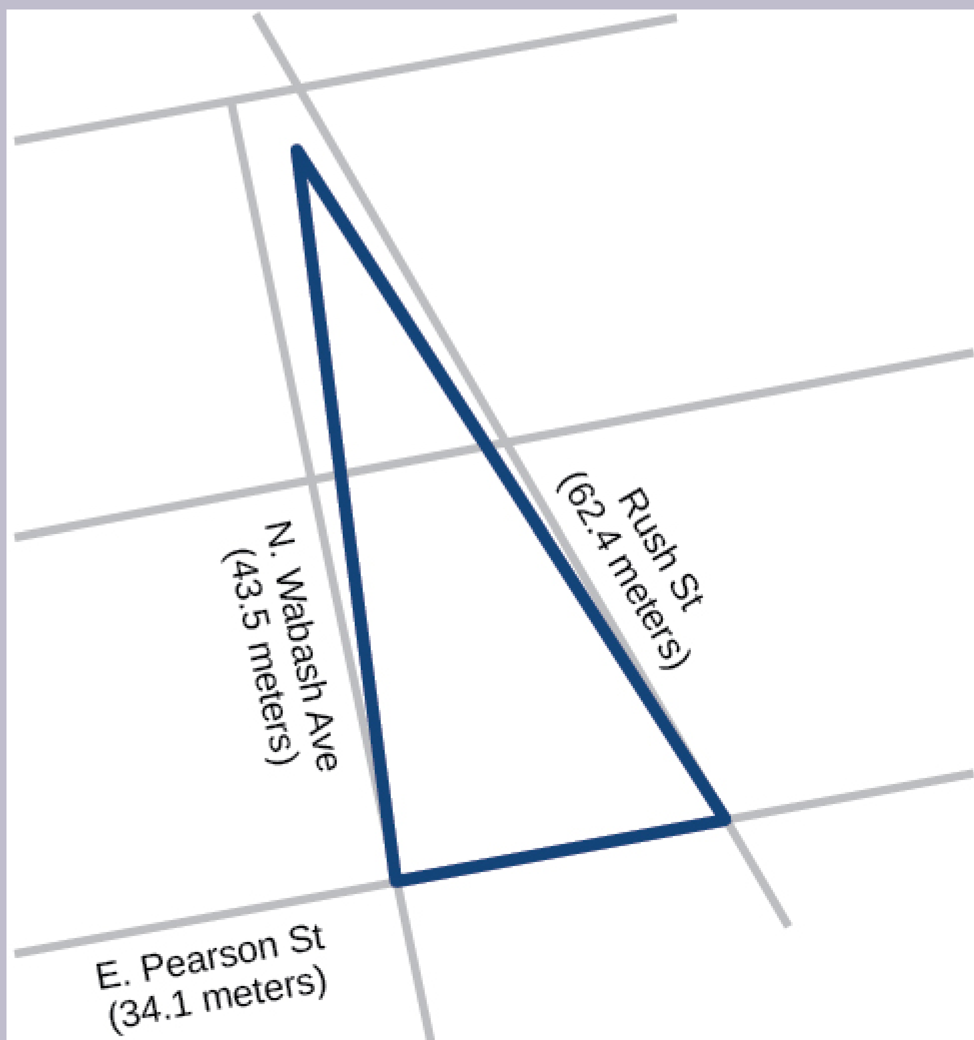


Figure 10.

Show Solution

Find the measurement for s, which is one-half of the perimeter.

$$s = \frac{(62.4+43.5+34.1)}{2}$$
$$s = 70\,\mathrm{m}$$

Apply Heron's formula.

$$\text{Area} = \sqrt{70\,(70-62.4)\,(70-43.5)\,(70-34.1)}$$
$$\text{Area} = \sqrt{506,118.2}$$
$$\text{Area} \approx 711.4$$

The developer has about 711.4 square meters.

Try It

Find the area of a triangle given $a = 4.38\,\mathrm{ft}$, $b = 3.79\,\mathrm{ft}$, and $c = 5.22\,\mathrm{ft}$.

Show Solution

about 8.15 square feet

Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines
- Law of Cosines: Applications
- Law of Cosines: Applications 2

Key Equations

Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$
Heron's formula	$\text{Area} = \sqrt{s\left(s-a\right)\left(s-b\right)\left(s-c\right)}$ where $s = \frac{(a+b+c)}{2}$

Key Concepts

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See (Figure) and (Figure).
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See (Figure) and (Figure).
- Heron's formula allows the calculation of area in oblique triangles. All three sides must be known to apply Heron's formula. See (Figure) and See (Figure).

Section Exercises

Verbal

1. If you are looking for a missing side of a triangle, what do you need to know when using the Law of Cosines?

Show Solution
two sides and the angle opposite the missing side.

2. If you are looking for a missing angle of a triangle, what do you need to know when using the Law of Cosines?

3. Explain what s represents in Heron's formula.

Show Solution
s is the semi-perimeter, which is half the perimeter of the triangle.

4. Explain the relationship between the Pythagorean Theorem and the Law of Cosines.

5. When must you use the Law of Cosines instead of the Pythagorean Theorem?

Show Solution
The Law of Cosines must be used for any oblique (non-right) triangle.

Algebraic

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. If possible, solve each triangle for the unknown side. Round to the nearest tenth.

6. $\gamma = 41.2°, a = 2.49, b = 3.13$

7. $\alpha = 120°, b = 6, c = 7$

Show Solution

11.3

8. $\beta = 58.7°, a = 10.6, c = 15.7$

9. $\gamma = 115°, a = 18, b = 23$

Show Solution
34.7

10. $\alpha = 119°, a = 26, b = 14$

11. $\gamma = 113°, b = 10, c = 32$

Show Solution
26.7

12. $\beta = 67°, a = 49, b = 38$

13. $\alpha = 43.1°, a = 184.2, b = 242.8$

Show Solution
257.4

14. $\alpha = 36.6°, a = 186.2, b = 242.2$

15. $\beta = 50°, a = 105, b = 45$

Show Solution
not possible

For the following exercises, use the Law of Cosines to solve for the missing angle of the oblique triangle. Round to the nearest tenth.

16. $a = 42, b = 19, c = 30$; find angle A.

17. $a = 14,\ \ b = 13,\ \ c = 20$; find angle C.

Show Solution
95.5°

18. $a = 16, b = 31, c = 20$; find angle B.

19. $a = 13,\ \ b = 22,\ \ c = 28$; find angle A.

Show Solution
26.9°

20. $a = 108,\ \ b = 132,\ \ c = 160$; find angle C.

For the following exercises, solve the triangle. Round to the nearest tenth.

21. $A = 35°, b = 8, c = 11$

Show Solution

$B \approx 45.9°, C \approx 99.1°, a \approx 6.4$

22. $B = 88°, a = 4.4, c = 5.2$

23. $C = 121°, a = 21, b = 37$

Show Solution
$A \approx 20.6°, B \approx 38.4°, c \approx 51.1$

24. $a = 13, b = 11, c = 15$

25. $a = 3.1, b = 3.5, c = 5$

Show Solution
$A \approx 37.8°, B \approx 43.8, C \approx 98.4°$

26. $a = 51, b = 25, c = 29$

For the following exercises, use Heron's formula to find the area of the triangle. Round to the nearest hundredth.

27. Find the area of a triangle with sides of length 18 in, 21 in, and 32 in. Round to the nearest tenth.

Show Solution
177.56 in^2

28. Find the area of a triangle with sides of length 20 cm, 26 cm, and 37 cm. Round to the nearest tenth.

29. $a = \frac{1}{2}\,\mathrm{m}, b = \frac{1}{3}\,\mathrm{m}, c = \frac{1}{4}\,\mathrm{m}$

Show Solution
0.04 m^2

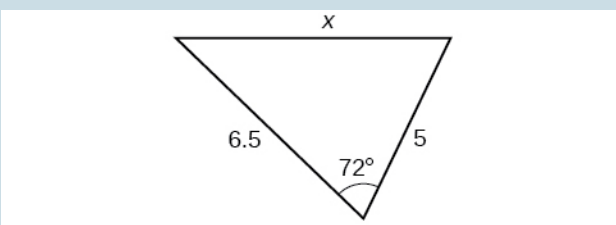
30. $a = 12.4\ \mathrm{ft},\ \ b = 13.7\ \mathrm{ft},\ \ c = 20.2\ \mathrm{ft}$

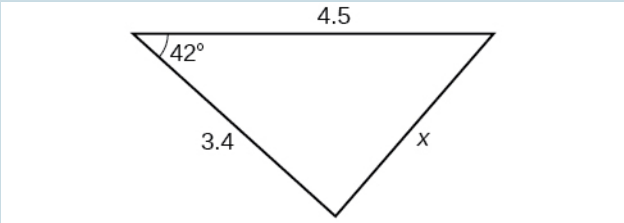
31. $a = 1.6\ \mathrm{yd},\ \ b = 2.6\ \mathrm{yd},\ \ c = 4.1\ \mathrm{yd}$

Show Solution
0.91 yd^2

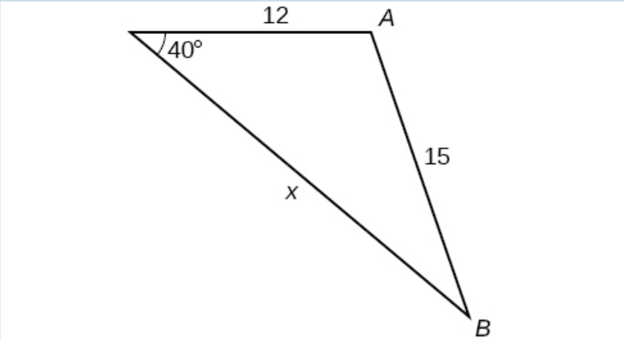
Graphical

For the following exercises, find the length of side x. Round to the nearest tenth.

32. 

33. 

Show Solution
3.0

34. 

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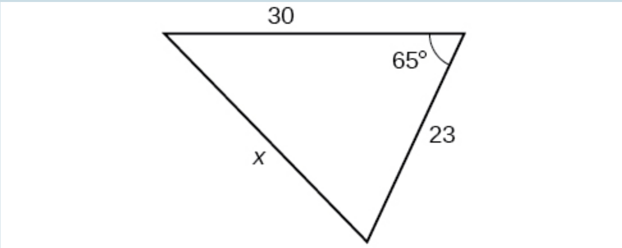
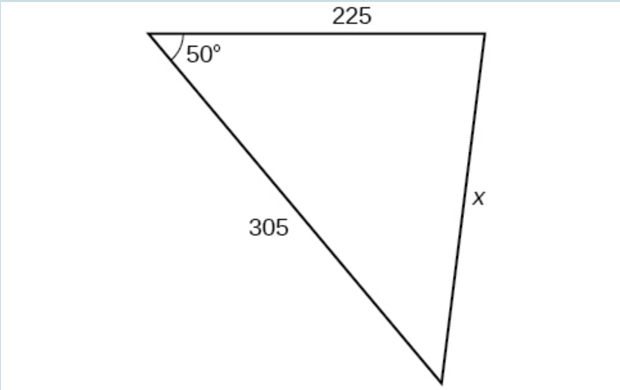
35. 

Show Solution
29.1

36. 

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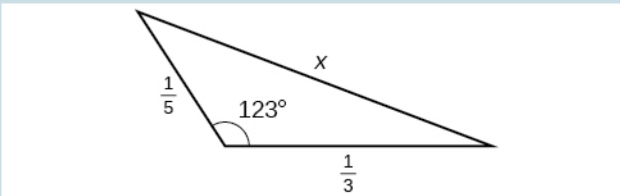
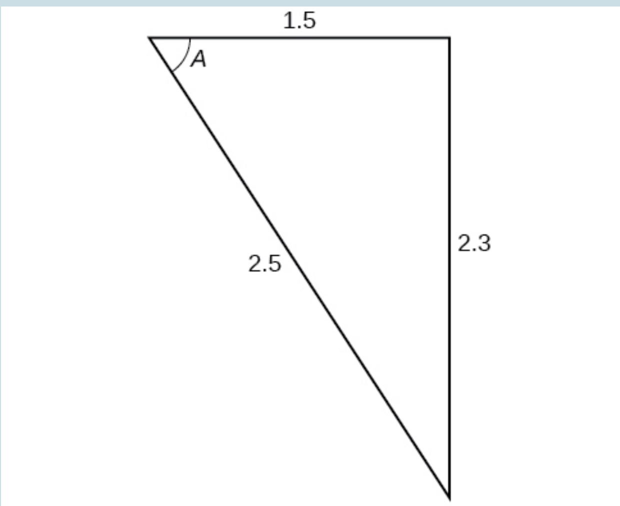
37. 

Show Solution

0.5

For the following exercises, find the measurement of angle A.

38. 

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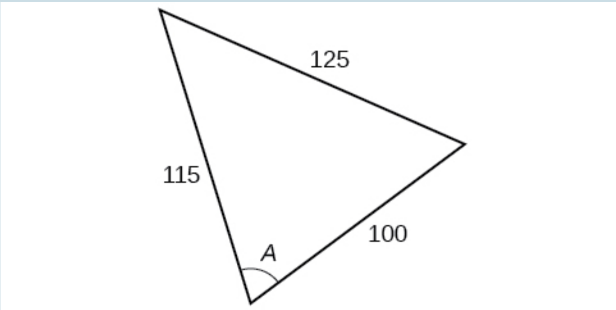
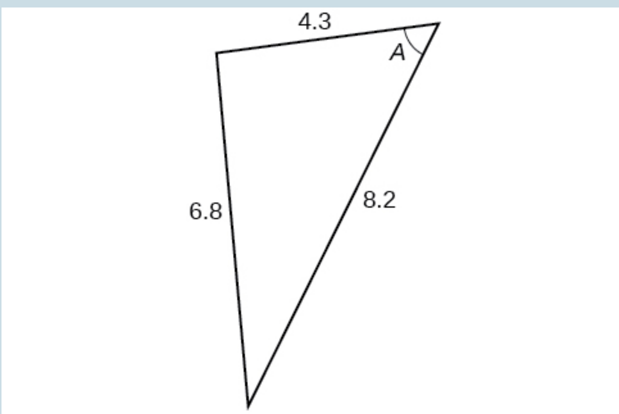
39. 

Show Solution

70.7°

40. 

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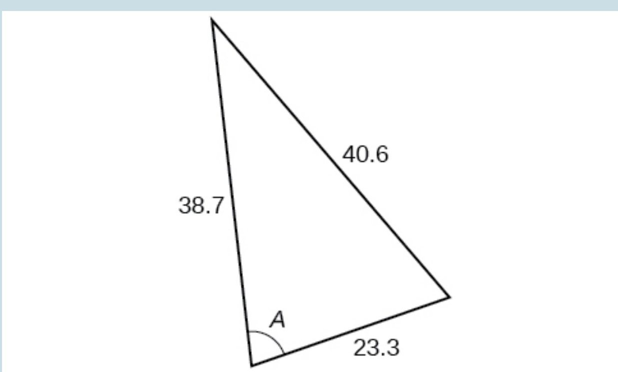
41. 

Show Solution

77.4°

42. Find the measure of each angle in the triangle shown in (Figure). Round to the nearest tenth.

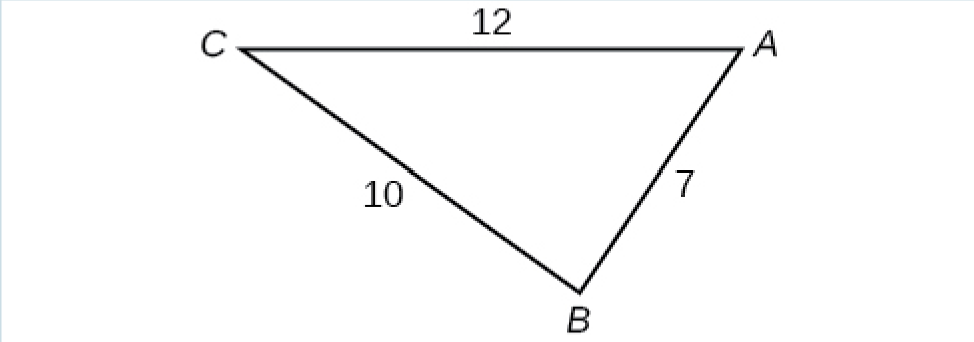
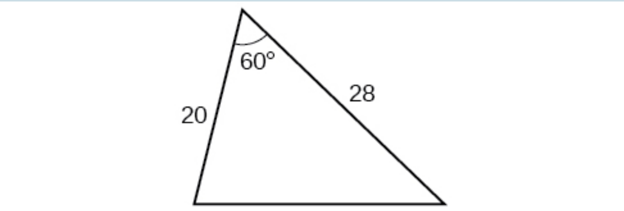


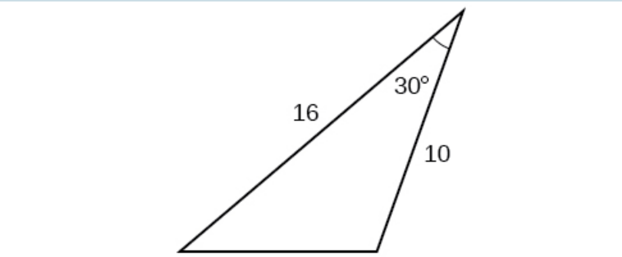
Figure 11.

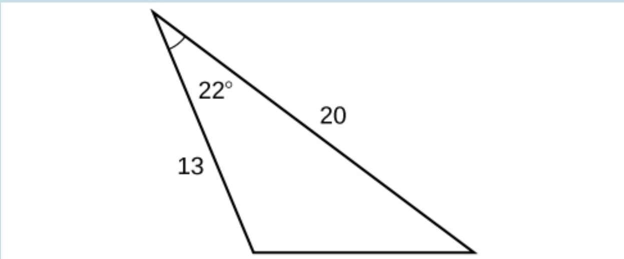
For the following exercises, solve for the unknown side. Round to the nearest tenth.

43. 

Show Solution

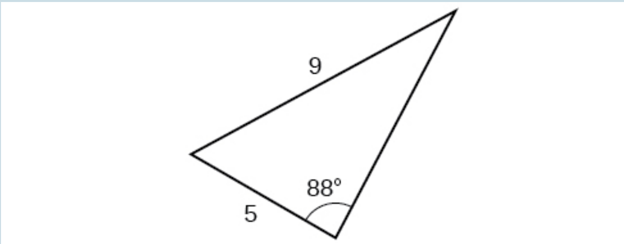
25.0

44. 

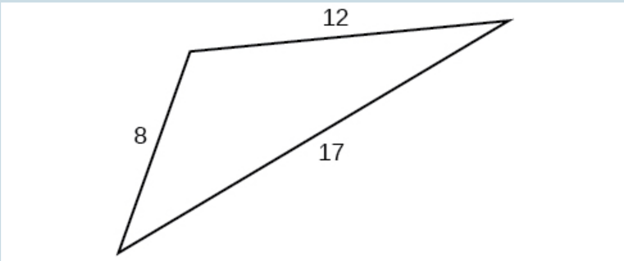
45. 

Show Solution

9.3

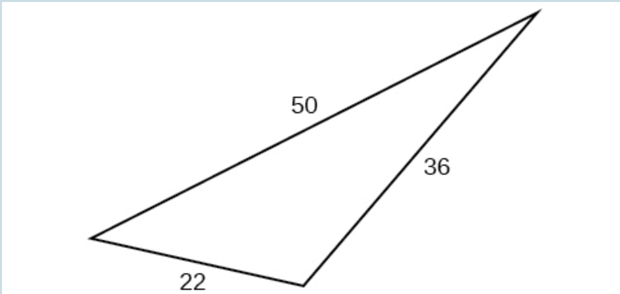
46. 

For the following exercises, find the area of the triangle. Round to the nearest hundredth.

47. 

Show Solution

43.52

48. 

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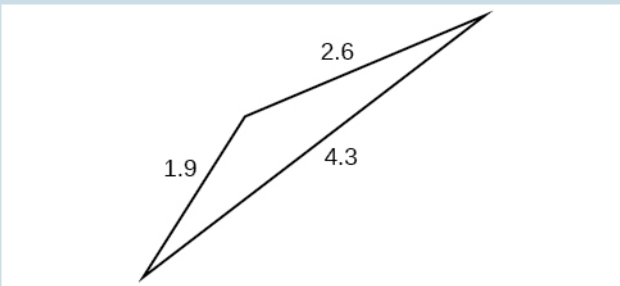
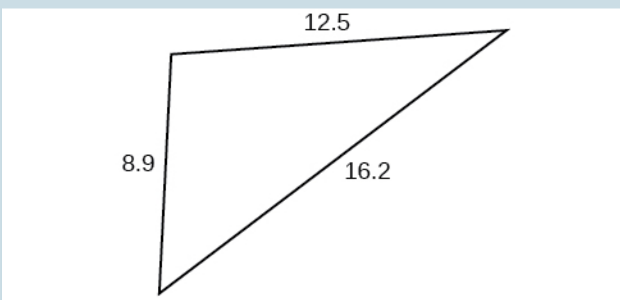
49. 

Show Solution
1.41

50. 

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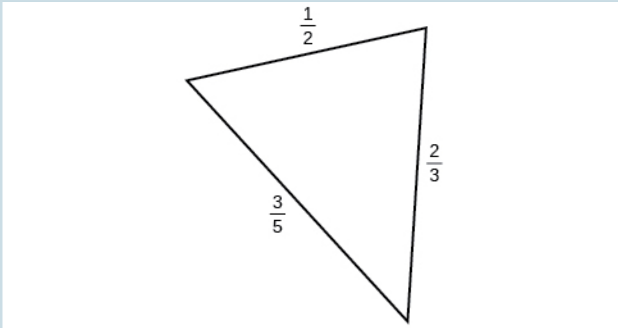
51. 

Show Solution

0.14

Extensions

52. A parallelogram has sides of length 16 units and 10 units. The shorter diagonal is 12 units. Find the measure of the longer diagonal.

53. The sides of a parallelogram are 11 feet and 17 feet. The longer diagonal is 22 feet. Find the length of the shorter diagonal.

Show Solution

18.3

54. The sides of a parallelogram are 28 centimeters and 40 centimeters. The measure of the larger angle is 100°. Find the length of the shorter diagonal.

55. A regular octagon is inscribed in a circle with a radius of 8 inches. (See (Figure).) Find the perimeter of the octagon.

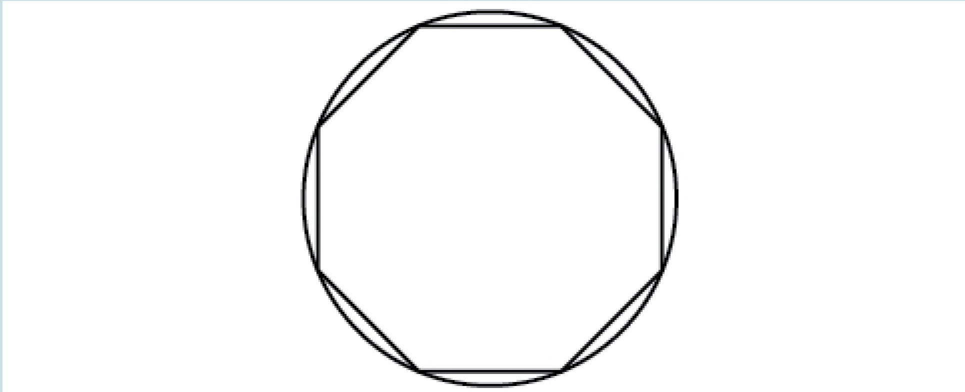


Figure 12.

Show Solution

48.98

56. A regular pentagon is inscribed in a circle of radius 12 cm. (See (Figure).) Find the perimeter of the pentagon. Round to the nearest tenth of a centimeter.

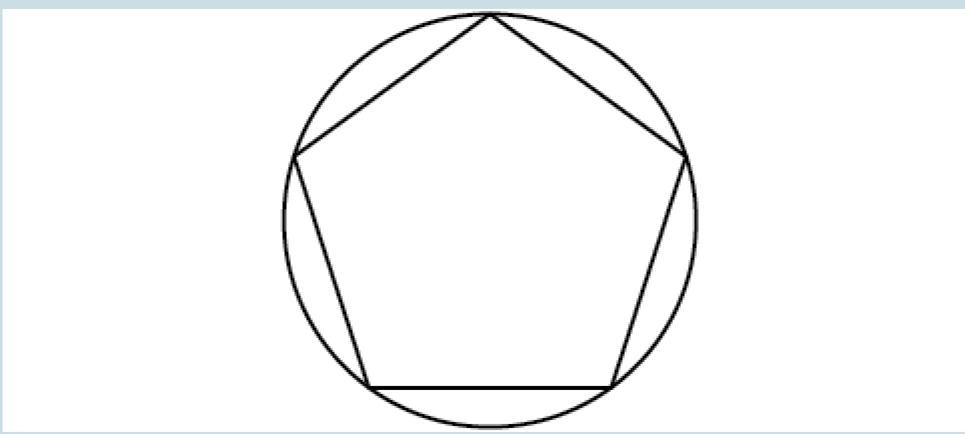
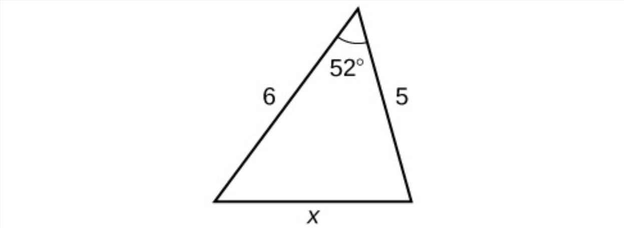


Figure 13.

For the following exercises, suppose that $x^2 = 25 + 36 - 60\cos\left(52\right)$ represents the relationship of three sides of a triangle and the cosine of an angle.

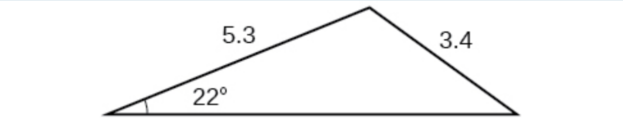
57. Draw the triangle.

Show Solution



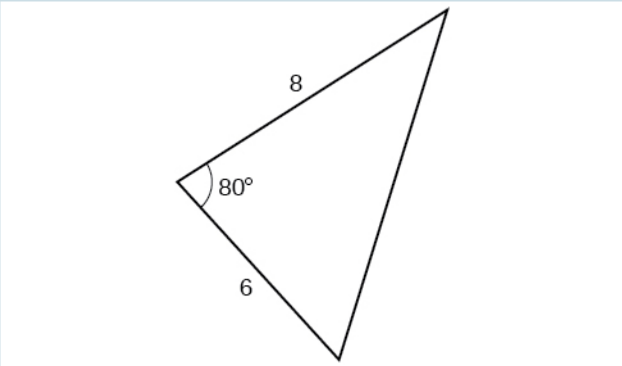
58. Find the length of the third side.

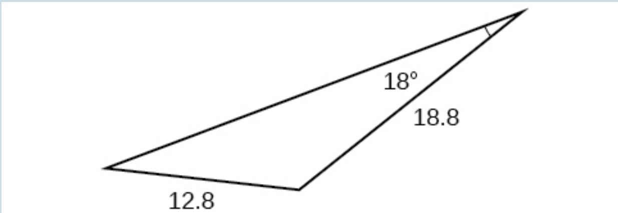
For the following exercises, find the area of the triangle.

59. 

Show Solution

7.62

60. 

61. 

Show Solution

85.1

Real-World Applications

62. A surveyor has taken the measurements shown in (Figure). Find the distance across the lake. Round answers to the nearest tenth.

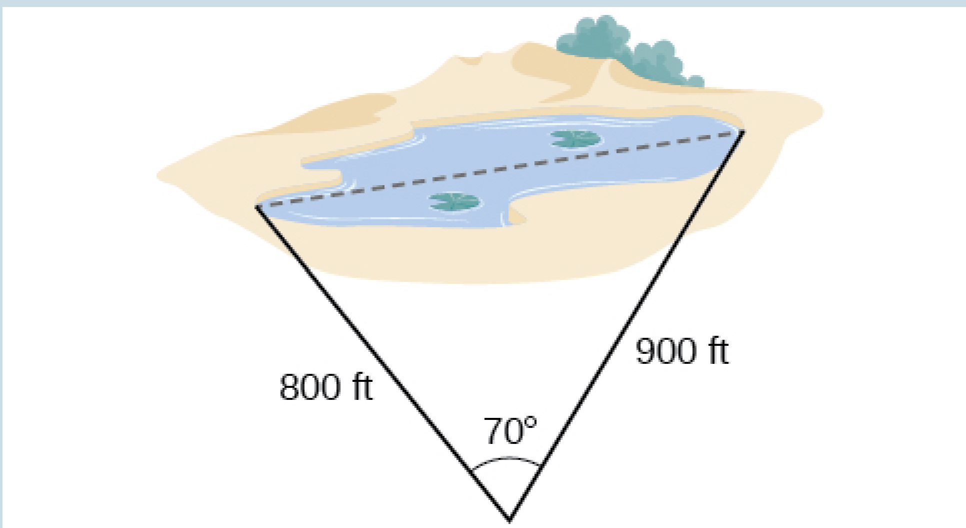


Figure 14.

63. A satellite calculates the distances and angle shown in (Figure) (not to scale). Find the distance between the two cities. Round answers to the nearest tenth.

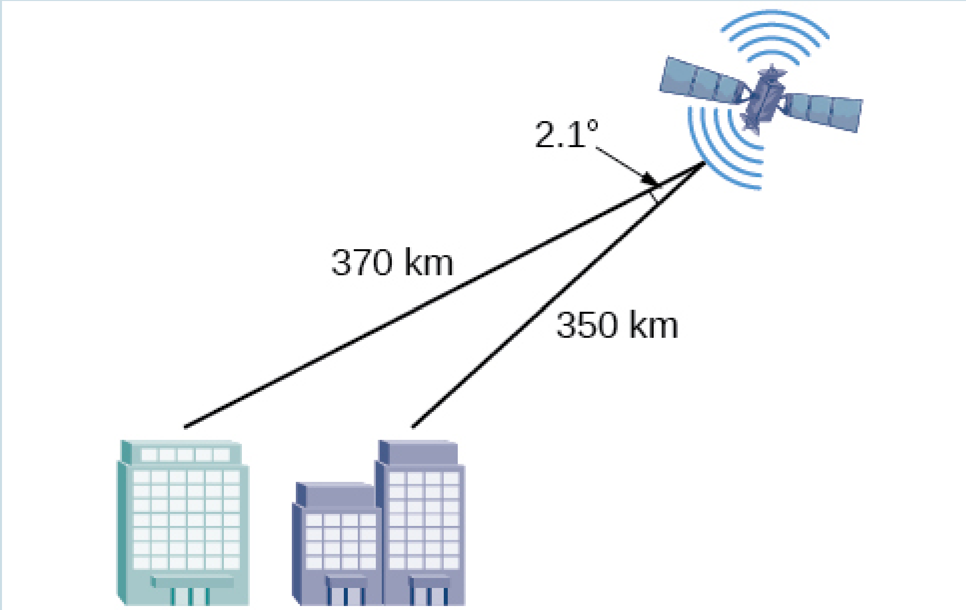


Figure 15.

Show Solution
24.0 km

64. An airplane flies 220 miles with a heading of 40°, and then flies 180 miles with a heading of 170°. How far is the plane from its starting point, and at what heading? Round answers to the nearest tenth.

65. A 113-foot tower is located on a hill that is inclined 34° to the horizontal, as shown in (Figure). A guy-wire is to be attached to the top of the tower and anchored at a point 98 feet uphill from the base of the tower. Find the length of wire needed.

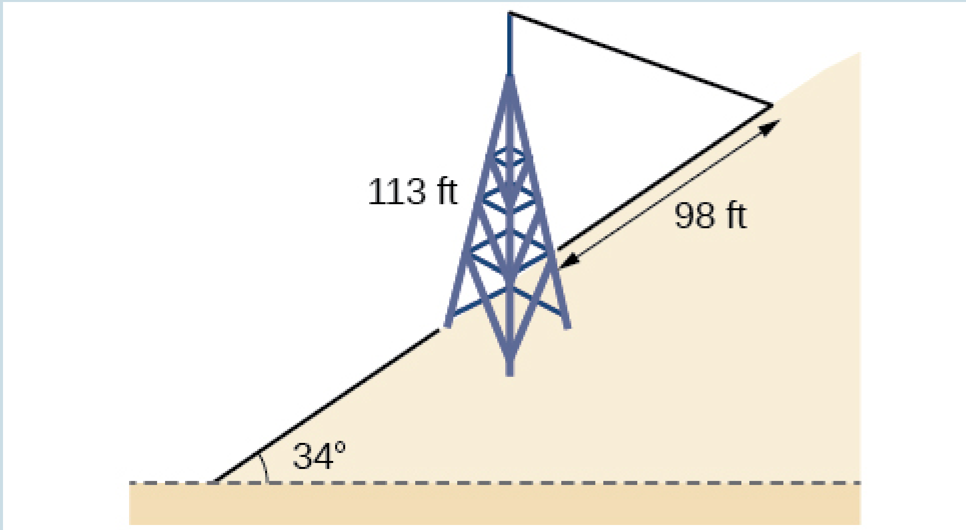


Figure 16.

Show Solution
99.9 ft

66. Two ships left a port at the same time. One ship traveled at a speed of 18 miles per hour at a heading of 320°. The other ship traveled at a speed of 22 miles per hour at a heading of 194°. Find the distance between the two ships after 10 hours of travel.

67. The graph in (Figure) represents two boats departing at the same time from the same dock. The first boat is traveling at 18 miles per hour at a heading of 327° and the second boat is traveling at 4 miles per hour at a heading of 60°. Find the distance between the two boats after 2 hours.

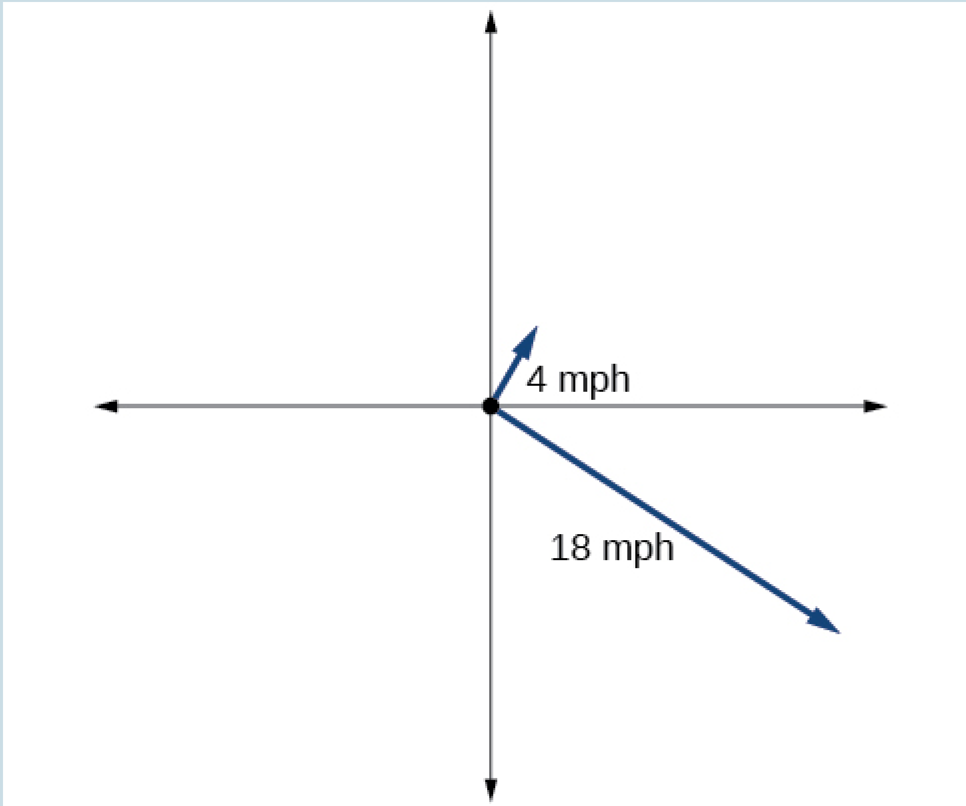


Figure 17.

Show Solution

37.3 miles

68. A triangular swimming pool measures 40 feet on one side and 65 feet on another side. These sides form an angle that measures 50°. How long is the third side (to the nearest tenth)?

69. A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

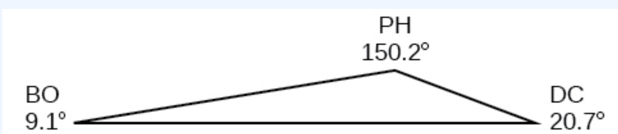
Show Solution

2371 miles

70. Los Angeles is 1,744 miles from Chicago, Chicago is 714 miles from New York, and New York is 2,451 miles from Los Angeles. Draw a triangle connecting these three cities, and find the angles in the triangle.

71. Philadelphia is 140 miles from Washington, D.C., Washington, D.C. is 442 miles from Boston, and Boston is 315 miles from Philadelphia. Draw a triangle connecting these three cities and find the angles in the triangle.

Show Solution



72. Two planes leave the same airport at the same time. One flies at 20° east of north at 500 miles per hour. The second flies at 30° east of south at 600 miles per hour. How far apart are the planes after 2 hours?

73. Two airplanes take off in different directions. One travels 300 mph due west and the other travels 25° north of west at 420 mph. After 90 minutes, how far apart are they, assuming they are flying at the same altitude?

Show Solution
599.8 miles

74. A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.

75. The four sequential sides of a quadrilateral have lengths 4.5 cm, 7.9 cm, 9.4 cm, and 12.9 cm. The angle between the two smallest sides is 117°. What is the area of this quadrilateral?

Show Solution
65.4 cm^2

76. The four sequential sides of a quadrilateral have lengths 5.7 cm, 7.2 cm, 9.4 cm, and 12.8 cm. The angle between the two smallest sides is 106°. What is the area of this quadrilateral?

77. Find the area of a triangular piece of land that measures 30 feet on one side and 42 feet on another; the included angle measures 132°. Round to the nearest whole square foot.

Show Solution
468 ft^2

78. Find the area of a triangular piece of land that measures 110 feet on one side and 250 feet on another; the included angle measures 85°. Round to the nearest whole square foot.

Glossary

Law of Cosines
: states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle

Generalized Pythagorean Theorem
: an extension of the Law of Cosines; relates the sides of an oblique triangle and is used for SAS and SSS triangles

CHAPTER 4.4: POLAR COORDINATES

Learning Objectives

In this section, you will:

- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see (Figure)). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.

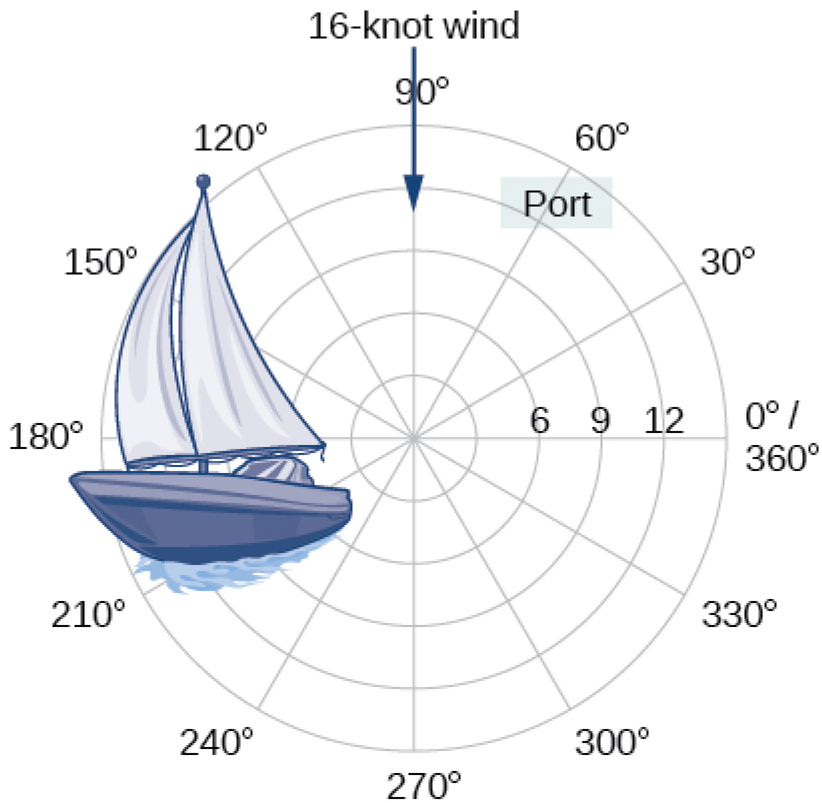


Figure 1.

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of rectangular coordinates (x, y) in the Cartesian coordinate plane. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to polar coordinates, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the pole, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive *x*-axis now viewed as the polar axis and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, measured in radians, indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ, and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the *r*-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$, we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in (Figure).

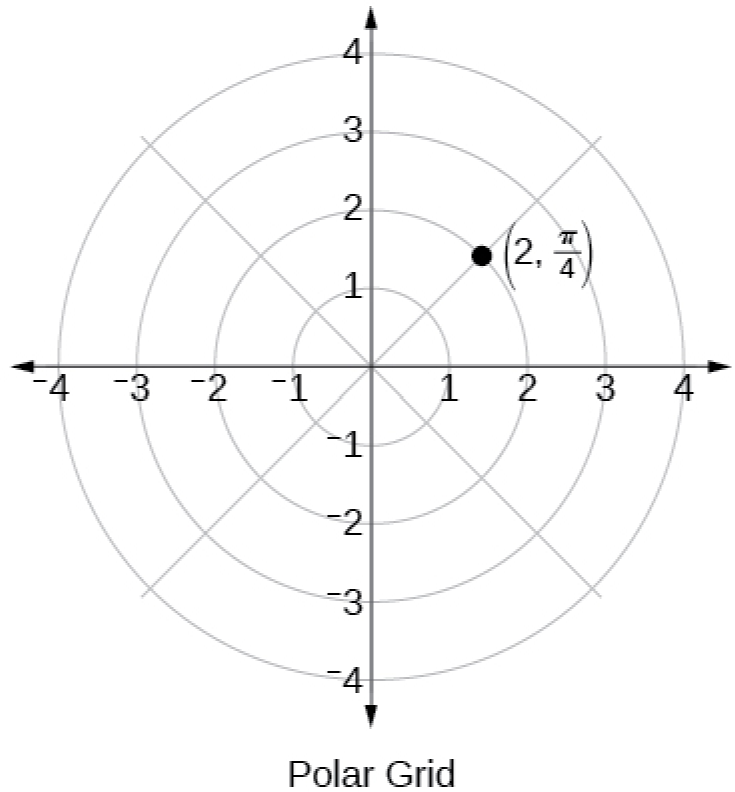


Figure 2.

Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2}\right)$ on the polar grid.

Show Solution

The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in (Figure).

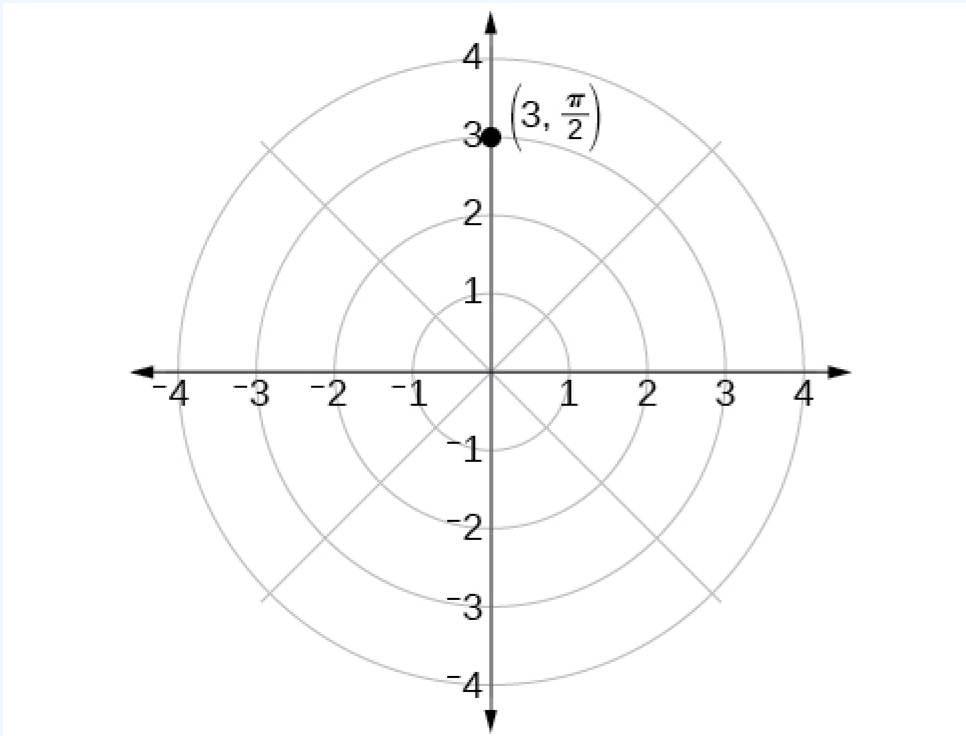
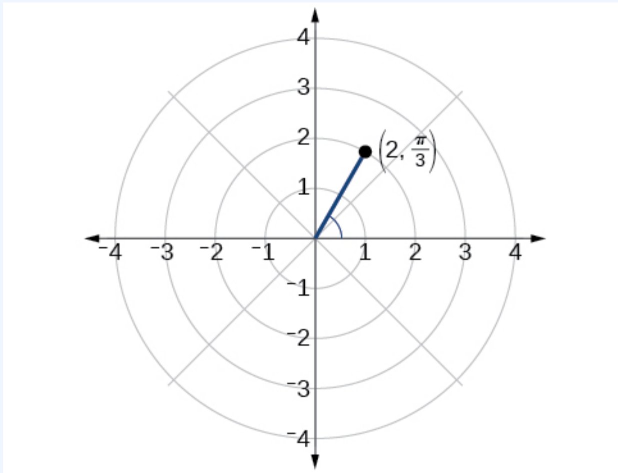


Figure 3.

Try It

Plot the point $\left(2, \frac{\pi}{3}\right)$ in the polar grid.

Show Solution



Plotting a Point in the Polar Coordinate System with a Negative Component

Plot the point $\left(-2, \frac{\pi}{6}\right)$ on the polar grid.

Show Solution

We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2.$ We can approach plotting a point with a negative r in two ways:

1. Plot the point $\left(2, \frac{\pi}{6}\right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;
2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See (Figure)(a). Compare this to the graph of the polar coordinate $\left(2, \frac{\pi}{6}\right)$ shown in (Figure)(b).

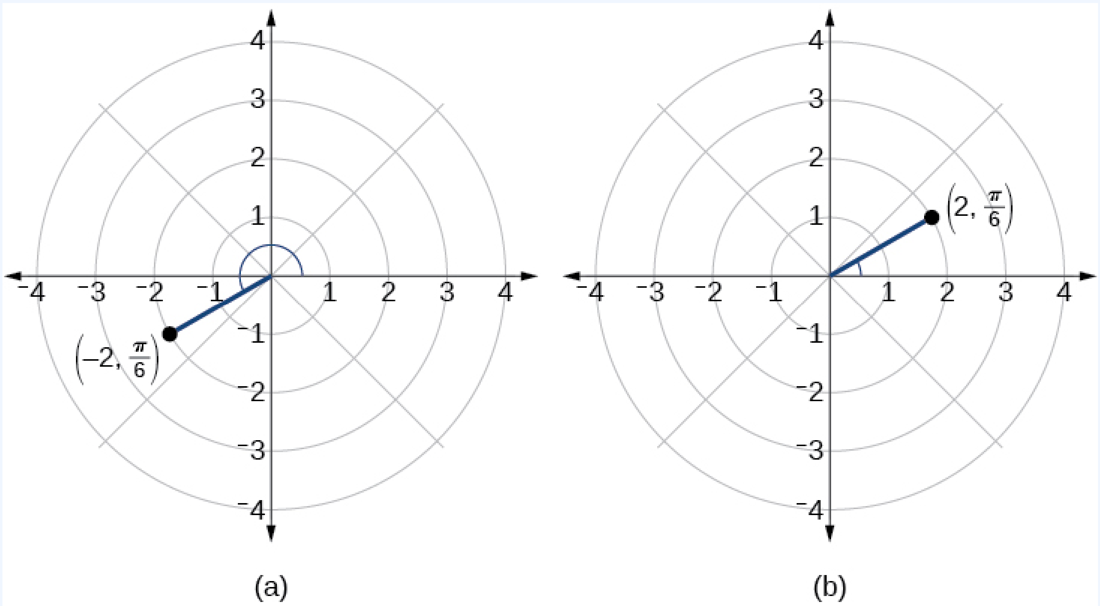
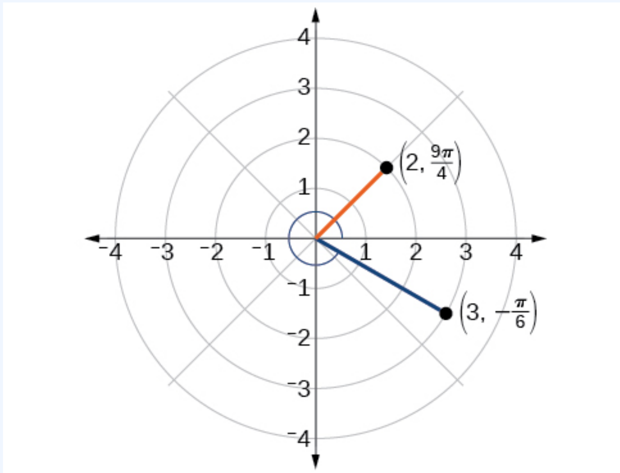


Figure 4.

Try It

Plot the points $\left(3, -\frac{\pi}{6}\right)$ and $\left(2, \frac{9\pi}{4}\right)$ on the same polar grid.

Show Solution



Converting from Polar Coordinates to Rectangular Coordinates

When given a set of polar coordinates, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables $x,\ y,\ r$, and θ.

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$$

Dropping a perpendicular from the point in the plane to the *x*-axis forms a right triangle, as illustrated in (Figure). An easy way to remember the equations above is to think of $\cos\theta$ as the adjacent side over the hypotenuse and $\sin\theta$ as the opposite side over the hypotenuse.

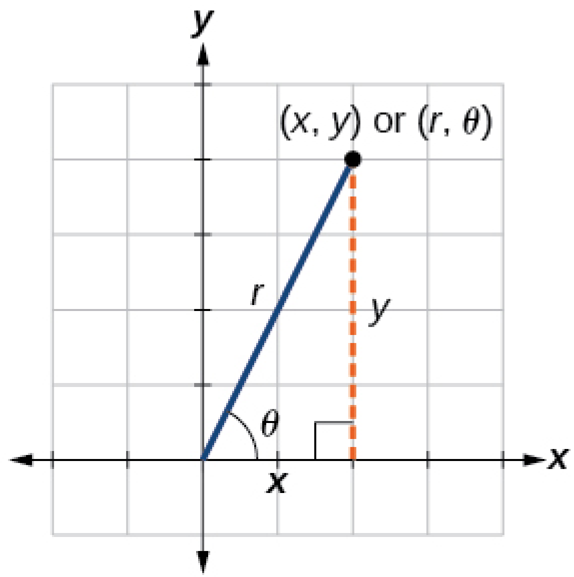


Figure 5.

Converting from Polar Coordinates to Rectangular Coordinates

To convert polar coordinates $(r,\ \theta)$ to rectangular coordinates $(x,\ y)$, let

$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$
$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$

How To

Given polar coordinates, convert to rectangular coordinates.

1. Given the polar coordinate (r, θ) , write $x = r\cos\theta$ and $y = r\sin\theta$.
2. Evaluate $\cos\theta$ and $\sin\theta$.
3. Multiply $\cos\theta$ by r to find the *x*-coordinate of the rectangular form.
4. Multiply $\sin\theta$ by r to find the *y*-coordinate of the rectangular form.

Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $\left(3,\frac{\pi}{2}\right)$ as rectangular coordinates.

Show Solution

Use the equivalent relationships.

$$x = r\cos\theta$$
$$x = 3\cos\frac{\pi}{2} = 0$$
$$y = r\sin\theta$$
$$y = 3\sin\frac{\pi}{2} = 3$$

The rectangular coordinates are $(0,3)$. See (Figure).

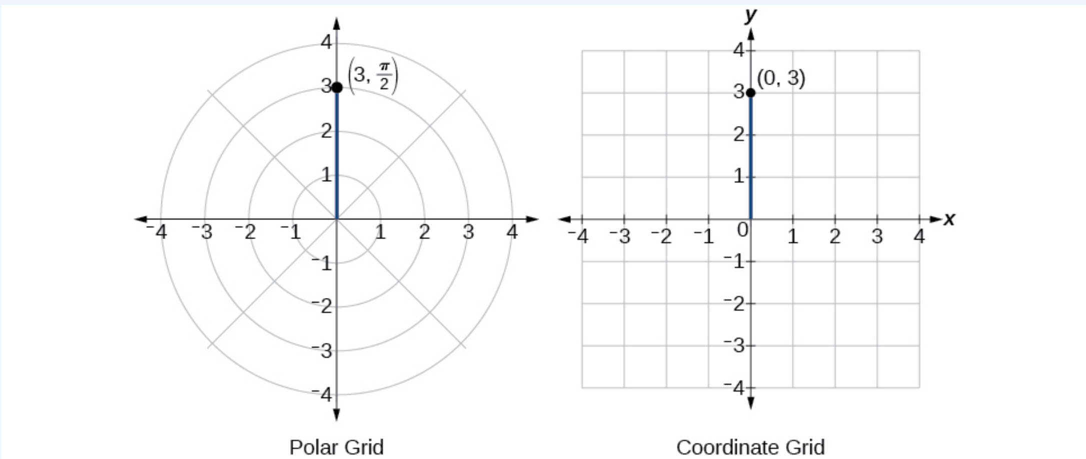


Figure 6.

Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $(-2,0)$ as rectangular coordinates.

Show Solution

See (Figure). Writing the polar coordinates as rectangular, we have

$$x = r\cos\theta$$
$$x = -2\cos(0) = -2$$

$$y = r\sin\theta$$
$$y = -2\sin(0) = 0$$

The rectangular coordinates are also $(-2, 0)$.

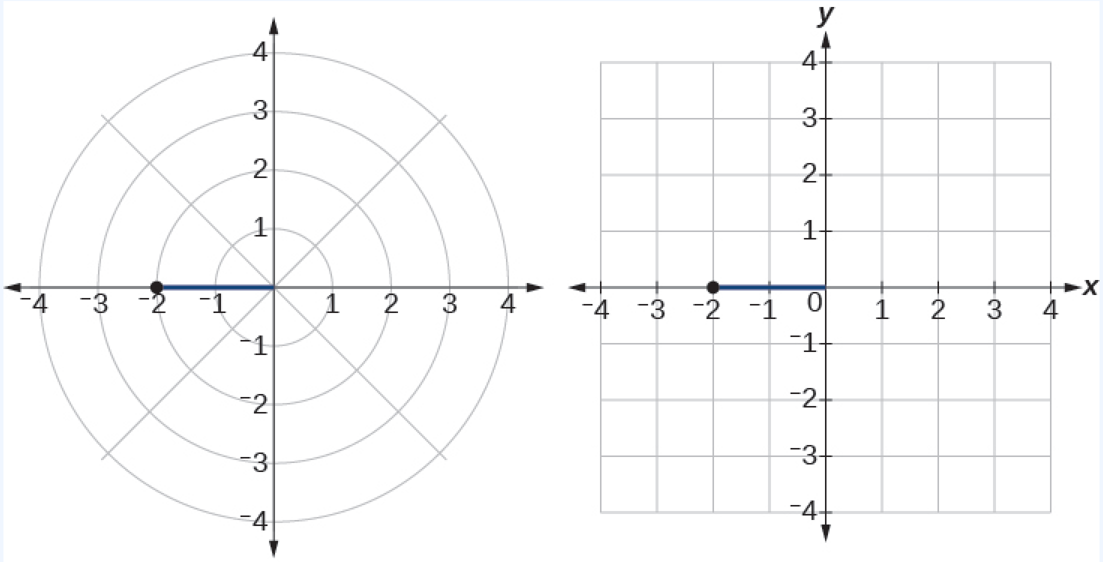


Figure 7.

Try It

Write the polar coordinates $\left(-1, \frac{2\pi}{3}\right)$ as rectangular coordinates.

Show Solution

$(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

Converting from Rectangular Coordinates to Polar Coordinates

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in (Figure).

$$\cos\theta = \frac{x}{r} \quad \text{or} \quad x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \quad \text{or} \quad y = r\sin\theta$$
$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$

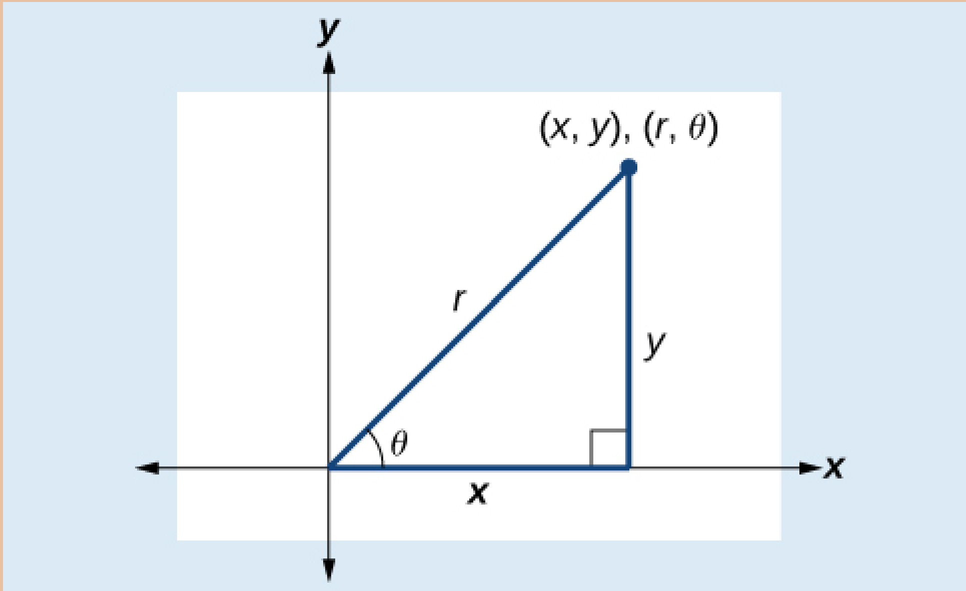


Figure 8.

Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

Show Solution

We see that the original point $(3, 3)$ is in the first quadrant. To find θ, use the formula $\tan\theta = \frac{y}{x}$. This gives

$$\tan\theta = \frac{3}{3}$$
$$\tan\theta = 1$$
$$\tan^{-1}(1) = \frac{\pi}{4}$$

To find r, we substitute the values for x and y into the formula $r = \sqrt{x^2 + y^2}$. We know that r must be positive, as $\frac{\pi}{4}$ is in the first quadrant. Thus

$$r = \sqrt{3^2 + 3^2}$$
$$r = \sqrt{9 + 9}$$
$$r = \sqrt{18} = 3\sqrt{2}$$

So, $r = 3\sqrt{2}$ and $\theta = \frac{\pi}{4}$, giving us the polar point $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. See (Figure).

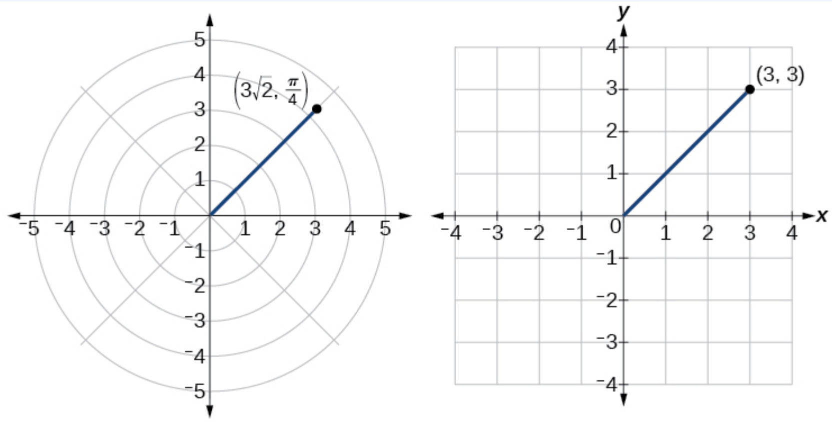


Figure 9.

Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ and $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ will coincide with the original solution of $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. The point $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ indicates a move further counterclockwise by π, which is directly opposite $\frac{\pi}{4}$. The radius is expressed as $-3\sqrt{2}$. However, the angle $\frac{5\pi}{4}$ is located in the third quadrant and, as r is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. The point $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ is a move further clockwise by $-\frac{7\pi}{4}$, from $\frac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How To

Given an equation in polar form, graph it using a graphing calculator.

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: $r_1,\ r_2,\ .\ .\ .\ ,\ r_6.$
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Writing a Cartesian Equation in Polar Form

Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

Show Solution

The goal is to eliminate x and y from the equation and introduce r and $\theta.$ Ideally, we would write the equation r as a function of $\theta.$ To obtain the polar form, we will use the relationships between (x, y) and (r, θ) . Since $x = r\cos\theta$ and $y = r\sin\theta,$ we can substitute and solve for $r.$

$$\begin{array}{ll}
(r\cos\theta)^2 + (r\sin\theta)^2 = 9 & \\
r^2\cos^2\theta + r^2\sin^2\theta = 9 & \\
r^2\left(\cos^2\theta + \sin^2\theta\right) = 9 & \\
r^2\,(1) = 9 & \text{Substitute } \cos^2\theta + \sin^2\theta = 1. \\
r = 3 & \text{Use the square root property.}
\end{array}$$

Thus, $x^2 + y^2 = 9, r = 3,$ and $r = -3$ should generate the same graph. See (Figure).

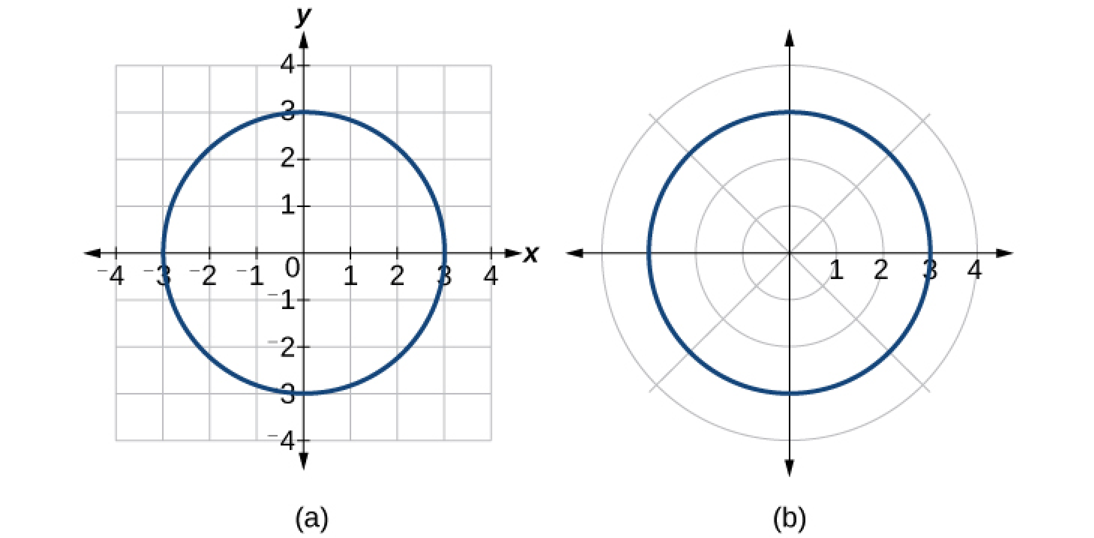


Figure 10. (a) Cartesian form $x^2 + y^2 = 9$ (b) Polar form $r = 3$

To graph a circle in rectangular form, we must first solve for y.

$$x^2 + y^2 = 9$$
$$y^2 = 9 - x^2$$
$$y = \sqrt{9 - x^2}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1 = \sqrt{9 - x^2}$ and $Y_2 = -\sqrt{9 - x^2}$. Press **GRAPH.**

Rewriting a Cartesian Equation as a Polar Equation

Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

Show Solution

This equation appears similar to the previous example, but it requires different steps to convert the equation.

We can still follow the same procedures we have already learned and make the following substitutions:

$$\begin{array}{ll} r^2 = 6y & \text{Use } x^2+y^2=r^2. \\ r^2 = 6r\sin\theta & \text{Substitute } y = r\sin\theta. \\ r^2 - 6r\sin\theta = 0 & \text{Set equal to 0.} \\ r(r - 6\sin\theta) = 0 & \text{Factor and solve.} \\ r = 0 & \text{We reject } r=0\text{, as it only represents one point, } (0,0). \\ \text{or } r = 6\sin\theta & \end{array}$$

Therefore, the equations $x^2 + y^2 = 6y$ and $r = 6\sin\theta$ should give us the same graph. See (Figure).

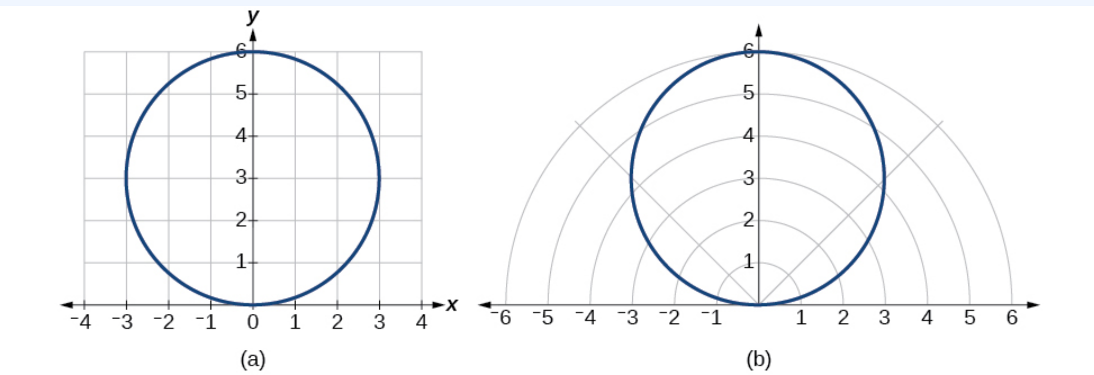


Figure 11. (a) Cartesian form $x^2 + y^2 = 6y$ (b) polar form $r = 6\sin\theta$

The Cartesian or rectangular equation is plotted on the rectangular grid, and the polar equation is plotted on the polar grid. Clearly, the graphs are identical.

Rewriting a Cartesian Equation in Polar Form

Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

Show Solution

We will use the relationships $x = r\cos\theta$ and $y = r\sin\theta$.

$$
\begin{array}{ll}
y = 3x + 2 & \\
r\sin\theta = 3r\cos\theta + 2 & \\
r\sin\theta - 3r\cos\theta = 2 & \\
r\left(\sin\theta - 3\cos\theta\right) = 2 & \text{Isolate } r. \\
r = \frac{2}{\sin\theta - 3\cos\theta} & \text{Solve for } r.
\end{array}
$$

Try It

Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

Show Solution

$r = \sqrt{3}$

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Graphing a Polar Equation by Converting to a Rectangular Equation

Covert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

Show Solution

The conversion is

$$\begin{aligned} r &= 2\sec\theta \\ r &= \frac{2}{\cos\theta} \\ r\cos\theta &= 2 \\ x &= 2 \end{aligned}$$

Notice that the equation $r = 2\sec\theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see (Figure)). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\theta$ is the standard form for a vertical line in polar form.

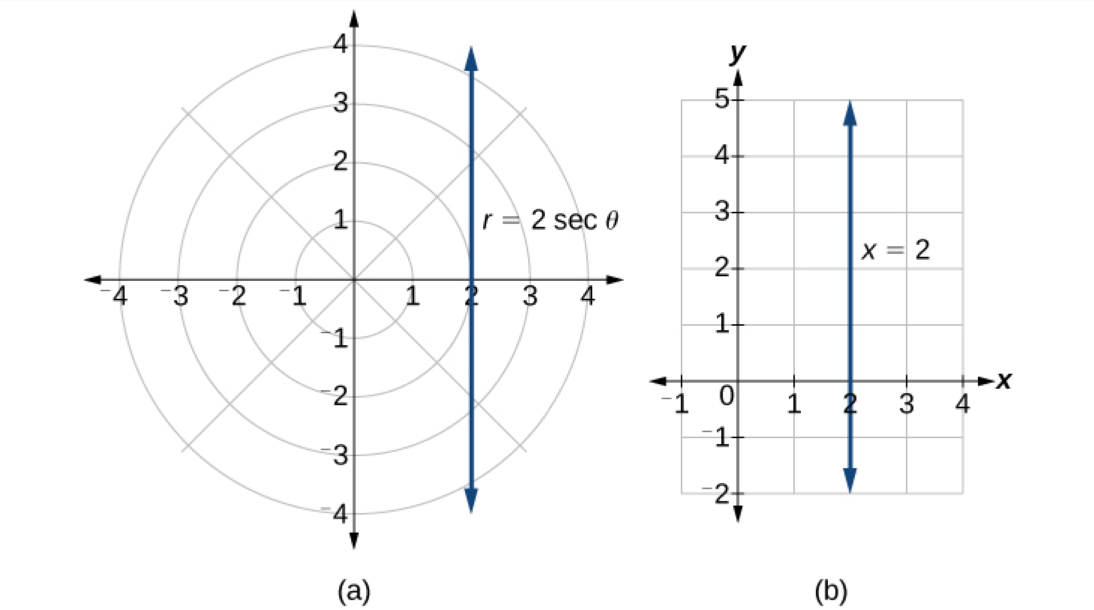


Figure 12. (a) Polar grid (b) Rectangular coordinate system

A similar discussion would demonstrate that the graph of the function $r = 2\csc\theta$ will be the horizontal line $y = 2$. In fact, $r = c\csc\theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \frac{3}{1-2\cos\theta}$ as a Cartesian equation.

Show Solution

The goal is to eliminate θ and r, and introduce x and y. We clear the fraction, and then use substitution. In order to replace r with x and y, we must use the expression $x^2 + y^2 = r^2$.

$$\begin{array}{ll} r = \frac{3}{1-2\cos\theta} & \\ r\left(1 - 2\cos\theta\right) = 3 & \\ r\left(1 - 2\left(\frac{x}{r}\right)\right) = 3 & \text{Use } \cos\theta = \frac{x}{r} \text{ to eliminate } \theta. \\ r - 2x = 3 & \\ r = 3 + 2x & \text{Isolate } r. \\ r^2 = (3 + 2x)^2 & \text{Square both sides.} \\ x^2 + y^2 = (3 + 2x)^2 & \text{Use } x^2 + y^2 = r^2. \end{array}$$

The Cartesian equation is $x^2 + y^2 = (3 + 2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate y.

$$\begin{array}{l} x^2 + y^2 = (3 + 2x)^2 \\ y^2 = (3 + 2x)^2 - x^2 \\ y = \sqrt{(3 + 2x)^2 - x^2} \end{array}$$

When our entire equation has been changed from r and θ to x and y, we can stop, unless asked to solve for y or simplify. See (Figure).

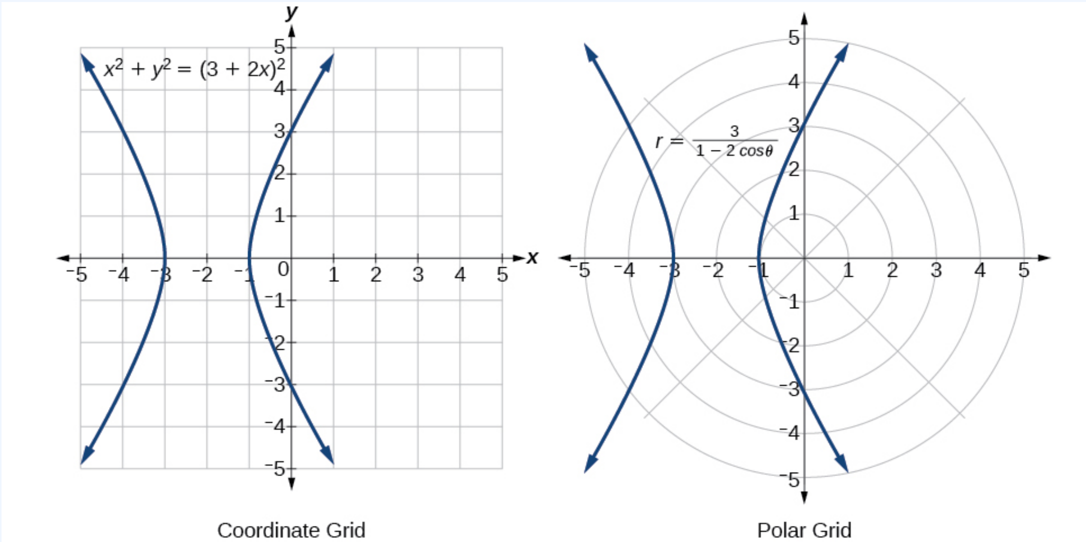


Figure 13.

The “hour-glass” shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in Analytic Geometry.

Analysis

In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola’s standard form. To do this, we can start with the initial equation.

$$\begin{array}{ll} x^2+y^2=(3+2x)^2 & \\ x^2+y^2-(3+2x)^2=0 & \\ x^2+y^2-\left(9+12x+4x^2\right)=0 & \\ x^2+y^2-9-12x-4x^2=0 & \\ -3x^2-12x+y^2=9 & \text{Multiply through by } -1. \\ 3x^2+12x-y^2=-9 & \\ 3\left(x^2+4x+\quad\right)-y^2=-9 & \text{Organize terms to complete the square for } x. \\ 3\left(x^2+4x+4\right)-y^2=-9+12 & \\ 3(x+2)^2-y^2=3 & \\ (x+2)^2-\frac{y^2}{3}=1 & \end{array}$$

Try It

Rewrite the polar equation $r = 2\sin\theta$ in Cartesian form.

Show Solution

$x^2 + y^2 = 2y$ or, in the standard form for a circle, $x^2 + (y-1)^2 = 1$

Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

Show Solution

$$\begin{array}{ll} r = \sin(2\theta) & \text{Use the double angle identity for sine.} \\ r = 2\sin\theta\cos\theta & \text{Use } \cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}. \\ r = 2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right) & \text{Simplify.} \\ r = \frac{2xy}{r^2} & \text{Multiply both sides by } r^2. \\ r^3 = 2xy & \\ \left(\sqrt{x^2+y^2}\right)^3 = 2xy & \text{As } x^2+y^2 = r^2, r = \sqrt{x^2+y^2}. \end{array}$$

This equation can also be written as

$$\left(x^2+y^2\right)^{\frac{3}{2}} = 2xy \text{ or } x^2+y^2 = (2xy)^{\frac{2}{3}}$$

Access these online resources for additional instruction and practice with polar coordinates.

- Introduction to Polar Coordinates
- Comparing Polar and Rectangular Coordinates

Key Equations

Conversion formulas	$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$ $\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$ $r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

Key Concepts

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form (r, θ) , $\theta > 0$, move in a counterclockwise direction from the polar axis by an angle of θ, and then extend a directed line segment from the pole the length of r in the direction of θ. If θ is negative, move in a clockwise direction, and extend a directed line segment the length of r in the direction of θ. See (Figure).
- If r is negative, extend the directed line segment in the opposite direction of θ. See (Figure).
- To convert from polar coordinates to rectangular coordinates, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$. See (Figure) and (Figure).
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $\tan\theta = \frac{y}{x}$, and $r = \sqrt{x^2 + y^2}$. See (Figure).
- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See (Figure), (Figure), and (Figure).
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See (Figure), (Figure), and (Figure).

Section Exercises

Verbal

1. How are polar coordinates different from rectangular coordinates?

Show Solution

For polar coordinates, the point in the plane depends on the angle from the positive *x*-axis and distance from the origin, while in Cartesian coordinates, the point represents the horizontal and vertical distances from the origin. For each point in the coordinate plane, there is one representation, but for each point in the polar plane, there are infinite representations.

2. How are the polar axes different from the *x*– and *y*-axes of the Cartesian plane?

3. Explain how polar coordinates are graphed.

Show Solution

Determine θ for the point, then move r units from the pole to plot the point. If r is negative, move r units from the pole in the opposite direction but along the same angle. The point is a distance of r away from the origin at an angle of θ from the polar axis.

4. How are the points $\left(3, \frac{\pi}{2}\right)$ and $\left(-3, \frac{\pi}{2}\right)$ related?

5. Explain why the points $\left(-3, \frac{\pi}{2}\right)$ and $\left(3, -\frac{\pi}{2}\right)$ are the same.

Show Solution

The point $\left(-3, \frac{\pi}{2}\right)$ has a positive angle but a negative radius and is plotted by moving to an angle of $\frac{\pi}{2}$ and then moving 3 units in the negative direction. This places the point 3 units down the negative *y*-axis. The point $\left(3, -\frac{\pi}{2}\right)$ has a negative angle and a positive radius and is plotted by first moving to an angle of $-\frac{\pi}{2}$ and then moving 3 units down, which is the positive direction for a negative angle. The point is also 3 units down the negative *y*-axis.

Algebraic

For the following exercises, convert the given polar coordinates to Cartesian coordinates with

$r > 0$ and $0 \leq \theta \leq 2\pi.$ Remember to consider the quadrant in which the given point is located when determining θ for the point.

6. $\left(7, \frac{7\pi}{6}\right)$

7. $(5, \pi)$

Show Solution

$(-5, 0)$

8. $\left(6, -\frac{\pi}{4}\right)$

9. $\left(-3, \frac{\pi}{6}\right)$

Show Solution

$\left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

10. $\left(4, \frac{7\pi}{4}\right)$

For the following exercises, convert the given Cartesian coordinates to polar coordinates with $r > 0,\ 0 \leq \theta < 2\pi.$ Remember to consider the quadrant in which the given point is located.

11. $(4, 2)$

Show Solution

$(2\sqrt{5}, 0.464)$

12. $(-4, 6)$

13. $(3, -5)$

Show Solution
$\left(\sqrt{34}, 5.253\right)$

14. $(-10, -13)$

15. $(8, 8)$

Show Solution
$\left(8\sqrt{2}, \frac{\pi}{4}\right)$

For the following exercises, convert the given Cartesian equation to a polar equation.

16. $x = 3$

17. $y = 4$

Show Solution
$r = 4\csc\theta$

18. $y = 4x^2$

19. $y = 2x^4$

Show Solution
$r = \sqrt[3]{\frac{sin\theta}{2cos^4\theta}}$

20. $x^2 + y^2 = 4y$

21. $x^2 + y^2 = 3x$

Show Solution
$r = 3\cos\theta$

22. $x^2 - y^2 = x$

23. $x^2 - y^2 = 3y$

Show Solution
$r = \frac{3\sin\theta}{\cos(2\theta)}$

24. $x^2 + y^2 = 9$

25. $x^2 = 9y$

Show Solution
$r = \frac{9\sin\theta}{\cos^2\theta}$

26. $y^2 = 9x$

27. $9xy = 1$

Show Solution
$r = \sqrt{\frac{1}{9\cos\theta\sin\theta}}$

For the following exercises, convert the given polar equation to a Cartesian equation. Write in the standard form of a conic if possible, and identify the conic section represented.

28. $r = 3\sin\theta$

29. $r = 4\cos\theta$

Show Solution

$x^2 + y^2 = 4x$ or $\frac{(x-2)^2}{4} + \frac{y^2}{4} = 1$; circle

30. $r = \frac{4}{\sin\theta + 7\cos\theta}$

31. $r = \frac{6}{\cos\theta + 3\sin\theta}$

Show Solution

$3y + x = 6$; line

32. $r = 2\sec\theta$

33. $r = 3\csc\theta$

Show Solution

$y = 3$; line

34. $r = \sqrt{r\cos\theta + 2}$

35. $r^2 = 4\sec\theta\csc\theta$

Show Solution

$xy = 4$; hyperbola

36. $r = 4$

37. $r^2 = 4$

Show Solution

$x^2 + y^2 = 4$; circle

38. $r = \frac{1}{4\cos\theta - 3\sin\theta}$

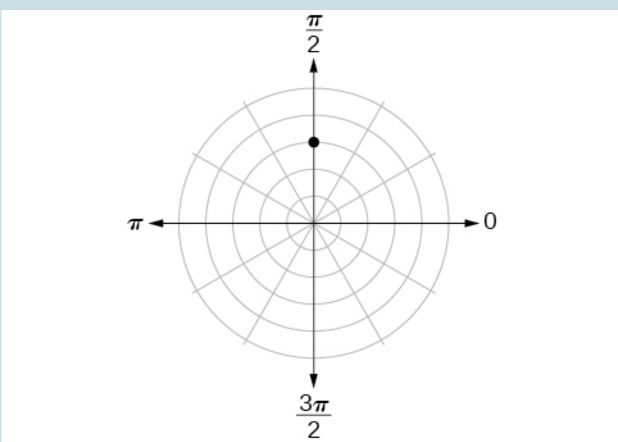
39. $r = \frac{3}{\cos\theta - 5\sin\theta}$

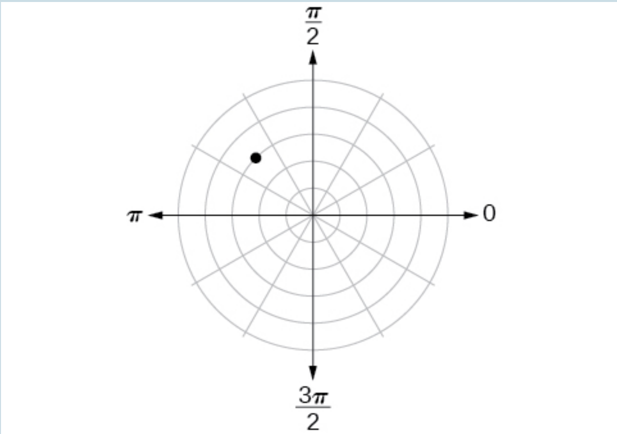
Show Solution

$x - 5y = 3$; line

Graphical

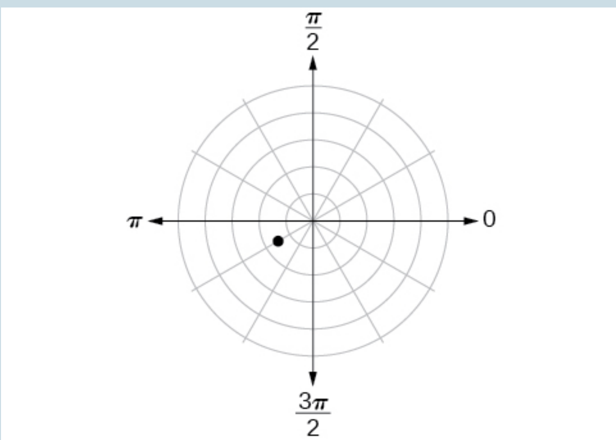
For the following exercises, find the polar coordinates of the point.

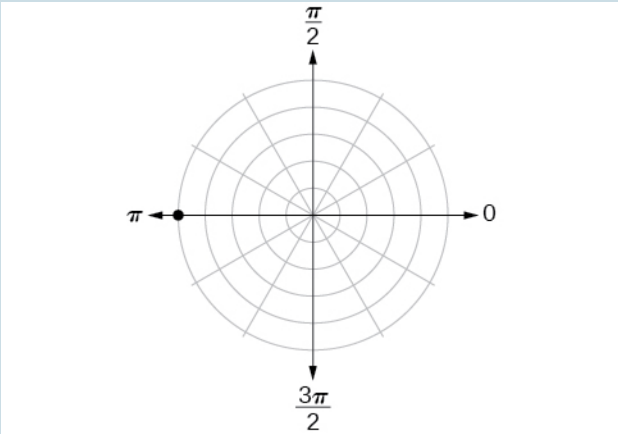
40. 

41. 

Show Solution

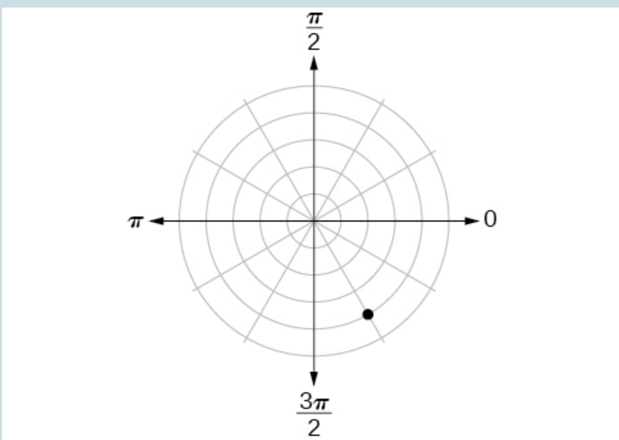
$\left(3, \frac{3\pi}{4}\right)$

42. 

43. 

Show Solution

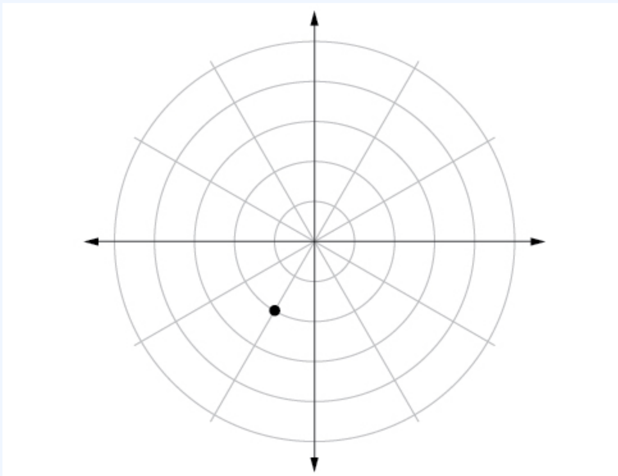
$(5, \pi)$

44. 

For the following exercises, plot the points.

45. $\left(-2, \frac{\pi}{3}\right)$

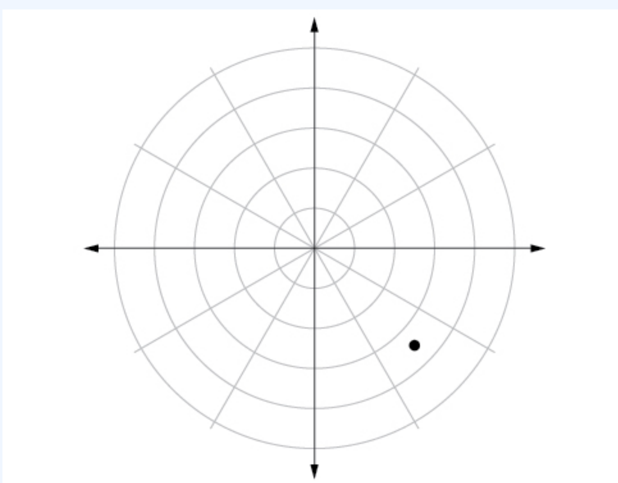
Show Solution



46. $\left(-1, -\frac{\pi}{2}\right)$

47. $\left(3.5, \frac{7\pi}{4}\right)$

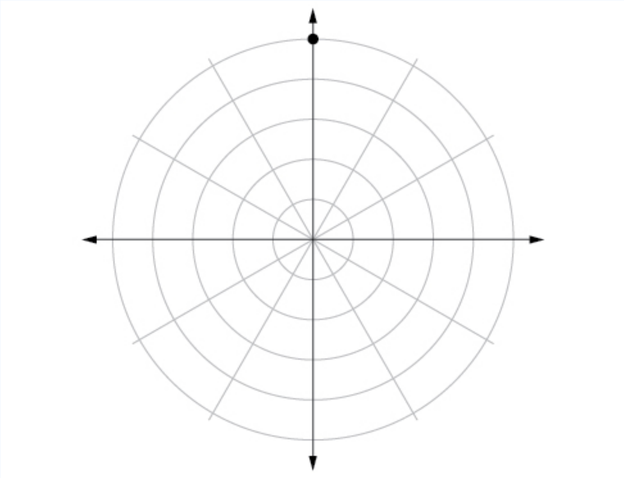
Show Solution



48. $\left(-4, \frac{\pi}{3}\right)$

49. $\left(5, \frac{\pi}{2}\right)$

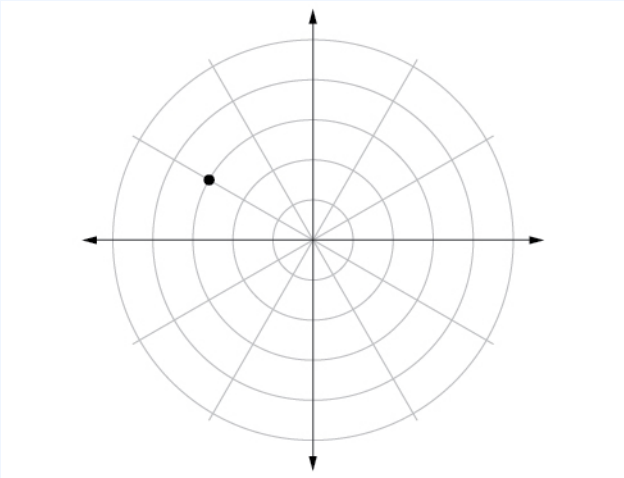
Show Solution



50. $\left(4, \frac{-5\pi}{4}\right)$

51. $\left(3, \frac{5\pi}{6}\right)$

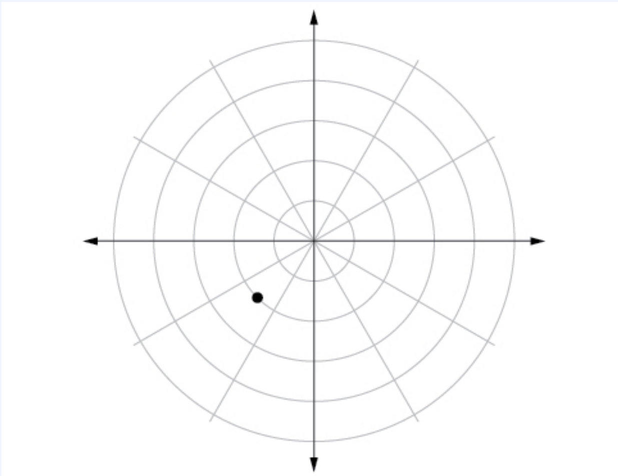
Show Solution



52. $\left(-1.5, \frac{7\pi}{6}\right)$

53. $\left(-2, \frac{\pi}{4}\right)$

Show Solution



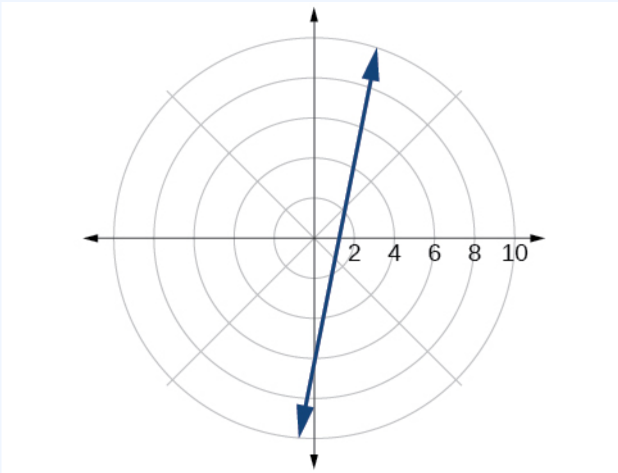
54. $\left(1, \frac{3\pi}{2}\right)$

For the following exercises, convert the equation from rectangular to polar form and graph on the polar axis.

55. $5x - y = 6$

Show Solution

$r = \frac{6}{5\cos\theta - \sin\theta}$

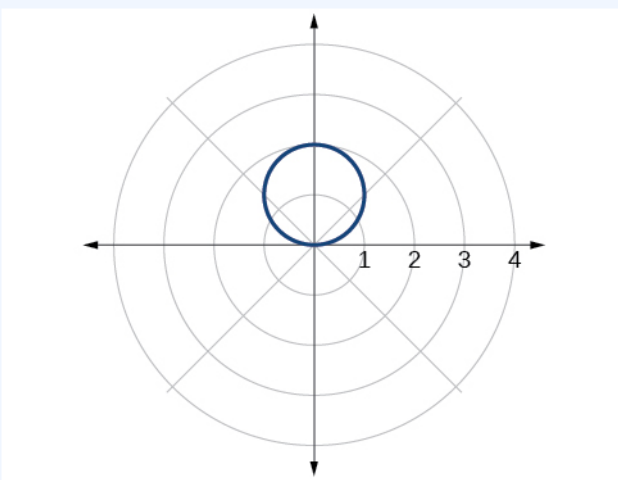


56. $2x + 7y = -3$

57. $x^2 + (y-1)^2 = 1$

Show Solution

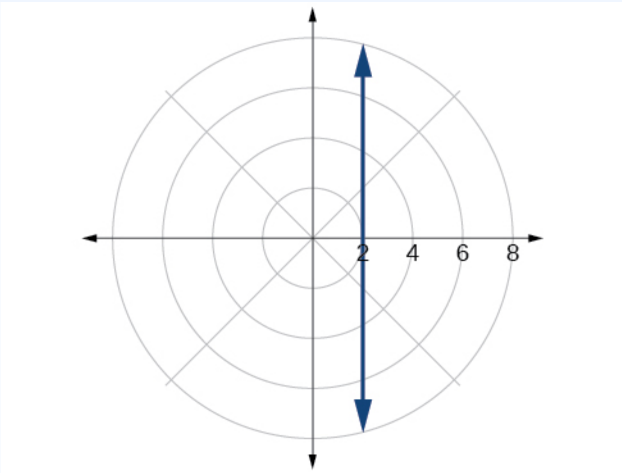
$r = 2\sin\theta$



58. $(x+2)^2 + (y+3)^2 = 13$

59. $x = 2$

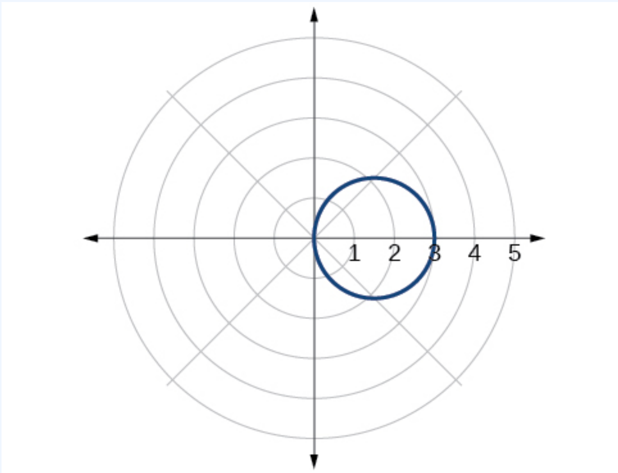
Show Solution
$r = \frac{2}{\cos\theta}$



60. $x^2 + y^2 = 5y$

61. $x^2 + y^2 = 3x$

Show Solution
$r = 3\cos\theta$

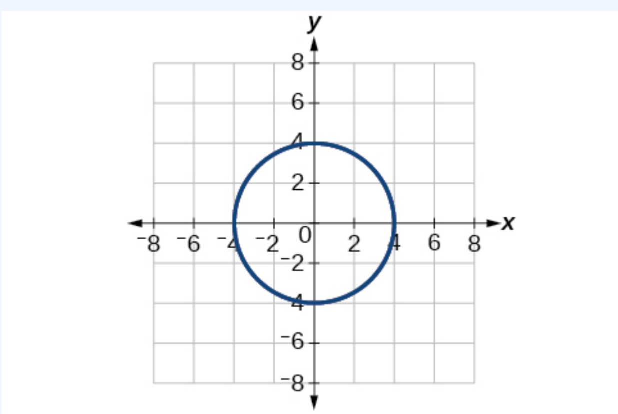


For the following exercises, convert the equation from polar to rectangular form and graph on the rectangular plane.

62. $r = 6$

63. $r = -4$

Show Solution
$x^2 + y^2 = 16$

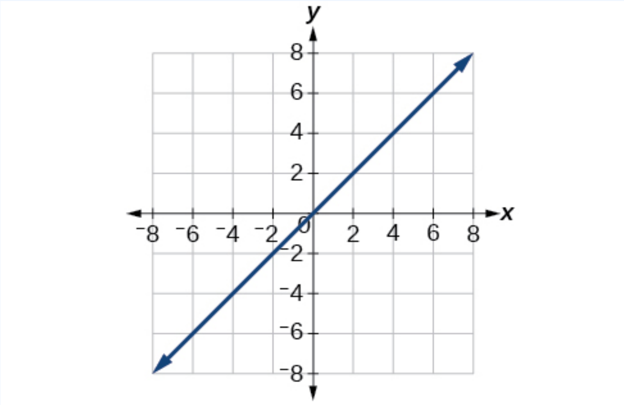


64. $\theta = -\frac{2\pi}{3}$

65. $\theta = \frac{\pi}{4}$

Show Solution

$y = x$

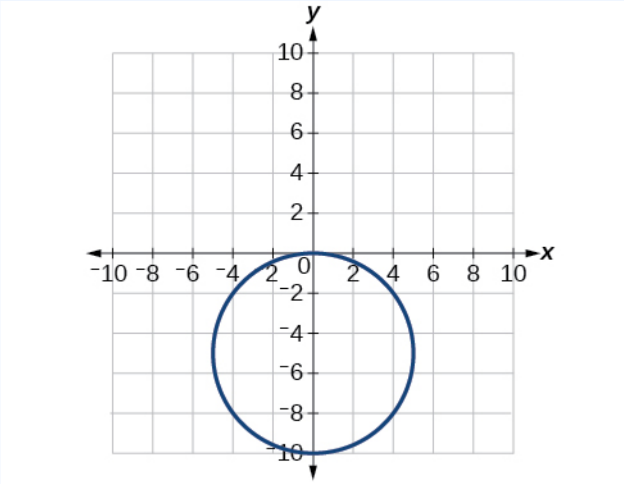


66. $r = \sec\theta$

67. $r = -10\sin\theta$

Show Solution

$x^2 + (y+5)^2 = 25$



68. $r = 3\cos\theta$

Technology

69. Use a graphing calculator to find the rectangular coordinates of $\left(2, -\frac{\pi}{5}\right)$. Round to the nearest thousandth.

Show Solution
$(1.618, -1.176)$

70. Use a graphing calculator to find the rectangular coordinates of $\left(-3, \frac{3\pi}{7}\right)$. Round to the nearest thousandth.

71. Use a graphing calculator to find the polar coordinates of $(-7, 8)$ in degrees. Round to the nearest thousandth.

Show Solution
$(10.630, 131.186)$

72. Use a graphing calculator to find the polar coordinates of $(3, -4)$ in degrees. Round to the nearest hundredth.

73. Use a graphing calculator to find the polar coordinates of $(-2, 0)$ in radians. Round to the nearest hundredth.

Show Solution
$(2, 3.14)\, or\, (2, \pi)$

Extensions

74. Describe the graph of $r = a\sec\theta; a > 0.$

75. Describe the graph of $r = a\sec\theta; a < 0.$

Show Solution

A vertical line with a units left of the *y*-axis.

76. Describe the graph of $r = a\csc\theta; a > 0.$

77. Describe the graph of $r = a\csc\theta; a < 0.$

Show Solution

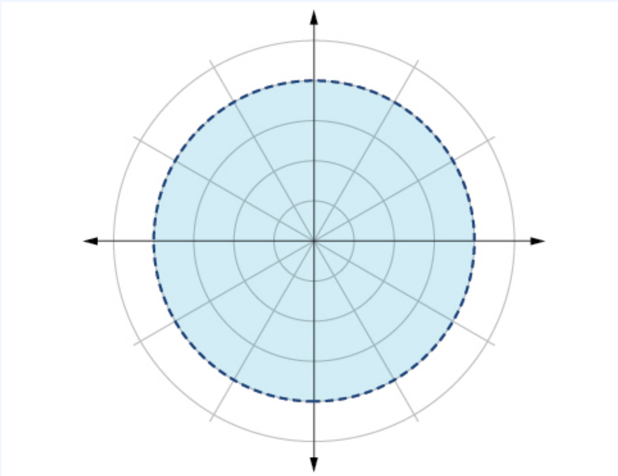
A horizontal line with a units below the *x*-axis.

78. What polar equations will give an oblique line?

For the following exercise, graph the polar inequality.

79. $r < 4$

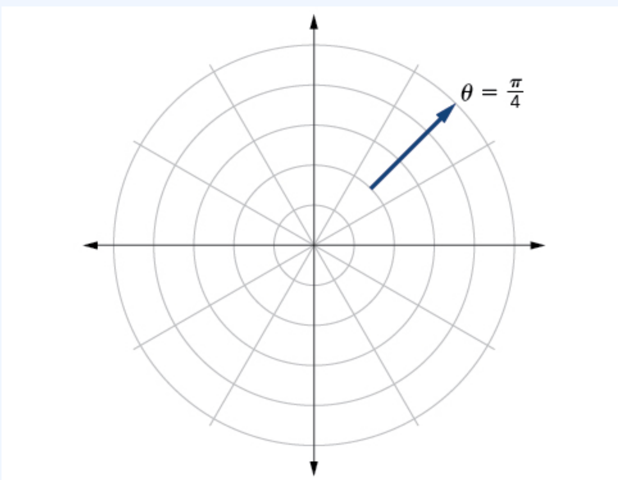
Show Solution



80. $0 \leq \theta \leq \frac{\pi}{4}$

81. $\theta = \frac{\pi}{4},\ r\ \geq\ 2$

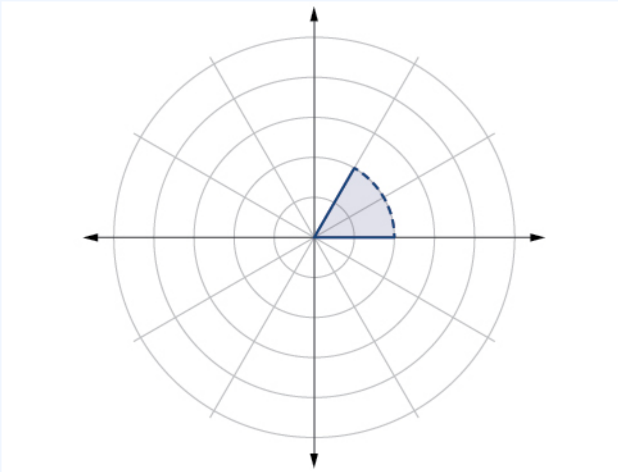
Show Solution



82. $\theta = \frac{\pi}{4},\ r\ \geq -3$

83. $0 \leq \theta \leq \frac{\pi}{3},\ r\ <\ 2$

Show Solution



84. $\frac{-\pi}{6} < \theta \leq \frac{\pi}{3}, -3 < r < 2$

Glossary

polar axis
: on the polar grid, the equivalent of the positive *x*-axis on the rectangular grid

polar coordinates
: on the polar grid, the coordinates of a point labeled (r, θ) , where θ indicates the angle of rotation from the polar axis and r represents the radius, or the distance of the point from the pole in the direction of θ

pole
: the origin of the polar grid

CHAPTER 4.5: POLAR COORDINATES: GRAPHS

Learning Objectives

In this section you will:

- Test polar equations for symmetry.
- Graph polar equations by plotting points.

The planets move through space in elliptical, periodic orbits about the sun, as shown in (Figure). They are in constant motion, so fixing an exact position of any planet is valid only for a moment. In other words, we can fix only a planet's *instantaneous* position. This is one application of polar coordinates, represented as (r, θ) . We interpret r as the distance from the sun and θ as the planet's angular bearing, or its direction from a fixed point on the sun. In this section, we will focus on the polar system and the graphs that are generated directly from polar coordinates.

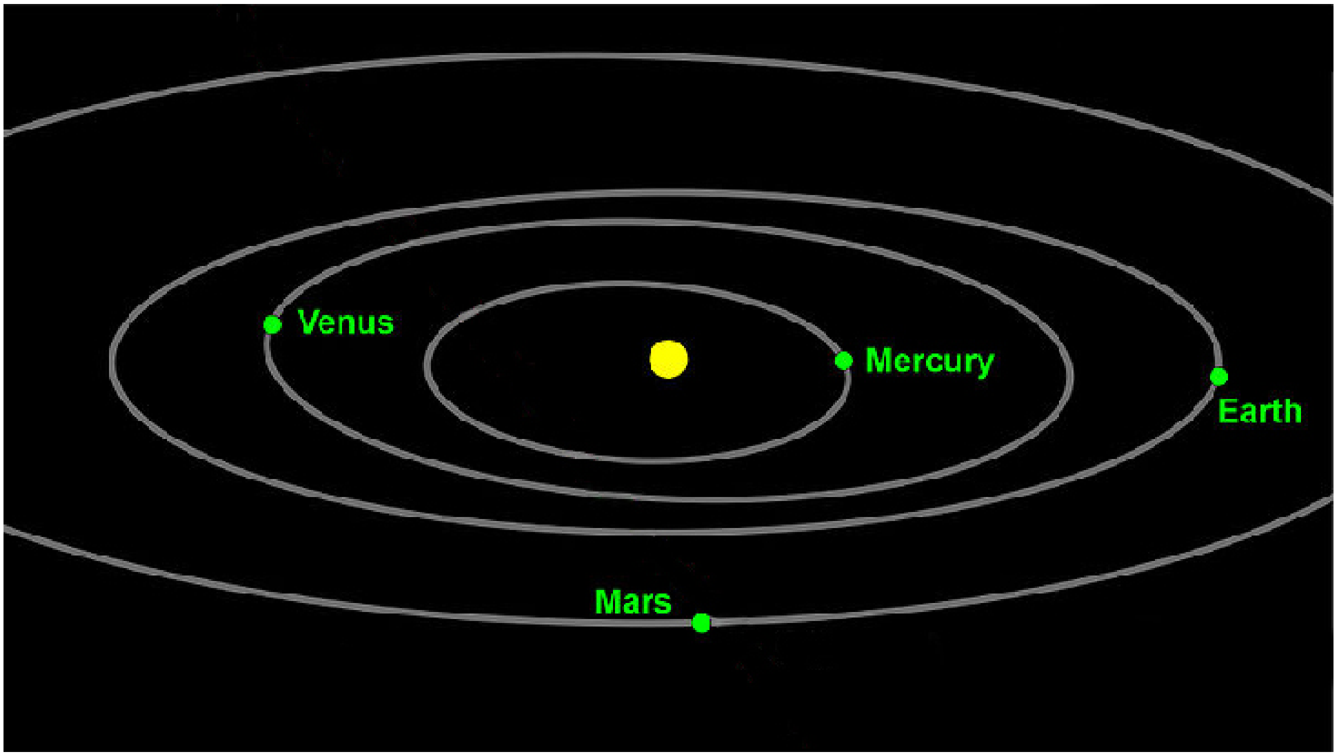


Figure 1. Planets follow elliptical paths as they orbit around the Sun. (credit: modification of work by NASA/ JPL-Caltech)

Testing Polar Equations for Symmetry

Just as a rectangular equation such as $y = x^2$ describes the relationship between x and y on a Cartesian grid, a **polar equation** describes a relationship between r and θ on a polar grid. Recall that the coordinate pair (r, θ) indicates that we move counterclockwise from the polar axis (positive *x*-axis) by an angle of θ, and extend a ray from the pole (origin) r units in the direction of θ. All points that satisfy the polar equation are on the graph.

Symmetry is a property that helps us recognize and plot the graph of any equation. If an equation has a graph that is symmetric with respect to an axis, it means that if we folded the graph in half over that axis, the portion of the graph on one side would coincide with the portion on the other side. By performing three tests, we will see how to apply the properties of symmetry to polar equations. Further, we will use symmetry (in addition to plotting key points, zeros, and maximums of r) to determine the graph of a polar equation.

In the first test, we consider symmetry with respect to the line $\theta = \frac{\pi}{2}$ (*y*-axis). We replace (r, θ) with $(-r, -\theta)$ to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin\theta$;

$$\begin{aligned} r &= 2\sin\theta \\ -r &= 2\sin(-\theta) && \text{Replace } (r,\theta) \text{ with } (-r,-\theta). \\ -r &= -2\sin\theta && \text{Identity: } \sin(-\theta) = -\sin\theta. \\ r &= 2\sin\theta && \text{Multiply both sides by } -1. \end{aligned}$$

This equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$.

In the second test, we consider symmetry with respect to the polar axis (x -axis). We replace (r,θ) with $(r,-\theta)$ or $(-r,\pi-\theta)$ to determine equivalency between the tested equation and the original. For example, suppose we are given the equation $r = 1 - 2\cos\theta$.

$$\begin{aligned} r &= 1 - 2\cos\theta \\ r &= 1 - 2\cos(-\theta) && \text{Replace } (r,\theta) \text{ with } (r,-\theta). \\ r &= 1 - 2\cos\theta && \text{Even/Odd identity} \end{aligned}$$

The graph of this equation exhibits symmetry with respect to the polar axis.

In the third test, we consider symmetry with respect to the pole (origin). We replace (r,θ) with $(-r,\theta)$ to determine if the tested equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin(3\theta)$.

$$\begin{aligned} r &= 2\sin(3\theta) \\ -r &= 2\sin(3\theta) \end{aligned}$$

The equation has failed the symmetry test, but that does not mean that it is not symmetric with respect to the pole. Passing one or more of the symmetry tests verifies that symmetry will be exhibited in a graph. However, failing the symmetry tests does not necessarily indicate that a graph will not be symmetric about the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole. In these instances, we can confirm that symmetry exists by plotting reflecting points across the apparent axis of symmetry or the pole. Testing for symmetry is a technique that simplifies the graphing of polar equations, but its application is not perfect.

Symmetry Tests

A polar equation describes a curve on the polar grid. The graph of a polar equation can be evaluated for three types of symmetry, as shown in (Figure).

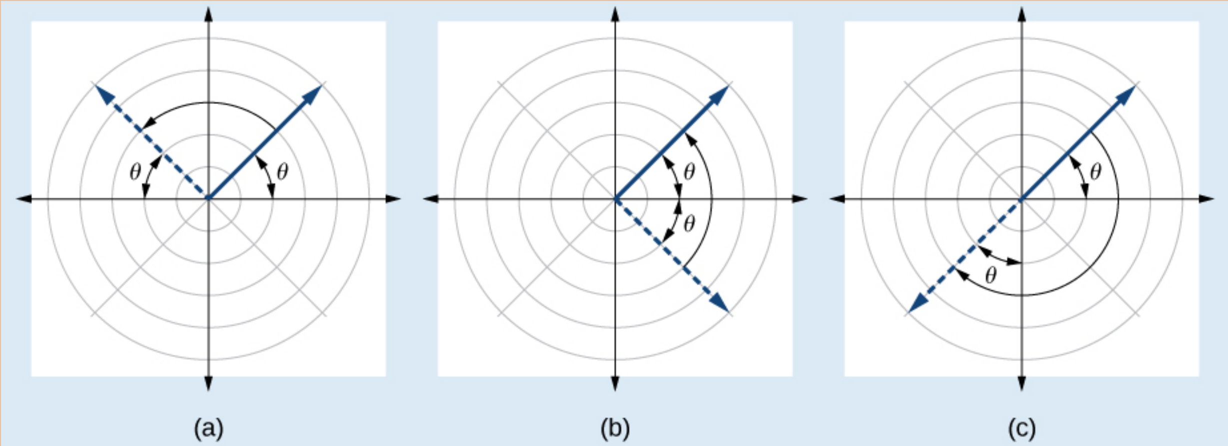


Figure 2. (a) A graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (y-axis) if replacing (r, θ) with $(-r, -\theta)$ yields an equivalent equation. (b) A graph is symmetric with respect to the polar axis (x-axis) if replacing (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ yields an equivalent equation. (c) A graph is symmetric with respect to the pole (origin) if replacing (r, θ) with $(-r, \theta)$ yields an equivalent equation.

How To

Given a polar equation, test for symmetry.

1. Substitute the appropriate combination of components for $(r, \theta) : (-r, -\theta)$ for $\theta = \frac{\pi}{2}$ symmetry; $(r, -\theta)$ for polar axis symmetry; and $(-r, \theta)$ for symmetry with respect to the pole.
2. If the resulting equations are equivalent in one or more of the tests, the graph produces the expected symmetry.

Testing a Polar Equation for Symmetry

Test the equation $r = 2\sin\theta$ for symmetry.

Show Solution

Test for each of the three types of symmetry.

1) Replacing (r, θ) with $(-r, -\theta)$ yields the same result. Thus, the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.	$-r = 2\sin(-\theta)$ $-r = -2\sin\theta$ Even-odd identity $r = 2\sin\theta$ Multiply by -1 Passed
2) Replacing θ with $-\theta$ does not yield the same equation. Therefore, the graph fails the test and may or may not be symmetric with respect to the polar axis.	$r = 2\sin(-\theta)$ $r = -2\sin\theta$ Even-odd identity $r = -2\sin\theta \neq 2\sin\theta$ Failed
3) Replacing r with $-r$ changes the equation and fails the test. The graph may or may not be symmetric with respect to the pole.	$-r = 2\sin\theta$ $r = -2\sin\theta \neq 2\sin\theta$ Failed

Analysis

Using a graphing calculator, we can see that the equation $r = 2\sin\theta$ is a circle centered at $(0, 1)$ with radius $r = 1$ and is indeed symmetric to the line $\theta = \frac{\pi}{2}$. We can also see that the graph is not symmetric with the polar axis or the pole. See (Figure).

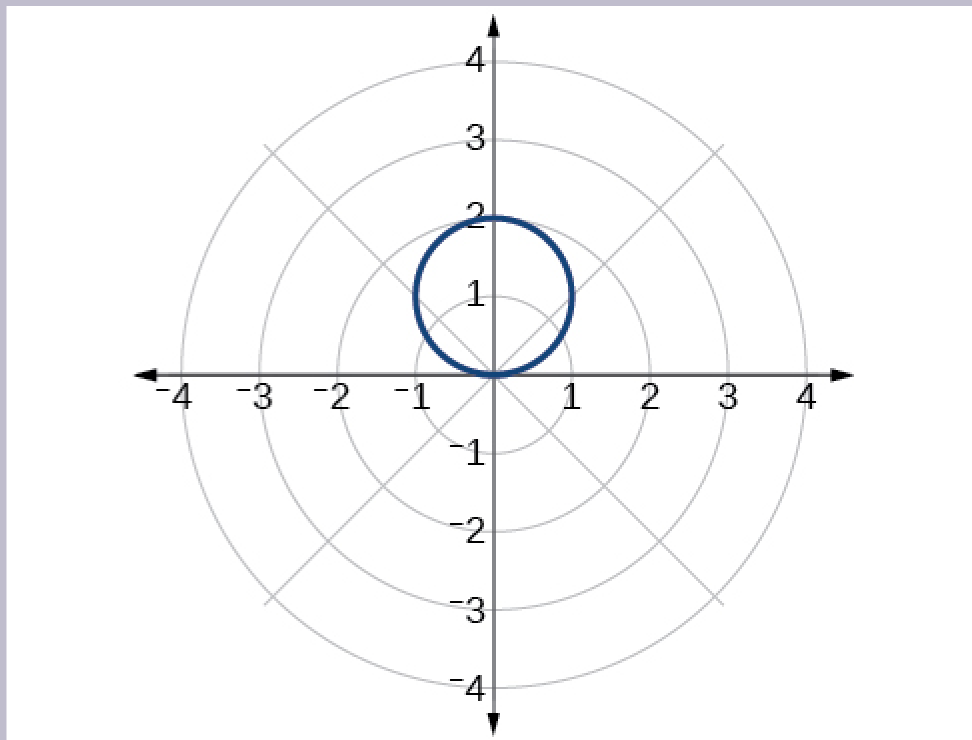


Figure 3.

Try It

Test the equation for symmetry: $r = -2\cos\theta$.

Show Solution

The equation fails the symmetry test with respect to the line $\theta = \frac{\pi}{2}$ and with respect to the pole. It passes the polar axis symmetry test.

Graphing Polar Equations by Plotting Points

To graph in the rectangular coordinate system we construct a table of x and y values. To graph in the polar coordinate system we construct a table of θ and r values. We enter values of θ into a polar equation and calculate r. However, using the properties of symmetry and finding key values of θ and r means fewer calculations will be needed.

Finding Zeros and Maxima

To find the zeros of a polar equation, we solve for the values of θ that result in $r = 0$. Recall that, to find the zeros of polynomial functions, we set the equation equal to zero and then solve for x. We use the same process for polar equations. Set $r = 0$, and solve for θ.

For many of the forms we will encounter, the maximum value of a polar equation is found by substituting those values of θ into the equation that result in the maximum value of the trigonometric functions. Consider $r = 5\cos\theta$; the maximum distance between the curve and the pole is 5 units. The maximum value of the cosine function is 1 when $\theta = 0$, so our polar equation is $5\cos\theta$, and the value $\theta = 0$ will yield the maximum $|r|$.

Similarly, the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, and if our polar equation is $r = 5\sin\theta$, the value $\theta = \frac{\pi}{2}$ will yield the maximum $|r|$. We may find additional information by calculating values of r when $\theta = 0$. These points would be polar axis intercepts, which may be helpful in drawing the graph and identifying the curve of a polar equation.

Finding Zeros and Maximum Values for a Polar Equation

Using the equation in (Figure), find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin\theta$.

Show Solution

To find the zeros, set r equal to zero and solve for θ.

$$\begin{aligned} 2\sin\theta &= 0 \\ \sin\theta &= 0 \\ \theta &= \sin^{-1}0 \\ \theta &= n\pi \qquad \text{where } n \text{ is an integer} \end{aligned}$$

Substitute any one of the θ values into the equation. We will use 0.

$$r = 2\sin(0)$$
$$r = 0$$

The points $(0, 0)$ and $(0, n\pi)$ are the zeros of the equation. They all coincide, so only one point is visible on the graph. This point is also the only polar axis intercept.

To find the maximum value of the equation, look at the maximum value of the trigonometric function $\sin\theta$, which occurs when $\theta = \frac{\pi}{2} 2k\pi$ resulting in $\sin\left(\frac{\pi}{2}\right) = 1.$ Substitute $\frac{\pi}{2}$ for $\theta.$

$$r = 2\sin\left(\frac{\pi}{2}\right)$$
$$r = 2(1)$$
$$r = 2$$

Analysis

The point $\left(2, \frac{\pi}{2}\right)$ will be the maximum value on the graph. Let's plot a few more points to verify the graph of a circle. See (Figure) and (Figure).

θ	$r = 2\sin\theta$	r
0	$r = 2\sin(0) = 0$	0
$\frac{\pi}{6}$	$r = 2\sin\left(\frac{\pi}{6}\right) = 1$	1
$\frac{\pi}{3}$	$r = 2\sin\left(\frac{\pi}{3}\right) \approx 1.73$	1.73
$\frac{\pi}{2}$	$r = 2\sin\left(\frac{\pi}{2}\right) = 2$	2
$\frac{2\pi}{3}$	$r = 2\sin\left(\frac{2\pi}{3}\right) \approx 1.73$	1.73
$\frac{5\pi}{6}$	$r = 2\sin\left(\frac{5\pi}{6}\right) = 1$	1
π	$r = 2\sin(\pi) = 0$	0

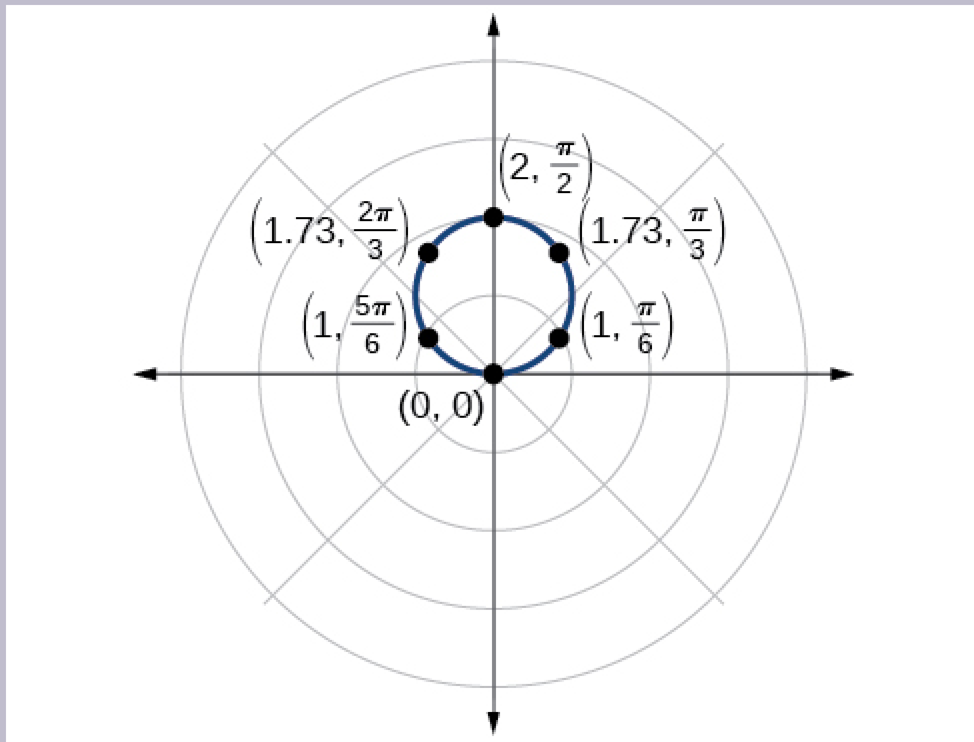


Figure 4.

Try It

Without converting to Cartesian coordinates, test the given equation for symmetry and find the zeros and maximum values of $|r| : r = 3\cos\theta$.

Show Solution

Tests will reveal symmetry about the polar axis. The zero is $\left(0, \frac{\pi}{2}\right)$, and the maximum value is $\left(3, 0\right)$.

Investigating Circles

Now we have seen the equation of a circle in the polar coordinate system. In the last two examples, the same equation was used to illustrate the properties of symmetry and demonstrate how to find the zeros, maximum values, and plotted points that produced the graphs. However, the circle is only one of many shapes in the set of polar curves.

There are five classic polar curves**: cardioids**, **limaçons, lemniscates, rose curves**, and **Archimedes' spirals**. We will briefly touch on the polar formulas for the circle before moving on to the classic curves and their variations.

Formulas for the Equation of a Circle

Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, where a is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the diameter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \frac{\pi}{2}\right)$. (Figure) shows the graphs of these four circles.

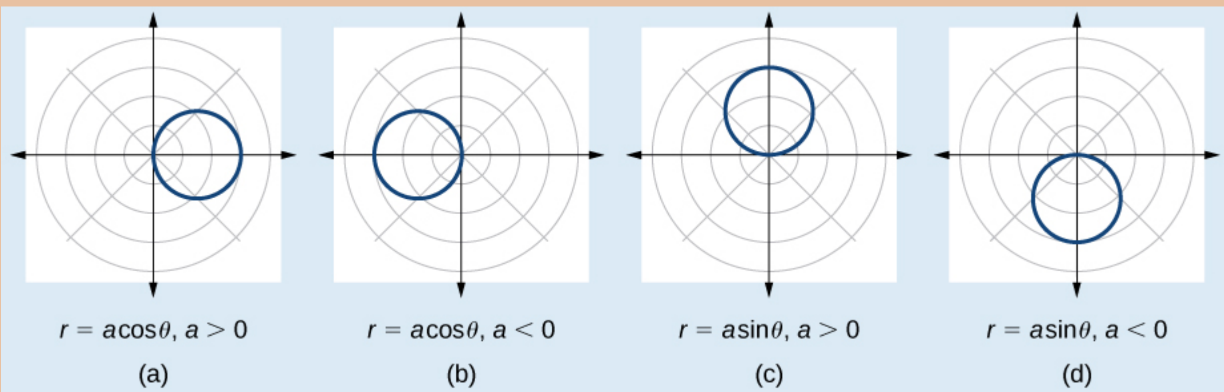


Figure 5.

Sketching the Graph of a Polar Equation for a Circle

Sketch the graph of $r = 4\cos\theta$.

Show Solution

First, testing the equation for symmetry, we find that the graph is symmetric about the polar axis. Next, we find the zeros and maximum $|r|$ for $r = 4\cos\theta.$ First, set $r = 0,$ and solve for θ . Thus, a zero occurs at $\theta = \frac{\pi}{2}k\pi.$ A key point to plot is $\left(0, \frac{\pi}{2}\right)$.

To find the maximum value of $r,$ note that the maximum value of the cosine function is 1 when $\theta = 02k\pi.$ Substitute $\theta = 0$ into the equation:

$$r = 4\cos\theta$$
$$r = 4\cos(0)$$
$$r = 4(1) = 4$$

The maximum value of the equation is 4. A key point to plot is $(4,\ 0)$.

As $r = 4\cos\theta$ is symmetric with respect to the polar axis, we only need to calculate *r*-values for θ over the interval $[0,\ \pi]$. Points in the upper quadrant can then be reflected to the lower quadrant. Make a table of values similar to (Figure). The graph is shown in (Figure).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	3.46	2.83	2	0	−2	−2.83	−3.46	4

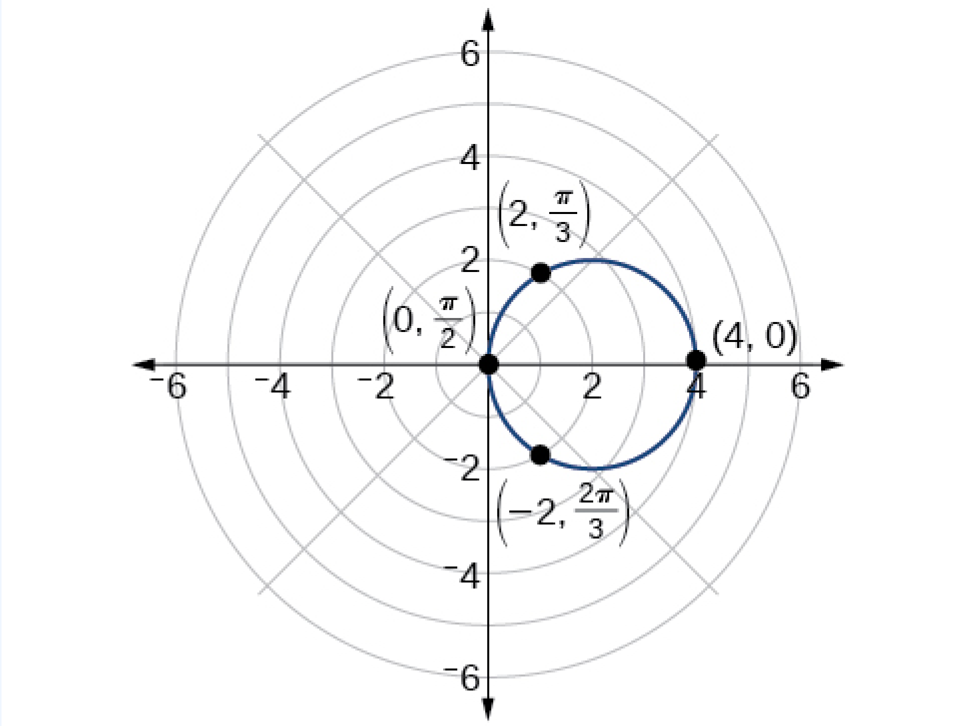


Figure 6.

Investigating Cardioids

While translating from polar coordinates to Cartesian coordinates may seem simpler in some instances, graphing the classic curves is actually less complicated in the polar system. The next curve is called a cardioid, as it resembles a heart. This shape is often included with the family of curves called limaçons, but here we will discuss the cardioid on its own.

Formulas for a Cardioid

The formulas that produce the graphs of a cardioid are given by $r = ab\cos\theta$ and $r = ab\sin\theta$

where $a > 0, \ b > 0$, and $\frac{a}{b} = 1$. The cardioid graph passes through the pole, as we can see in (Figure).

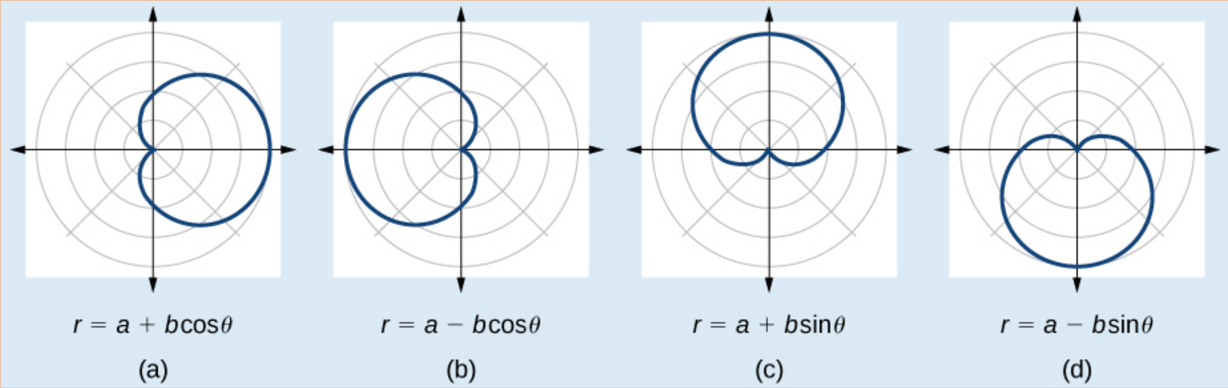


Figure 7.

How To

Given the polar equation of a cardioid, sketch its graph.

1. Check equation for the three types of symmetry.
2. Find the zeros. Set $r = 0$.
3. Find the maximum value of the equation according to the maximum value of the trigonometric expression.
4. Make a table of values for r and θ.
5. Plot the points and sketch the graph.

Sketching the Graph of a Cardioid

Sketch the graph of $r = 2 + 2\cos\theta$.

Show Solution

First, testing the equation for symmetry, we find that the graph of this equation will be symmetric about the polar axis. Next, we find the zeros and maximums. Setting $r = 0,$ we have $\theta = \pi + 2k\pi.$ The zero of the equation is located at $(0, \pi)$. The graph passes through this point.

The maximum value of $r = 2 + 2\cos\theta$ occurs when $\cos\theta$ is a maximum, which is when $\cos\theta = 1$ or when $\theta = 0.$ Substitute $\theta = 0$ into the equation, and solve for $r.$

$$r = 2 + 2\cos(0)$$
$$r = 2 + 2(1) = 4$$

The point $(4, 0)$ is the maximum value on the graph.

We found that the polar equation is symmetric with respect to the polar axis, but as it extends to all four quadrants, we need to plot values over the interval $[0,\ \pi]$. The upper portion of the graph is then reflected over the polar axis. Next, we make a table of values, as in (Figure), and then we plot the points and draw the graph. See (Figure).

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3.41	2	1	0

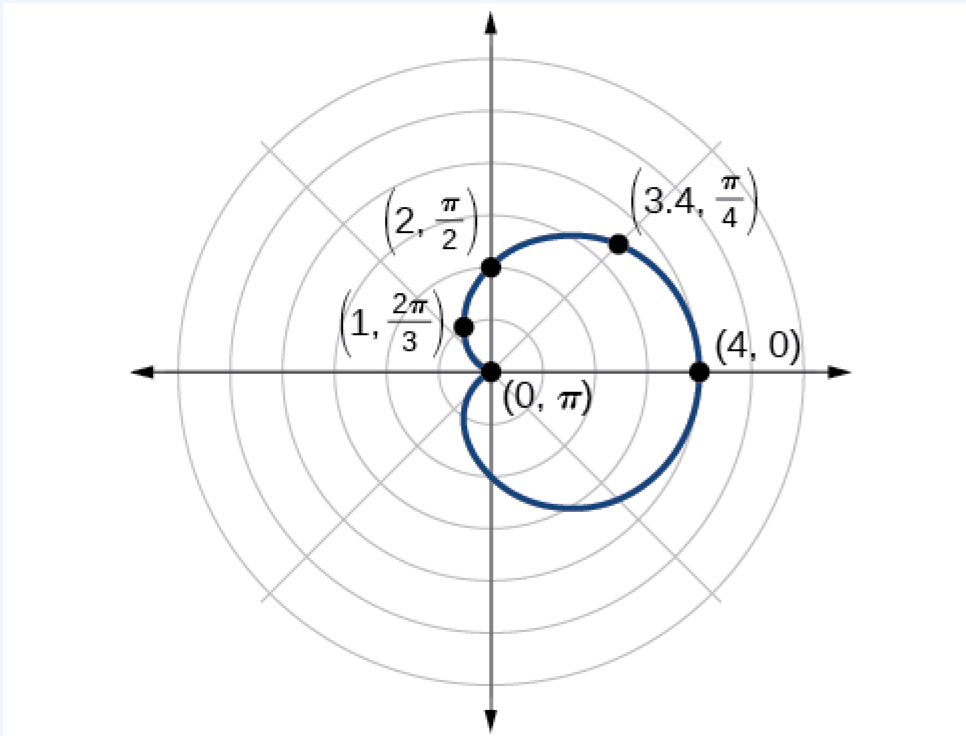


Figure 8.

Investigating Limaçons

The word *limaçon* is Old French for "snail," a name that describes the shape of the graph. As mentioned earlier, the cardioid is a member of the limaçon family, and we can see the similarities in the graphs. The other images in this category include the one-loop limaçon and the two-loop (or inner-loop) limaçon. **One-loop limaçons** are sometimes referred to as dimpled limaçons when $1 < \frac{a}{b} < 2$ and convex limaçons when $\frac{a}{b} \geq 2$.

Formulas for One-Loop Limaçons

The formulas that produce the graph of a dimpled one-loop limaçon are given by $r = ab\cos\theta$ and $r = ab\sin\theta$ where $a > 0,\ b > 0,\ \text{and}\ 1¡\frac{a}{b} < 2.$ All four graphs are shown in (Figure).

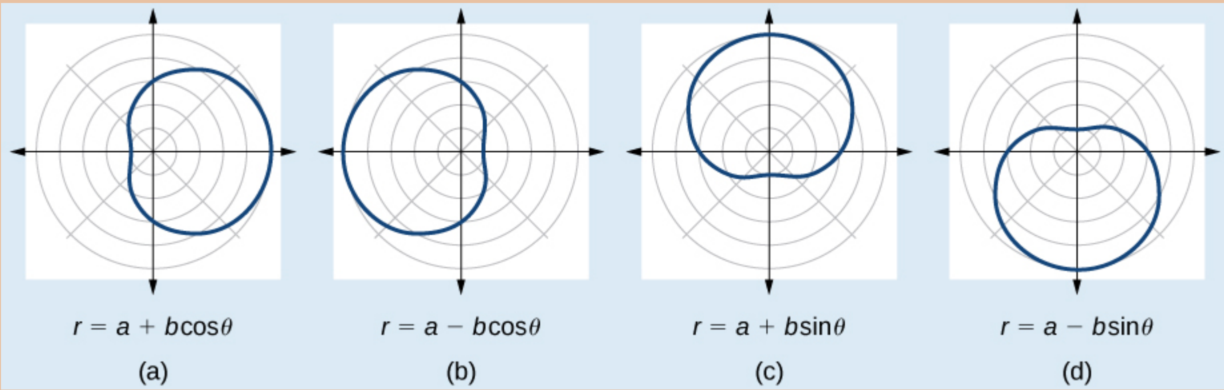


Figure 9. Dimpled limaçons

How To

Given a polar equation for a one-loop limaçon, sketch the graph.

1. Test the equation for symmetry. Remember that failing a symmetry test does not mean that the shape will not exhibit symmetry. Often the symmetry may reveal itself when the points are plotted.
2. Find the zeros.
3. Find the maximum values according to the trigonometric expression.
4. Make a table.
5. Plot the points and sketch the graph.

Sketching the Graph of a One-Loop Limaçon

Graph the equation $r = 4 - 3\sin\theta.$

Show Solution

First, testing the equation for symmetry, we find that it fails all three symmetry tests, meaning that the graph may or may not exhibit symmetry, so we cannot use the symmetry to help us graph it. However, this equation has a graph that clearly displays symmetry with respect to the line $\theta = \frac{\pi}{2}$, yet it fails all the three symmetry tests. A graphing calculator will immediately illustrate the graph's reflective quality.

Next, we find the zeros and maximum, and plot the reflecting points to verify any symmetry. Setting $r = 0$ results in θ being undefined. What does this mean? How could θ be undefined? The angle θ is undefined for any value of $\sin\theta > 1$. Therefore, θ is undefined because there is no value of θ for which $\sin\theta > 1$. Consequently, the graph does not pass through the pole. Perhaps the graph does cross the polar axis, but not at the pole. We can investigate other intercepts by calculating r when $\theta = 0$.

$$r\left(0\right) = 4 - 3\sin\left(0\right)$$
$$r = 4 - 3 \cdot 0 = 4$$

So, there is at least one polar axis intercept at $\left(4, 0\right)$.

Next, as the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, we will substitute $\theta = \frac{\pi}{2}$ into the equation and solve for r. Thus, $r = 1$.

Make a table of the coordinates similar to (Figure).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	4	2.5	1.4	1	1.4	2.5	4	5.5	6.6	7	6.6	5.5	4

The graph is shown in (Figure).

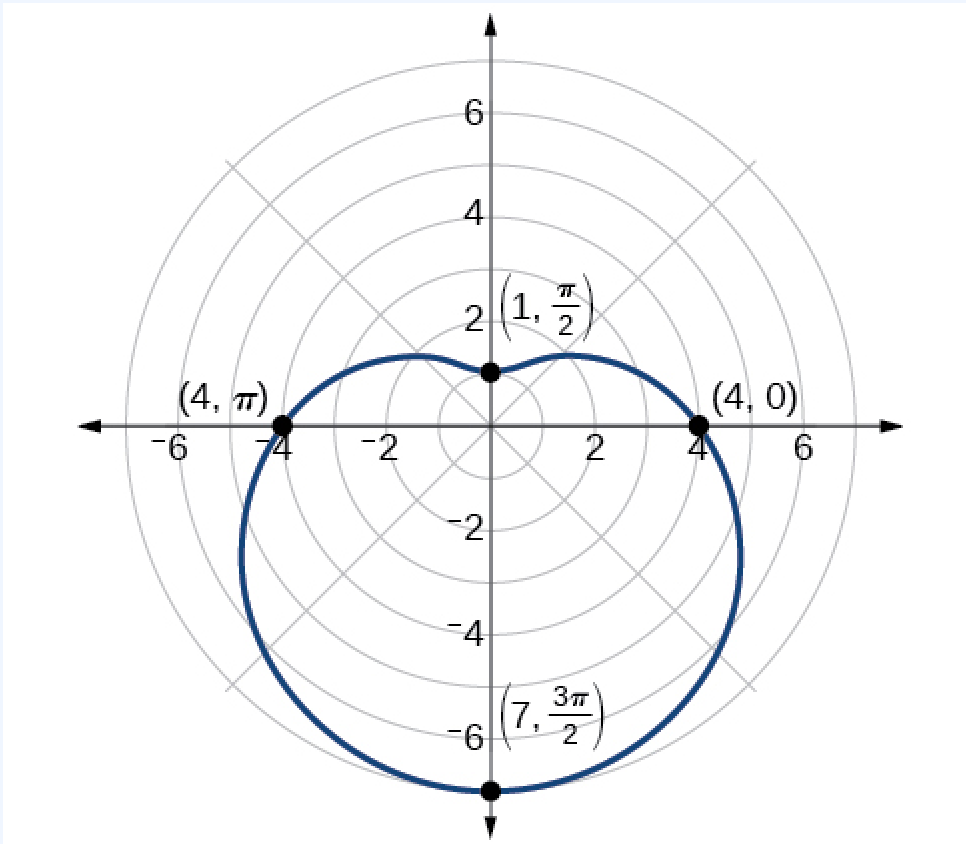


Figure 10.

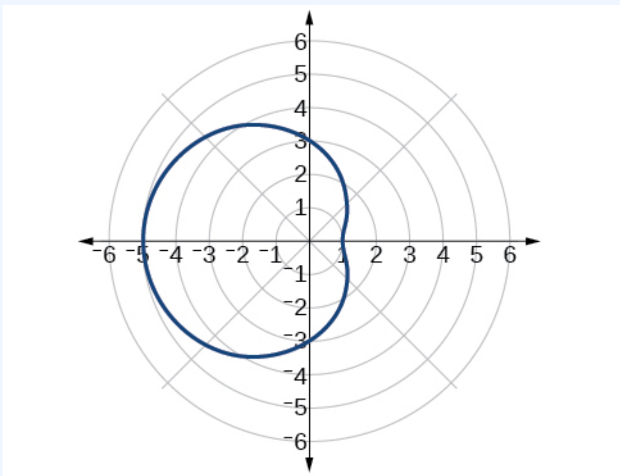
Analysis

This is an example of a curve for which making a table of values is critical to producing an accurate graph. The symmetry tests fail; the zero is undefined. While it may be apparent that an equation involving $\sin\theta$ is likely symmetric with respect to the line $\theta = \frac{\pi}{2}$, evaluating more points helps to verify that the graph is correct.

Try It

Sketch the graph of $r = 3 - 2\cos \theta.$

Show Solution



Another type of limaçon, the **inner-loop limaçon**, is named for the loop formed inside the general limaçon shape. It was discovered by the German artist Albrecht Dürer(1471-1528), who revealed a method for drawing the inner-loop limaçon in his 1525 book *Underweysung der Messing*. A century later, the father of mathematician Blaise Pascal, Étienne Pascal(1588-1651), rediscovered it.

Formulas for Inner-Loop Limaçons

The formulas that generate the inner-loop limaçons are given by $r = ab\cos \theta$ and $r = ab\sin \theta$ where $a > 0,\ b > 0,$ and $a < b.$ The graph of the inner-loop limaçon passes through the pole twice: once for the outer loop, and once for the inner loop. See (Figure) for the graphs.

<img src="https://cnx.org/resources/74560a4ec59151a6de935f75758119e17279c304/

CNX_Precalc_Figure_08_04_012new.jpg" alt="Graph of four inner loop limaçons side by side. (A) is r=a+bcos(θ),a<b. Extended to the right. (B) is a-bcos(θ), a<b. Extends to the left. (C) is r=a+bsin(θ), a<b. Extends up. (D) is r=a-bsin(θ), a **Figure 11.**

Sketching the Graph of an Inner-Loop Limaçon

Sketch the graph of $r=2+5\cos\theta.$

Show Solution

Testing for symmetry, we find that the graph of the equation is symmetric about the polar axis. Next, finding the zeros reveals that when $r=0,\ \theta=1.98.$

The maximum $|r|$ is found when $\cos\theta=1$ or when $\theta=0.$ Thus, the maximum is found at the point (7, 0).

Even though we have found symmetry, the zero, and the maximum, plotting more points will help to define the shape, and then a pattern will emerge.

See (Figure).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	7	6.3	4.5	2	−0.5	−2.3	−3	−2.3	−0.5	2	4.5	6.3	7

As expected, the values begin to repeat after $\theta=\pi.$ The graph is shown in (Figure).

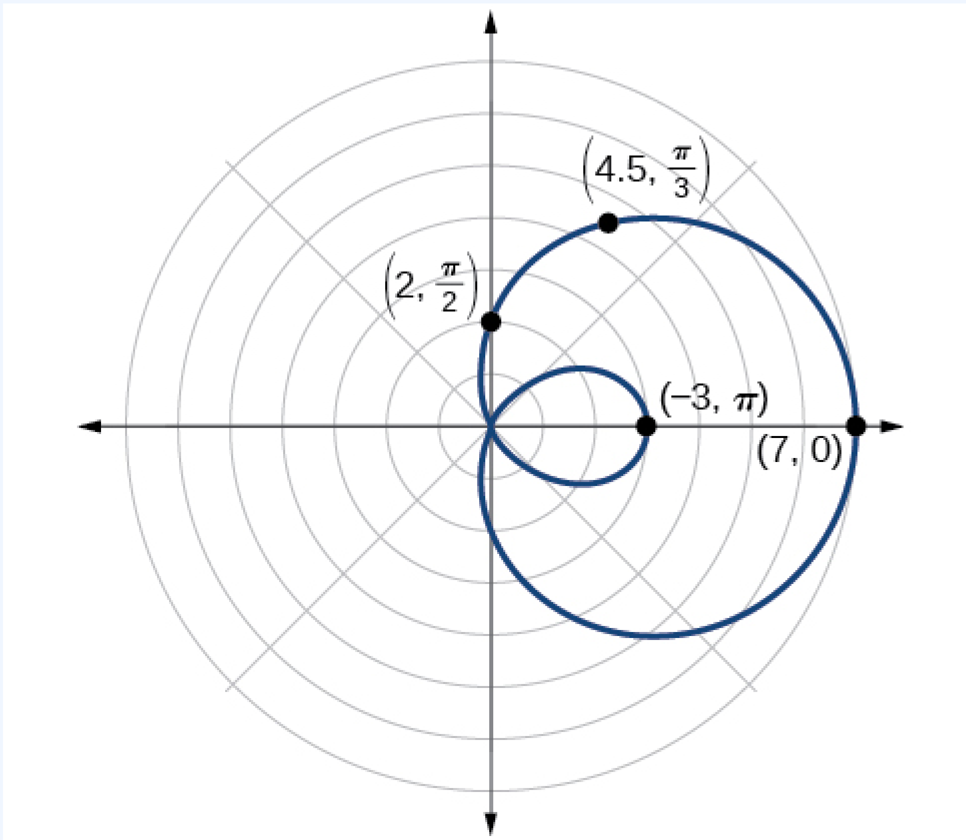


Figure 12. Inner-loop limaçon

Investigating Lemniscates

The lemniscate is a polar curve resembling the infinity symbol ∞ or a figure 8. Centered at the pole, a lemniscate is symmetrical by definition.

Formulas for Lemniscates

The formulas that generate the graph of a lemniscate are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$ where $a \neq 0$. The formula $r^2 = a^2\sin 2\theta$ is symmetric with respect to the pole. The formula $r^2 = a^2\cos 2\theta$ is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis. See (Figure) for the graphs.

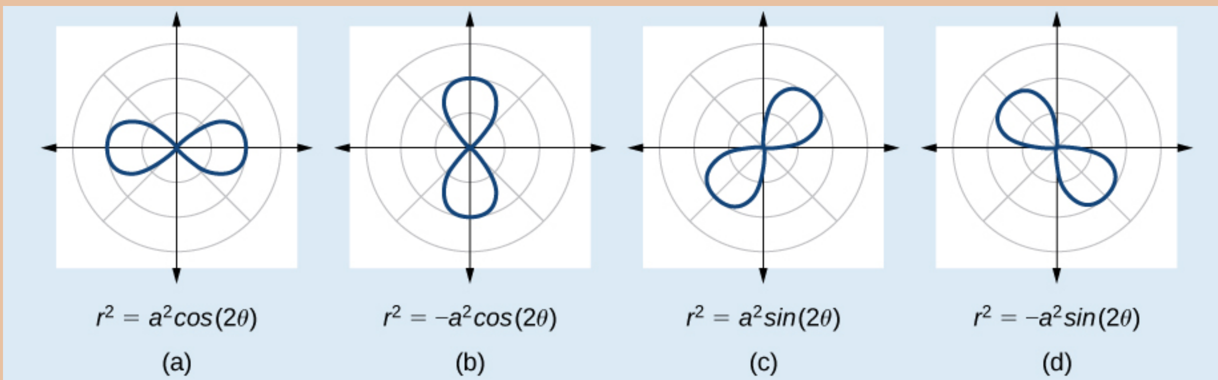


Figure 13.

Sketching the Graph of a Lemniscate

Sketch the graph of $r^2 = 4\cos 2\theta$.

Show Solution

The equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, and the pole.

Let's find the zeros. It should be routine by now, but we will approach this equation a little differently by making the substitution $u = 2\theta$.

$$\begin{aligned} 0 &= 4\cos 2\theta \\ 0 &= 4\cos u \\ 0 &= \cos u \\ \cos^{-1} 0 &= \tfrac{\pi}{2} \\ u &= \tfrac{\pi}{2} && \text{Substitute } 2\theta \text{ back in for } u. \\ 2\theta &= \tfrac{\pi}{2} \\ \theta &= \tfrac{\pi}{4} \end{aligned}$$

So, the point $\left(0, \frac{\pi}{4}\right)$ is a zero of the equation.

Now let's find the maximum value. Since the maximum of $\cos u = 1$ when $u = 0$, the maximum $\cos 2\theta = 1$ when $2\theta = 0$. Thus,

$$\begin{aligned} r^2 &= 4\cos(0) \\ r^2 &= 4(1) = 4 \\ r &= \sqrt{4} = 2 \end{aligned}$$

We have a maximum at (2, 0). Since this graph is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis, we only need to plot points in the first quadrant.

Make a table similar to (Figure).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$\sqrt{2}$	0

Plot the points on the graph, such as the one shown in (Figure).

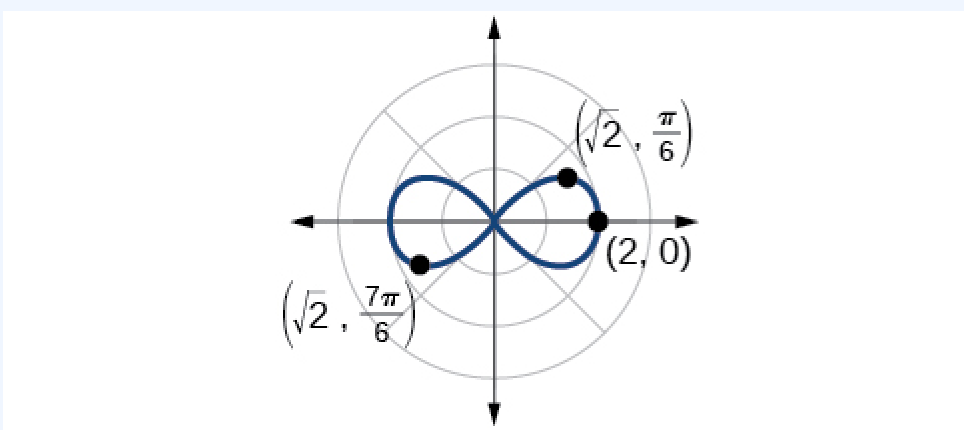


Figure 14. Lemniscate

Analysis

Making a substitution such as $u = 2\theta$ is a common practice in mathematics because it can make calculations simpler. However, we must not forget to replace the substitution term with the original term at the end, and then solve for the unknown.

Some of the points on this graph may not show up using the Trace function on the TI-84 graphing calculator, and the calculator table may show an error for these same points of r. This is because there are no real square roots for these values of θ. In other words, the corresponding r-values of

$$\sqrt{4\cos(2\theta)}$$

are complex numbers because there is a negative number under the radical.

Investigating Rose Curves

The next type of polar equation produces a petal-like shape called a rose curve. Although the graphs look complex, a simple polar equation generates the pattern.

Rose Curves

The formulas that generate the graph of a rose curve are given by $r = a\cos n\theta$ and $r = a\sin n\theta$ where $a \neq 0$. If n is even, the curve has $2n$ petals. If n is odd, the curve has n petals. See (Figure).

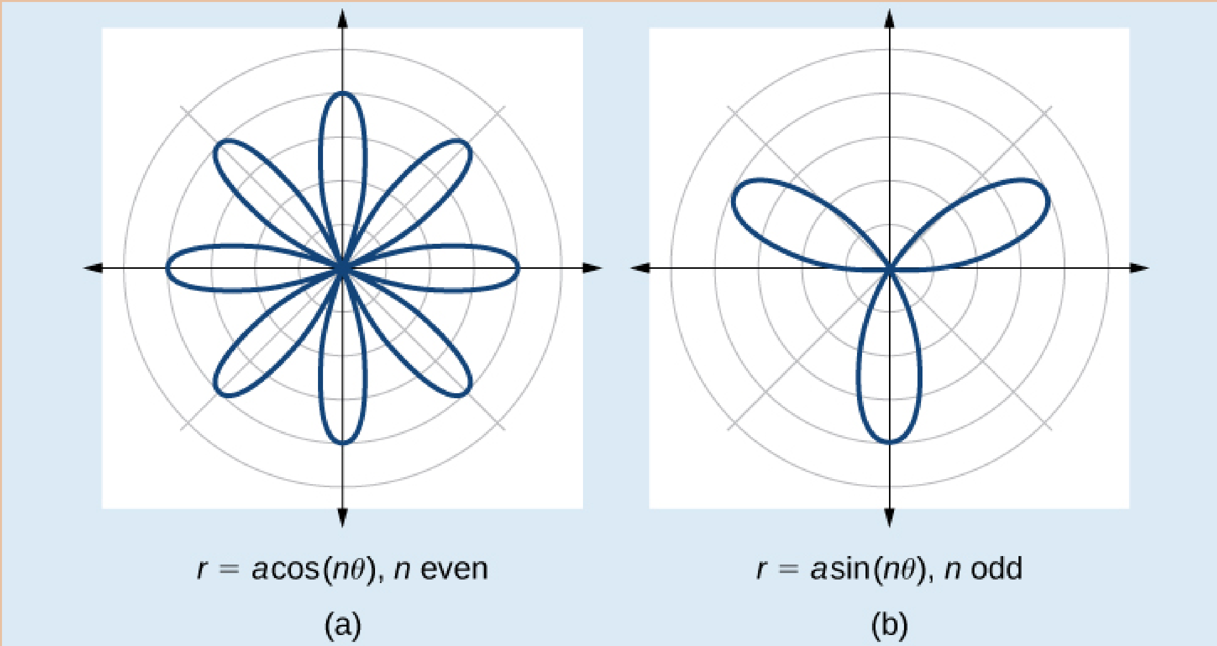


Figure 15.

Sketching the Graph of a Rose Curve (*n* Even)

Sketch the graph of $r = 2\cos 4\theta$.

Show Solution

Testing for symmetry, we find again that the symmetry tests do not tell the whole story. The graph is not only symmetric with respect to the polar axis, but also with respect to the line $\theta = \frac{\pi}{2}$ and the pole.

Now we will find the zeros. First make the substitution $u = 4\theta$.

$$\begin{aligned}0 &= 2\cos 4\theta \\ 0 &= \cos 4\theta \\ 0 &= \cos u \\ \cos^{-1} 0 &= u \\ u &= \frac{\pi}{2} \\ 4\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{8}\end{aligned}$$

The zero is $\theta = \frac{\pi}{8}$. The point $\left(0, \frac{\pi}{8}\right)$ is on the curve.

Next, we find the maximum $|r|$. We know that the maximum value of $\cos u = 1$ when $\theta = 0$. Thus,

$$\begin{aligned}r &= 2\cos\left(4 \cdot 0\right) \\ r &= 2\cos\left(0\right) \\ r &= 2\left(1\right) = 2\end{aligned}$$

The point $\left(2, 0\right)$ is on the curve.

The graph of the rose curve has unique properties, which are revealed in (Figure).

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$
r	2	0	−2	0	2	0	−2

As $r = 0$ when $\theta = \frac{\pi}{8}$, it makes sense to divide values in the table by $\frac{\pi}{8}$ units. A definite pattern emerges. Look at the range of *r*-values: 2, 0, −2, 0, 2, 0, −2, and so on. This represents the development of the curve one petal at a time. Starting at $r = 0$, each petal extends out a distance of $r = 2$, and then turns back to zero $2n$ times for a total of eight petals. See the graph in (Figure).

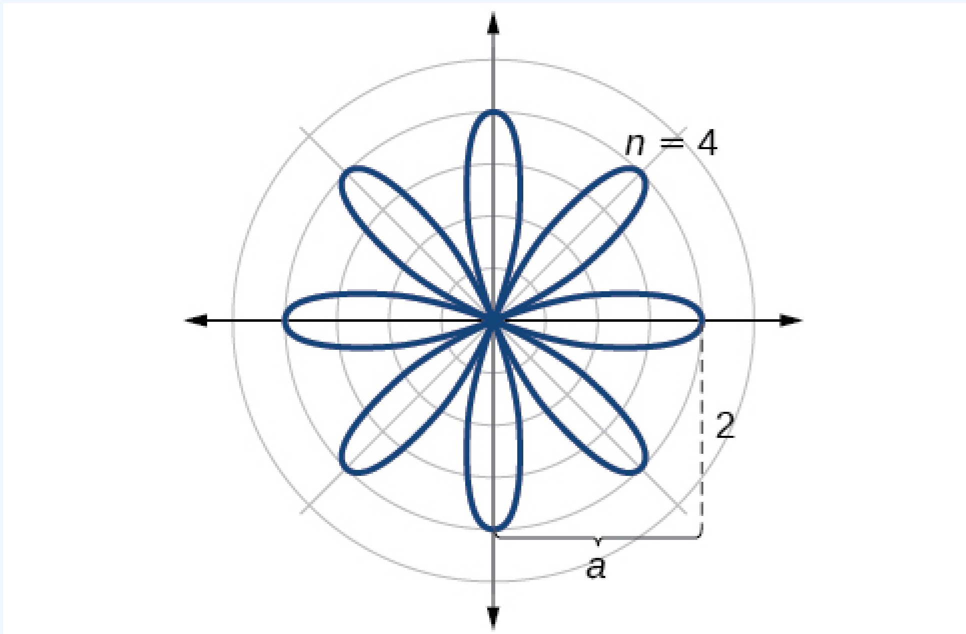


Figure 16. Rose curve, n even

Analysis

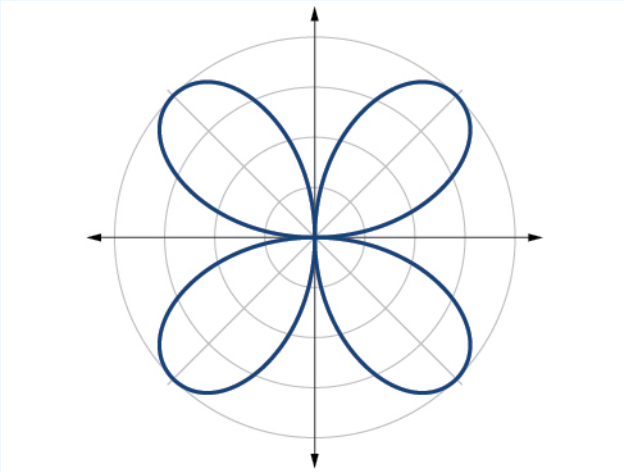
When these curves are drawn, it is best to plot the points in order, as in the (Figure). This allows us to see how the graph hits a maximum (the tip of a petal), loops back crossing the pole, hits the opposite maximum, and loops back to the pole. The action is continuous until all the petals are drawn.

Try It

Sketch the graph of $r = 4\sin\left(2\theta\right).$

Show Solution

The graph is a rose curve, n even



Sketching the Graph of a Rose Curve (*n* Odd)

Sketch the graph of $r = 2\sin(5\theta)$.

Show Solution

The graph of the equation shows symmetry with respect to the line $\theta = \frac{\pi}{2}$. Next, find the zeros and maximum. We will want to make the substitution $u = 5\theta$.

$$\begin{aligned} 0 &= 2\sin(5\theta) \\ 0 &= \sin u \\ \sin^{-1} 0 &= 0 \\ u &= 0 \\ 5\theta &= 0 \\ \theta &= 0 \end{aligned}$$

The maximum value is calculated at the angle where $\sin\theta$ is a maximum. Therefore,

$$r = 2\sin\left(5 \cdot \frac{\pi}{2}\right)$$
$$r = 2\left(1\right) = 2$$

Thus, the maximum value of the polar equation is 2. This is the length of each petal. As the curve for n odd yields the same number of petals as n, there will be five petals on the graph. See (Figure).

Create a table of values similar to (Figure).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	1	−1.73	2	−1.73	1	0

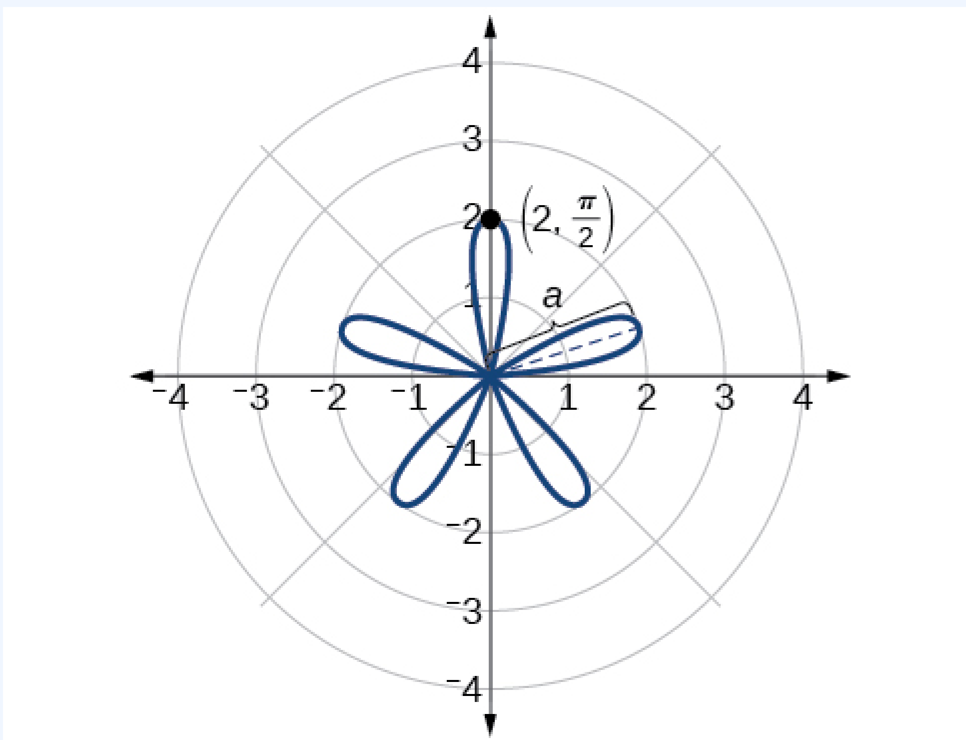
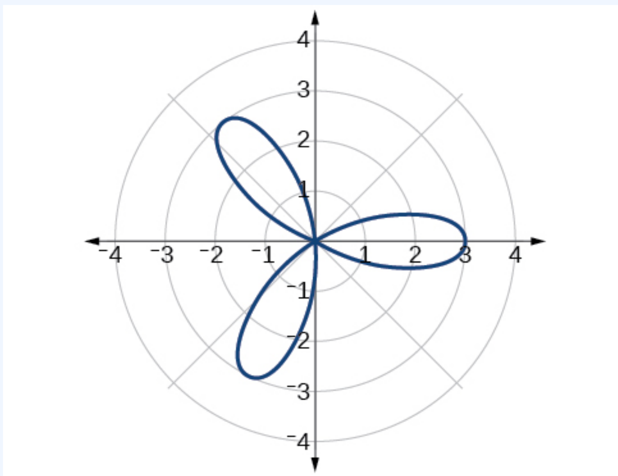


Figure 17. Rose curve, n odd

Try It

Sketch the graph of $r = 3\cos(3\theta)$.

Show Solution



Rose curve, n odd

Investigating the Archimedes' Spiral

The final polar equation we will discuss is the Archimedes' spiral, named for its discoverer, the Greek mathematician Archimedes (c. 287 BCE-c. 212 BCE), who is credited with numerous discoveries in the fields of geometry and mechanics.

Archimedes' Spiral

The formula that generates the graph of the Archimedes' spiral is given by $r = \theta$ for $\theta \geq 0.$ As θ increases, r increases at a constant rate in an ever-widening, never-ending, spiraling path. See (Figure).

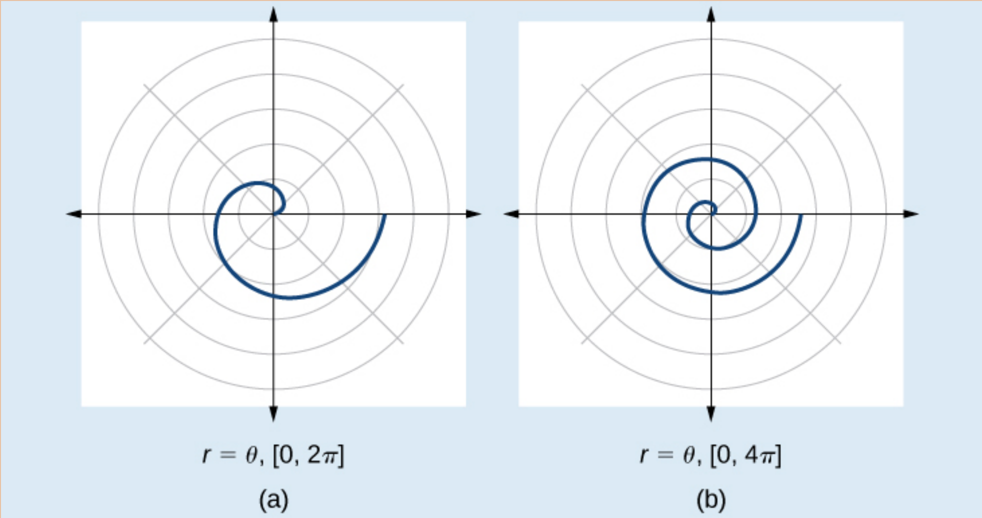


Figure 18.

How To

Given an Archimedes' spiral over $[0, 2\pi]$ **, sketch the graph.**

1. Make a table of values for r and θ over the given domain.
2. Plot the points and sketch the graph.

Sketching the Graph of an Archimedes' Spiral

Sketch the graph of $r = \theta$ over $[0, 2\pi]$.

Show Solution

As r is equal to θ, the plot of the Archimedes' spiral begins at the pole at the point (0, 0). While the graph hints of symmetry, there is no formal symmetry with regard to passing the symmetry tests. Further, there is no maximum value, unless the domain is restricted.

Create a table such as (Figure).

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0.785	1.57	3.14	4.71	5.50	6.28

Notice that the *r*-values are just the decimal form of the angle measured in radians. We can see them on a graph in (Figure).

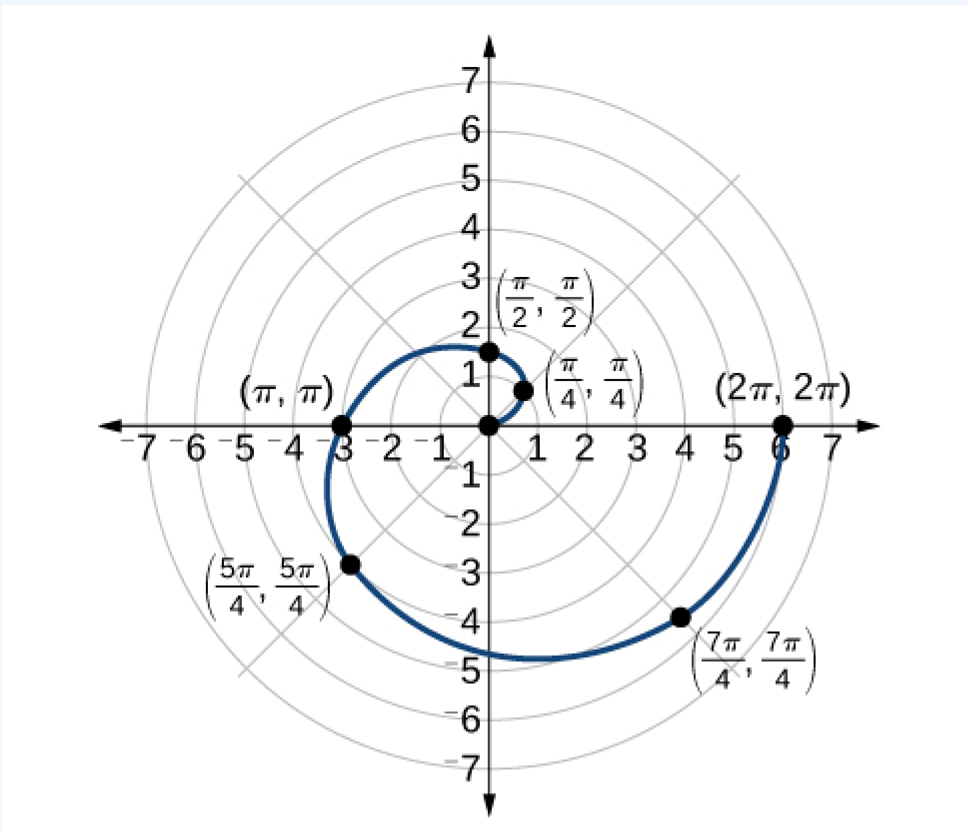


Figure 19. Archimedes' spiral

Analysis

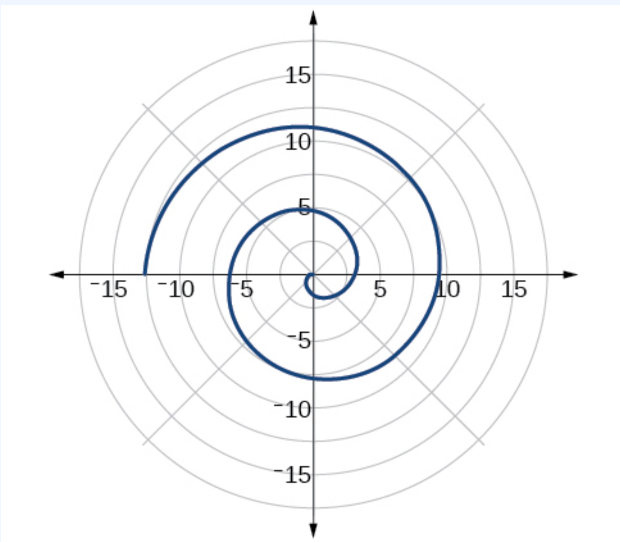
The domain of this polar curve is $[0, 2\pi]$. In general, however, the domain of this function is

$(-\infty, \infty)$. Graphing the equation of the Archimedes' spiral is rather simple, although the image makes it seem like it would be complex.

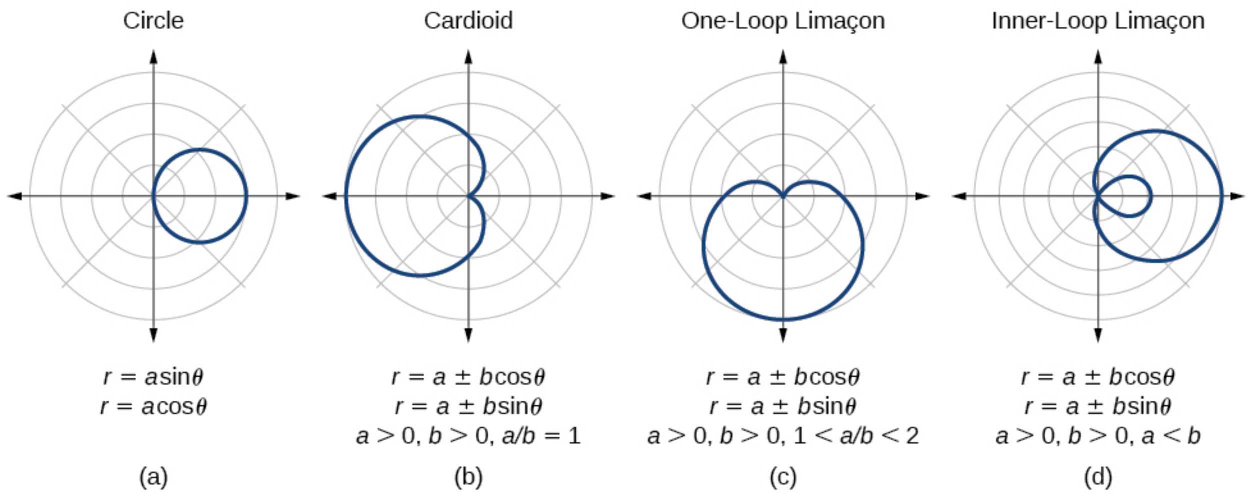
Try It

Sketch the graph of $r = -\theta$ over the interval $[0, 4\pi]$.

Show Solution



Summary of Curves



We have explored a number of seemingly complex polar curves in this section. (Figure) and (Figure) summarize the graphs and equations for each of these curves.

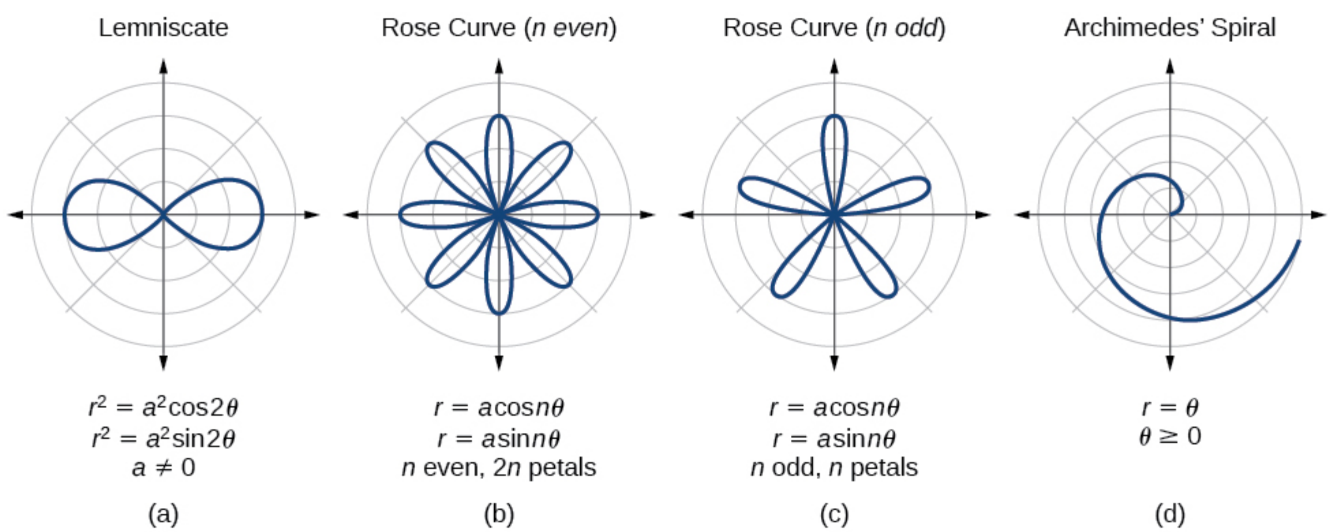


Figure 21.

Access these online resources for additional instruction and practice with graphs of polar coordinates.

- Graphing Polar Equations Part 1
- Graphing Polar Equations Part 2
- Animation: The Graphs of Polar Equations
- Graphing Polar Equations on the TI-84

Key Concepts

- It is easier to graph polar equations if we can test the equations for symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole.
- There are three symmetry tests that indicate whether the graph of a polar equation will exhibit symmetry. If an equation fails a symmetry test, the graph may or may not exhibit symmetry. See (Figure).
- Polar equations may be graphed by making a table of values for θ and r.
- The maximum value of a polar equation is found by substituting the value θ that leads to the maximum value of the trigonometric expression.
- The zeros of a polar equation are found by setting $r = 0$ and solving for θ. See (Figure).
- Some formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$. See (Figure).
- The formulas that produce the graphs of a cardioid are given by $r = ab\cos\theta$ and $r = ab\sin\theta$, for $a > 0,\ b > 0$, and $\frac{a}{b} = 1$. See (Figure).
- The formulas that produce the graphs of a one-loop limaçon are given by $r = ab\cos\theta$ and $r = ab\sin\theta$ for $1 < \frac{a}{b} < 2$. See (Figure).
- The formulas that produce the graphs of an inner-loop limaçon are given by $r = ab\cos\theta$ and $r = ab\sin\theta$ for $a > 0,\ b > 0$, and $a < b$. See (Figure).
- The formulas that produce the graphs of a lemniscates are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, where $a \neq 0$. See (Figure).
- The formulas that produce the graphs of rose curves are given by $r = a\cos n\theta$ and $r = a\sin n\theta$, where $a \neq 0$; if n is even, there are $2n$ petals, and if n is odd, there are n petals. See (Figure) and (Figure).
- The formula that produces the graph of an Archimedes' spiral is given by $r = \theta,\ \theta \geq 0$. See (Figure).

Section Exercises

Verbal

1. Describe the three types of symmetry in polar graphs, and compare them to the symmetry of the Cartesian plane.

Show Solution

Symmetry with respect to the polar axis is similar to symmetry about the x -axis, symmetry with respect to the pole is similar to symmetry about the origin, and symmetric with respect to the line $\theta = \frac{\pi}{2}$ is similar to symmetry about the y -axis.

2. Which of the three types of symmetries for polar graphs correspond to the symmetries with respect to the *x*-axis, *y*-axis, and origin?

3. What are the steps to follow when graphing polar equations?

Show Solution

Test for symmetry; find zeros, intercepts, and maxima; make a table of values. Decide the general type of graph, cardioid, limaçon, lemniscate, etc., then plot points at $\theta = 0,\ \frac{\pi}{2},\ \pi \text{ and } \frac{3\pi}{2},$ and sketch the graph.

4. Describe the shapes of the graphs of cardioids, limaçons, and lemniscates.

5. What part of the equation determines the shape of the graph of a polar equation?

Show Solution

The shape of the polar graph is determined by whether or not it includes a sine, a cosine, and constants in the equation.

Graphical

For the following exercises, test the equation for symmetry.

6. $r = 5\cos 3\theta$

7. $r = 3 - 3\cos \theta$

Show Solution
symmetric with respect to the polar axis

8. $r = 3 + 2\sin \theta$

9. $r = 3\sin 2\theta$

Show Solution
symmetric with respect to the polar axis, symmetric with respect to the line $\theta = \frac{\pi}{2}$, symmetric with respect to the pole

10. $r = 4$

11. $r = 2\theta$

Show Solution
no symmetry

12. $r = 4\cos \frac{\theta}{2}$

13. $r = \frac{2}{\theta}$

Show Solution

no symmetry

14. $r=3\sqrt{1-\cos^2\theta}$

15. $r=\sqrt{5\sin 2\theta}$

Show Solution

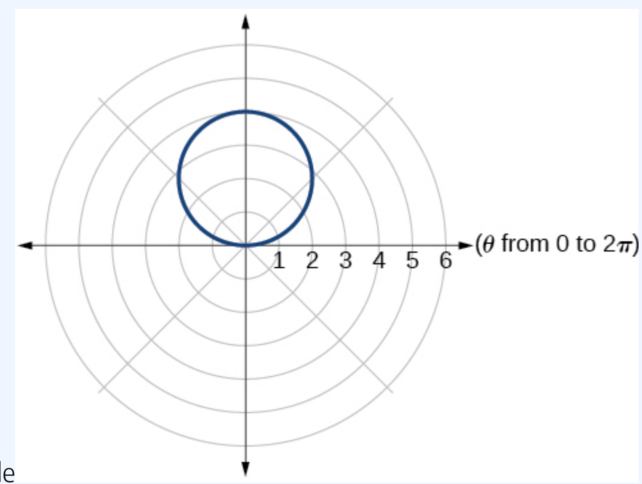
symmetric with respect to the pole

For the following exercises, graph the polar equation. Identify the name of the shape.

16. $r=3\cos\theta$

17. $r=4\sin\theta$

Show Solution



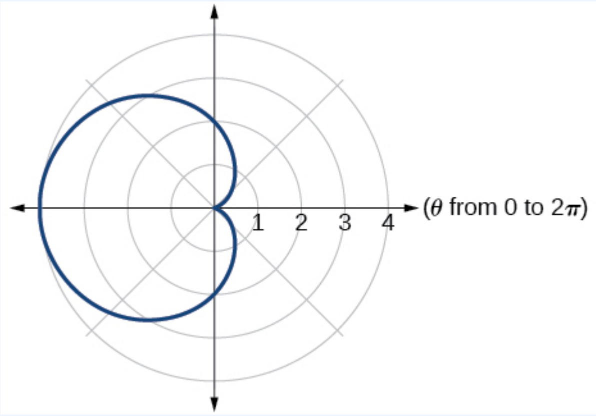
circle

18. $r=2+2\cos\theta$

19. $r = 2 - 2\cos\theta$

Show Solution

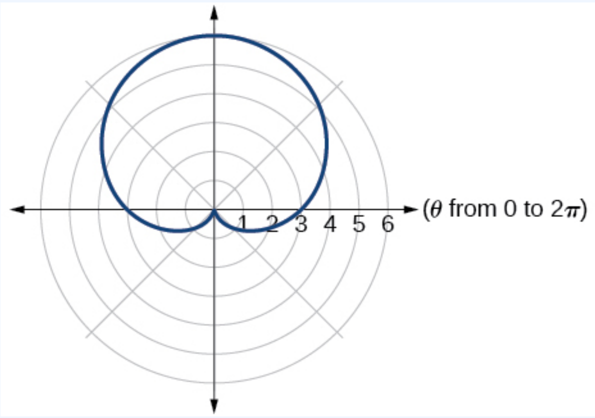
cardioid



20. $r = 5 - 5\sin\theta$

21. $r = 3 + 3\sin\theta$

Show Solution

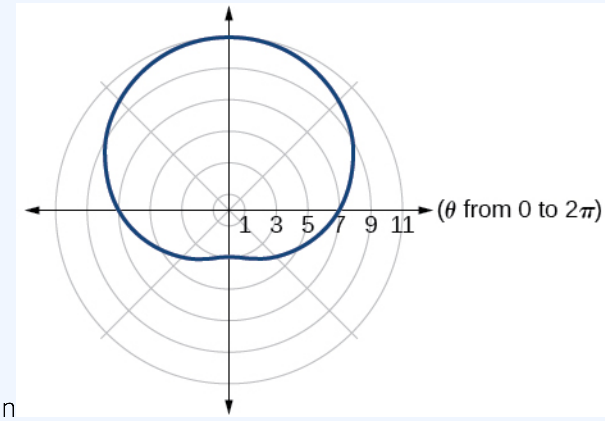


cardioid

22. $r = 3 + 2\sin\theta$

23. $r = 7 + 4\sin\theta$

Show Solution



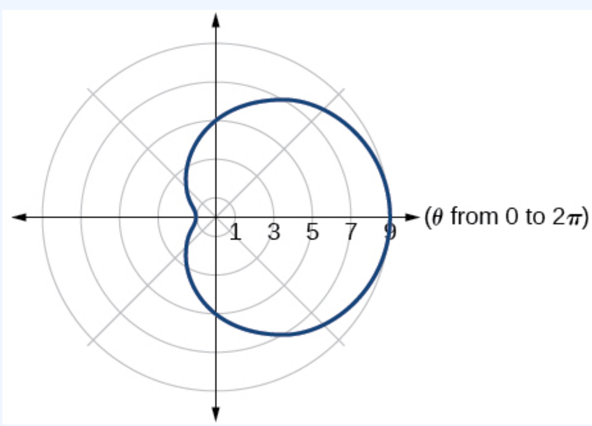
one-loop/dimpled limaçon

24. $r = 4 + 3\cos\theta$

25. $r = 5 + 4\cos\theta$

Show Solution

one-loop/dimpled limaçon

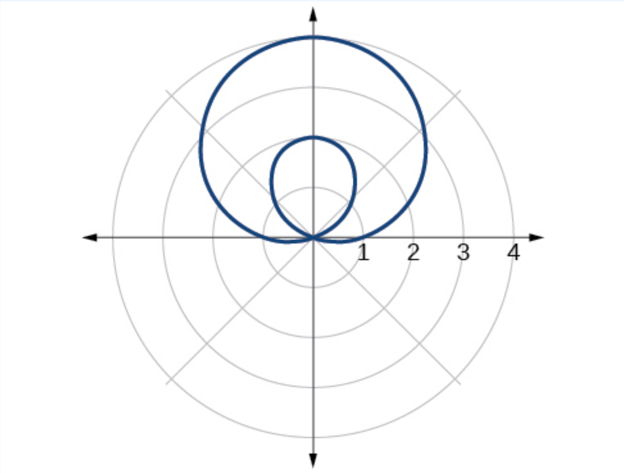


26. $r = 10 + 9\cos\theta$

27. $r = 1 + 3\sin\theta$

Show Solution

inner loop/two-loop limaçon

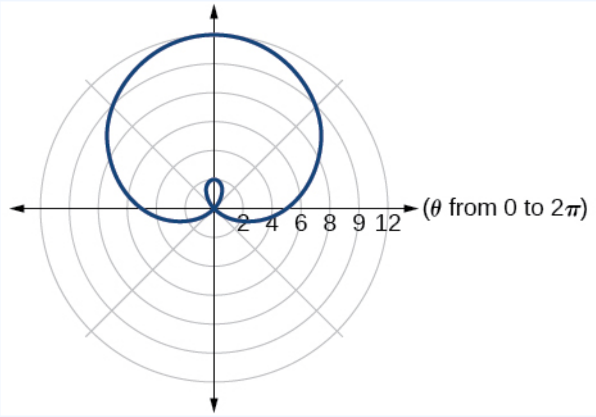


28. $r = 2 + 5\sin\theta$

29. $r = 5 + 7\sin\theta$

Show Solution

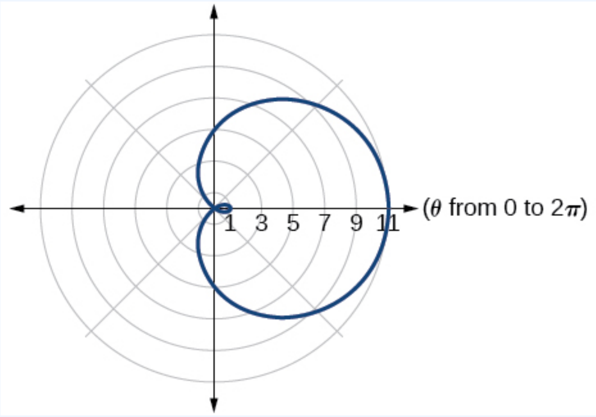
inner loop/two-loop limaçon



30. $r = 2 + 4\cos\theta$

31. $r = 5 + 6\cos\theta$

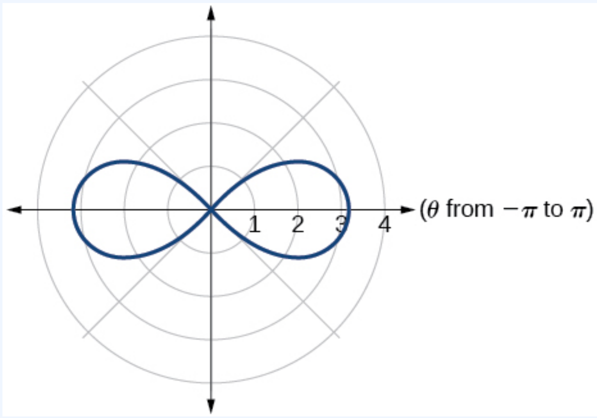
Show Solution
inner loop/two-loop limaçon



32. $r^2 = 36\cos(2\theta)$

33. $r^2 = 10\cos(2\theta)$

Show Solution
lemniscate

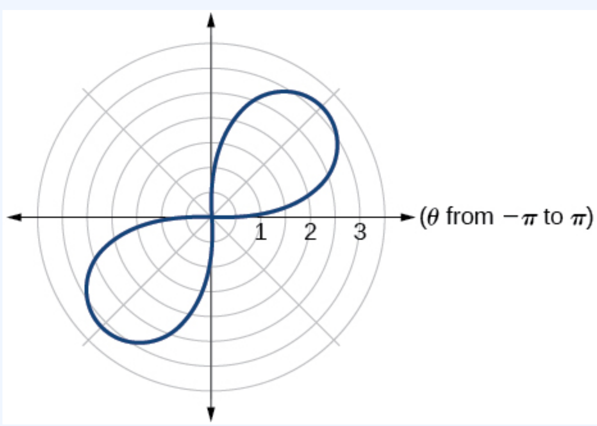


34. $r^2 = 4\sin(2\theta)$

35. $r^2 = 10\sin(2\theta)$

Show Solution

lemniscate

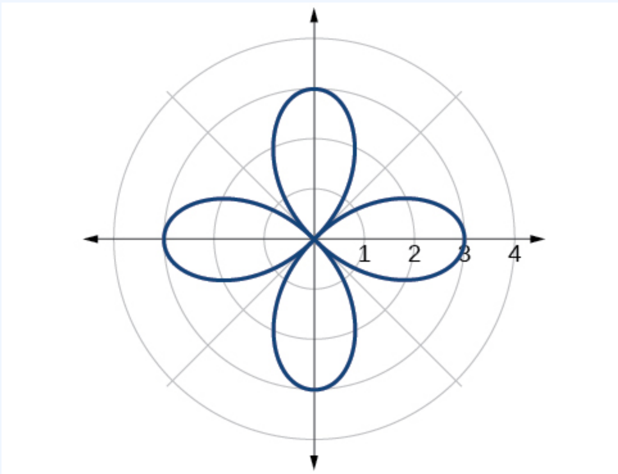


36. $r = 3\sin(2\theta)$

37. $r = 3\cos(2\theta)$

Show Solution

rose curve

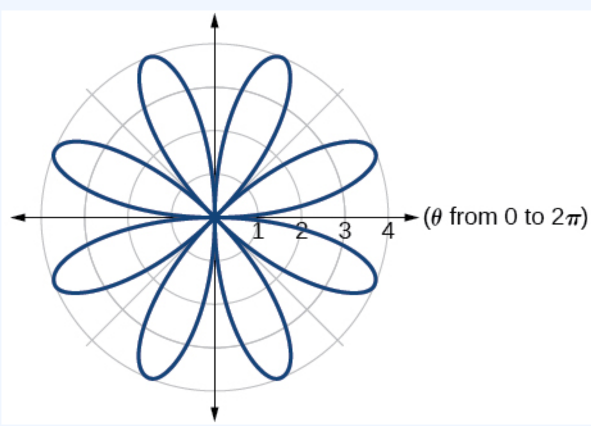


38. $r = 5\sin(3\theta)$

39. $r = 4\sin(4\theta)$

Show Solution

rose curve

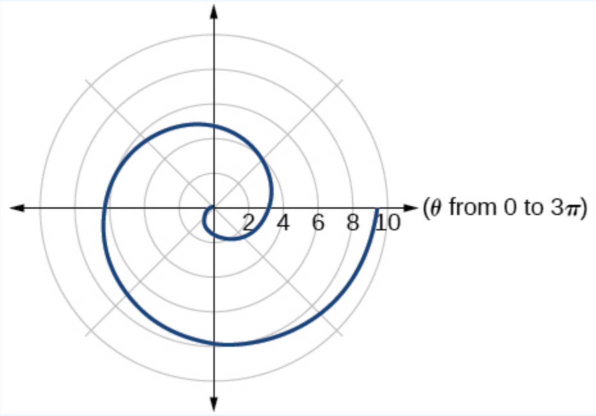


40. $r = 4\sin(5\theta)$

41. $r = -\theta$

Show Solution

Archimedes' spiral

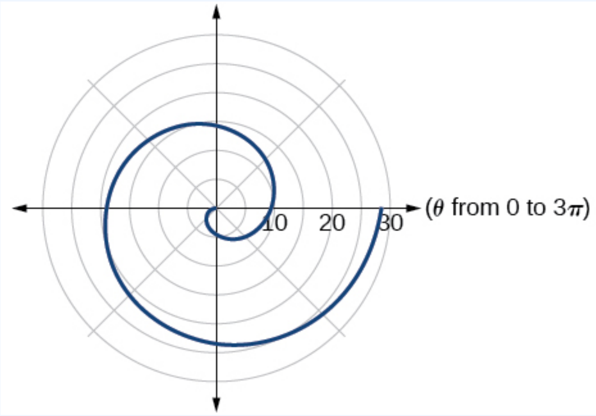


42. $r = 2\theta$

43. $r = -3\theta$

Show Solution

Archimedes' spiral



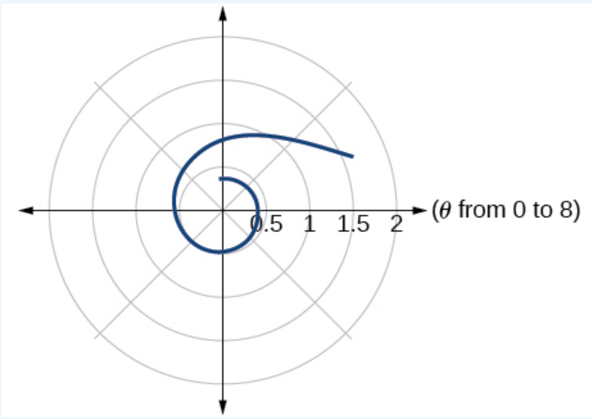
Technology

For the following exercises, use a graphing calculator to sketch the graph of the polar equation.

44. $r = \frac{1}{\theta}$

45. $r = \frac{1}{\sqrt{\theta}}$

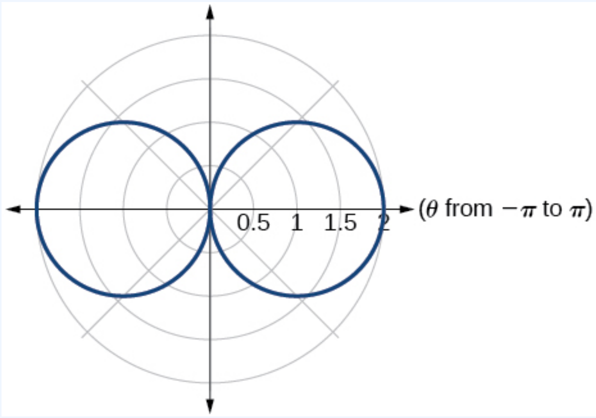
Show Solution



46. $r = 2\sin\theta\tan\theta$, a cissoid

47. $r = 2\sqrt{1 - \sin^2\theta}$, a hippopede

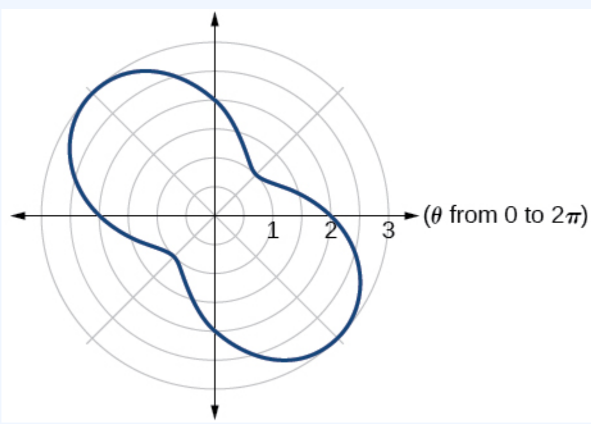
Show Solution



48. $r = 5 + \cos(4\theta)$

49. $r = 2 - \sin(2\theta)$

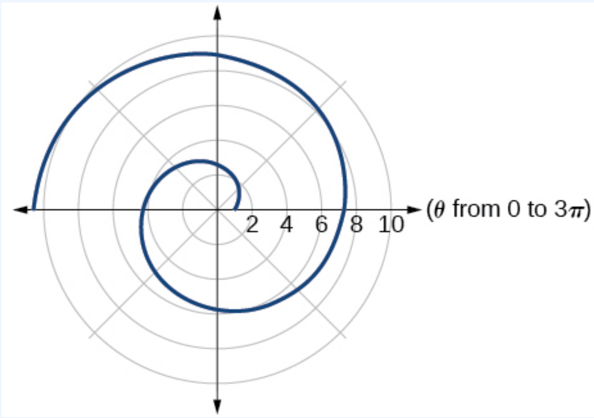
Show Solution



50. $r = \theta^2$

51. $r = \theta + 1$

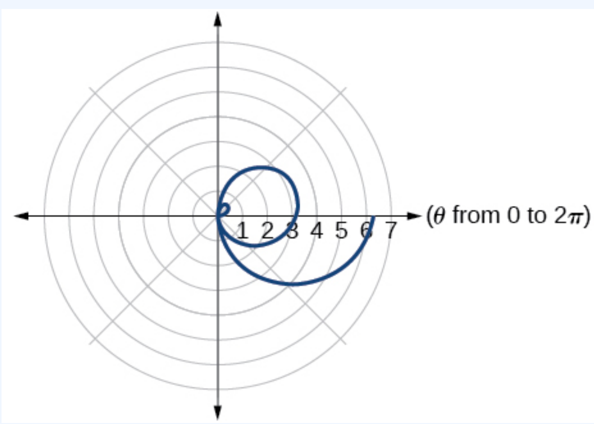
Show Solution



52. $r = \theta \sin \theta$

53. $r = \theta \cos \theta$

Show Solution



For the following exercises, use a graphing utility to graph each pair of polar equations on a domain of $[0, 4\pi]$ and then explain the differences shown in the graphs.

54. $r = \theta, r = -\theta$

55. $r=\theta, r=\theta+\sin\theta$

Show Solution
They are both spirals, but not quite the same.

56. $r=\sin\theta+\theta, r=\sin\theta-\theta$

57. $r=2\sin\left(\frac{\theta}{2}\right), r=\theta\sin\left(\frac{\theta}{2}\right)$

Show Solution
Both graphs are curves with 2 loops. The equation with a coefficient of θ has two loops on the left, the equation with a coefficient of 2 has two loops side by side. Graph these from 0 to 4π to get a better picture.

58. $r=\sin\left(\cos\left(3\theta\right)\right)\ \ r=\sin\left(3\theta\right)$

59. On a graphing utility, graph $r=\sin\left(\frac{16}{5}\theta\right)$ on $[0,4\pi]$, $[0,8\pi]$, $[0,12\pi]$, and $[0,16\pi]$. Describe the effect of increasing the width of the domain.

Show Solution
When the width of the domain is increased, more petals of the flower are visible.

60. On a graphing utility, graph and sketch $r=\sin\theta+\left(\sin\left(\frac{5}{2}\theta\right)\right)^3$ on $[0,4\pi]$.

61. On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$r_1=3\sin\left(3\theta\right)$
$r_2=2\sin\left(3\theta\right)$
$r_3=\sin\left(3\theta\right)$

Show Solution
The graphs are three-petal, rose curves. The larger the coefficient, the greater the curve's distance from the pole.

62. On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3 + 3\cos\theta$$
$$r_2 = 2 + 2\cos\theta$$
$$r_3 = 1 + \cos\theta$$

63. On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3\theta$$
$$r_2 = 2\theta$$
$$r_3 = \theta$$

Show Solution
The graphs are spirals. The smaller the coefficient, the tighter the spiral.

Extensions

For the following exercises, draw each polar equation on the same set of polar axes, and find the points of intersection.

64. $r_1 = 3 + 2\sin\theta,\ r_2 = 2$

65. $r_1 = 6 - 4\cos\theta,\ r_2 = 4$

Show Solution
$\left(4, \frac{\pi}{3}\right), \left(4, \frac{5\pi}{3}\right)$

66. $r_1 = 1 + \sin\theta,\ r_2 = 3\sin\theta$

67. $r_1 = 1 + \cos\theta,\ r_2 = 3\cos\theta$

Show Solution

$\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right)$

68. $r_1 = \cos(2\theta),\ r_2 = \sin(2\theta)$

69. $r_1 = \sin^2(2\theta),\ r_2 = 1 - \cos(4\theta)$

Show Solution

$\left(0, \frac{\pi}{2}\right), (0, \pi), \left(0, \frac{3\pi}{2}\right), (0, 2\pi)$

70. $r_1 = \sqrt{3},\ r_2 = 2\sin(\theta)$

71. ${r_1}^2 = \sin\theta, {r_2}^2 = \cos\theta$

Show Solution

$\left(\frac{\sqrt[4]{8}}{2}, \frac{\pi}{4}\right), \left(\frac{\sqrt[4]{8}}{2}, \frac{5\pi}{4}\right)$ and at $\theta = \frac{3\pi}{4},\ \frac{7\pi}{4}$ since r is squared

72. $r_1 = 1 + \cos\theta,\ r_2 = 1 - \sin\theta$

Glossary

Archimedes' spiral
: a polar curve given by $r = \theta$. When multiplied by a constant, the equation appears as $r = a\theta$. As $r = \theta$, the curve continues to widen in a spiral path over the domain.

cardioid
: a member of the limaçon family of curves, named for its resemblance to a heart; its equation is given as $r = ab\cos\theta$ and $r = ab\sin\theta$, where $\frac{a}{b} = 1$

convex limaçon
: a type of one-loop limaçon represented by $r = ab\cos\theta$ and $r = ab\sin\theta$ such that $\frac{a}{b} \geq 2$

dimpled limaçon
: a type of one-loop limaçon represented by $r = ab\cos\theta$ and $r = ab\sin\theta$ such that $1 < \frac{a}{b} < 2$

inner-loop limaçon
: a polar curve similar to the cardioid, but with an inner loop; passes through the pole twice; represented by $r = ab\cos\theta$ and $r = ab\sin\theta$ where $a < b$

lemniscate
: a polar curve resembling a figure 8 and given by the equation $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta,\ \ a \neq 0$

one-loop limaçon
: a polar curve represented by $r = ab\cos\theta$ and $r = ab\sin\theta$ such that $a > 0, b > 0,$ and $\frac{a}{b} > 1;$ may be dimpled or convex; does not pass through the pole

polar equation
: an equation describing a curve on the polar grid.

rose curve
: a polar equation resembling a flower, given by the equations $r = a\cos n\theta$ and $r = a\sin n\theta;$ when n is even there are $2n$ petals, and the curve is highly symmetrical; when n is odd there are n petals.

CHAPTER 4.6: POLAR FORM OF COMPLEX NUMBERS

Learning Objectives

In this section, you will:

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

"God made the integers; all else is the work of man." This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in Complex Numbers. In this section, we will focus on the mechanics of working with complex numbers: translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre's Theorem.

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a + bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

How To

Given a complex number $a + bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis.*
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

Plotting a Complex Number in the Complex Plane

Plot the complex number $2 - 3i$ in the complex plane.

Show Solution

From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See (Figure).

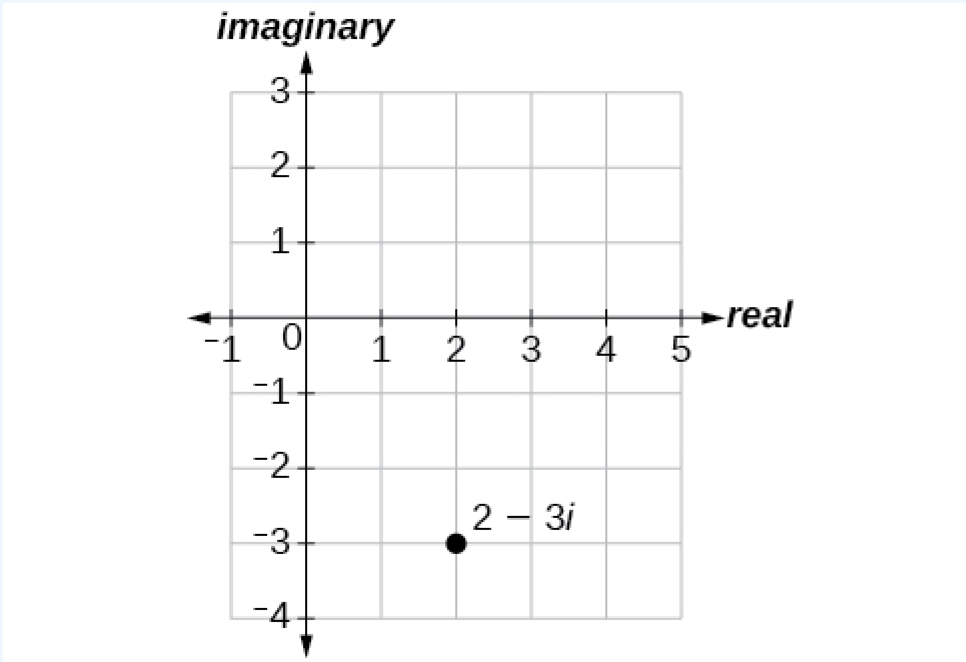
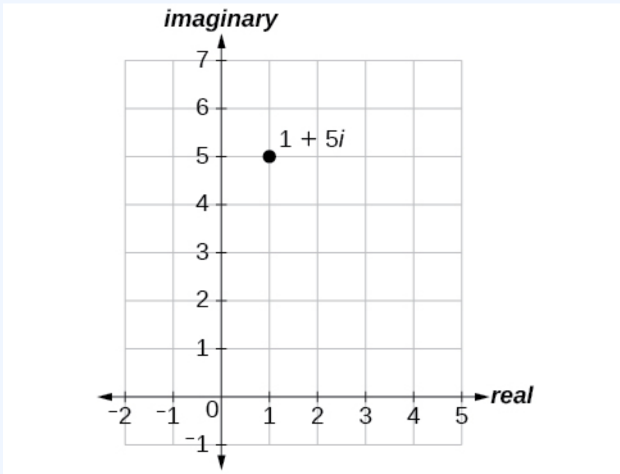


Figure 1.

Try It

Plot the point $1 + 5i$ in the complex plane.

Show Solution



Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z = 2 + 4i$, in (Figure), shows $|z|$.

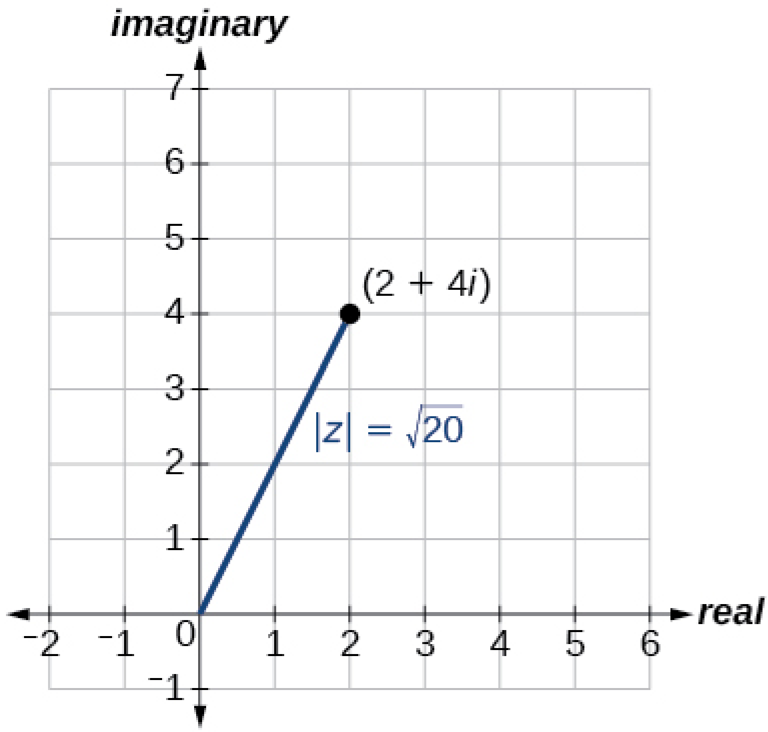


Figure 2.

Absolute Value of a Complex Number

Given $z = x + yi$, a complex number, the absolute value of z is defined as

$|z| = \sqrt{x^2 + y^2}$

It is the distance from the origin to the point (x, y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, $(0,\ 0)$.

Finding the Absolute Value of a Complex Number with a Radical

Find the absolute value of $z = \sqrt{5} - i$.

Show Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{\left(\sqrt{5}\right)^2 + (-1)^2}$$
$$|z| = \sqrt{5 + 1}$$
$$|z| = \sqrt{6}$$

See (Figure).

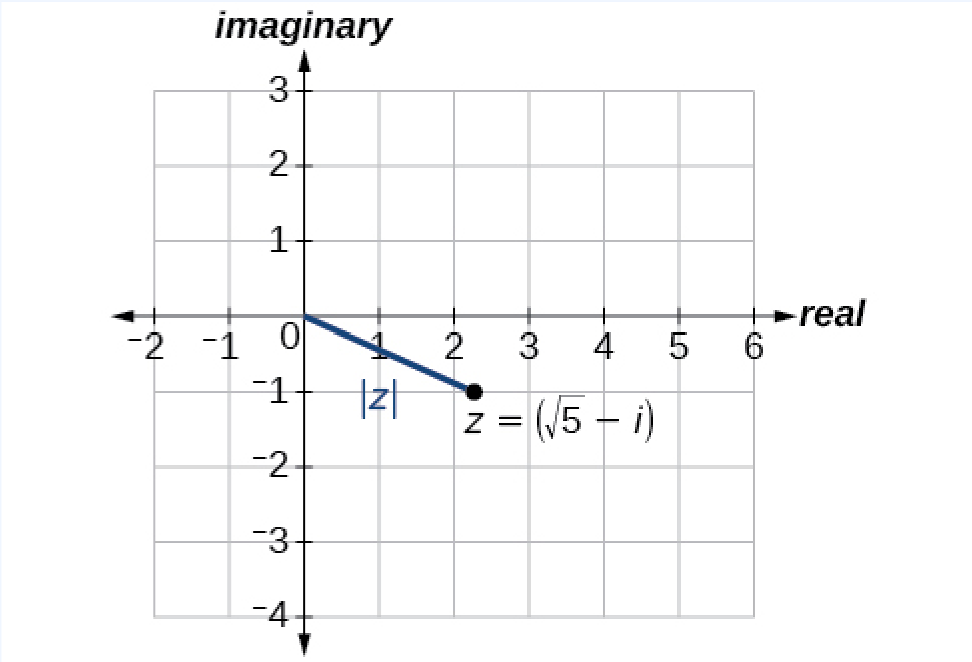


Figure 3.

Try It

Find the absolute value of the complex number $z = 12 - 5i$.

Show Solution

13

Finding the Absolute Value of a Complex Number

Given $z = 3 - 4i$, find $|z|$.

Show Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(3)^2 + (-4)^2}$$
$$|z| = \sqrt{9 + 16}$$
$$|z| = \sqrt{25}$$
$$|z| = 5$$

The absolute value z is 5. See (Figure).

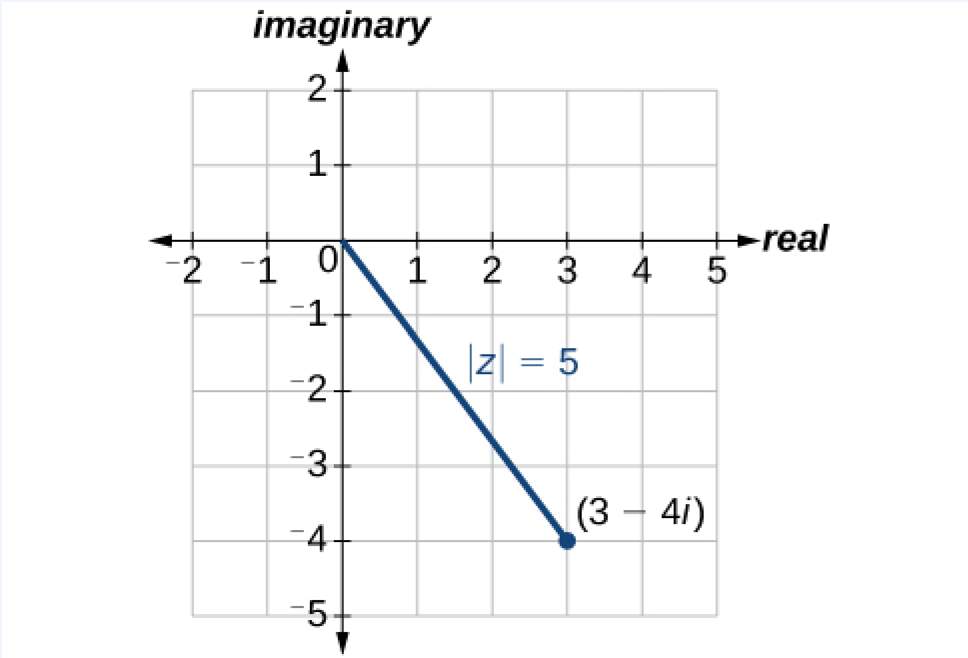


Figure 4.

Try It

Given $z = 1 - 7i$, find $|z|$.

Show Solution

$|z| = \sqrt{50} = 5\sqrt{2}$

Writing Complex Numbers in Polar Form

The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$

We review these relationships in (Figure).

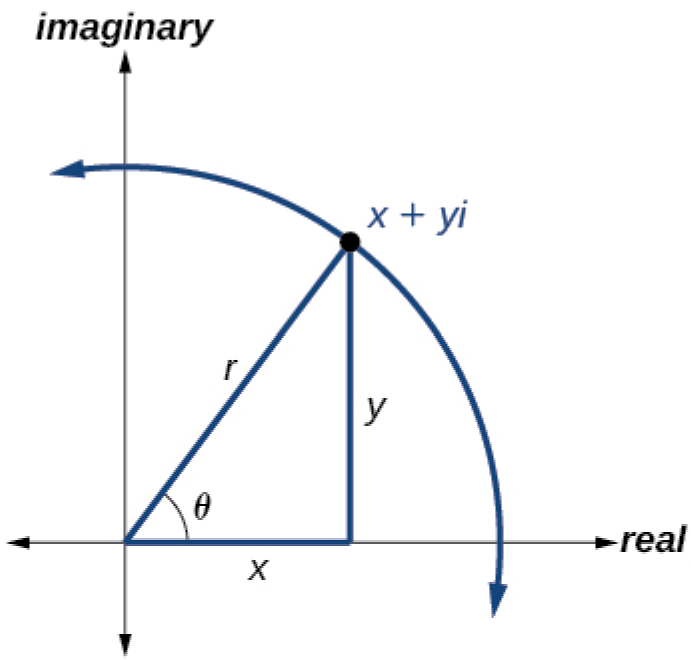


Figure 5.

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x, y) . The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$z = x + yi$$
$$z = r\cos\theta + (r\sin\theta)\,i$$
$$z = r\left(\cos\theta + i\sin\theta\right)$$

Polar Form of a Complex Number

Writing a complex number in polar form involves the following conversion formulas:

$x = r\cos\theta$
$y = r\sin\theta$
$r = \sqrt{x^2 + y^2}$

Making a direct substitution, we have
$z = x + yi$
$z = (r\cos\theta) + i\,(r\sin\theta)$
$z = r\,(\cos\theta + i\sin\theta)$

where r is the modulus and θ is the argument. We often use the abbreviation $r\text{cis}\,\theta$ to represent $r\,(\cos\theta + i\sin\theta)$.

Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Show Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$r = \sqrt{x^2 + y^2}$
$r = \sqrt{0^2 + 4^2}$
$r = \sqrt{16}$
$r = 4$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \frac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ or $4\text{cis}\left(\frac{\pi}{2}\right)$. See (Figure).

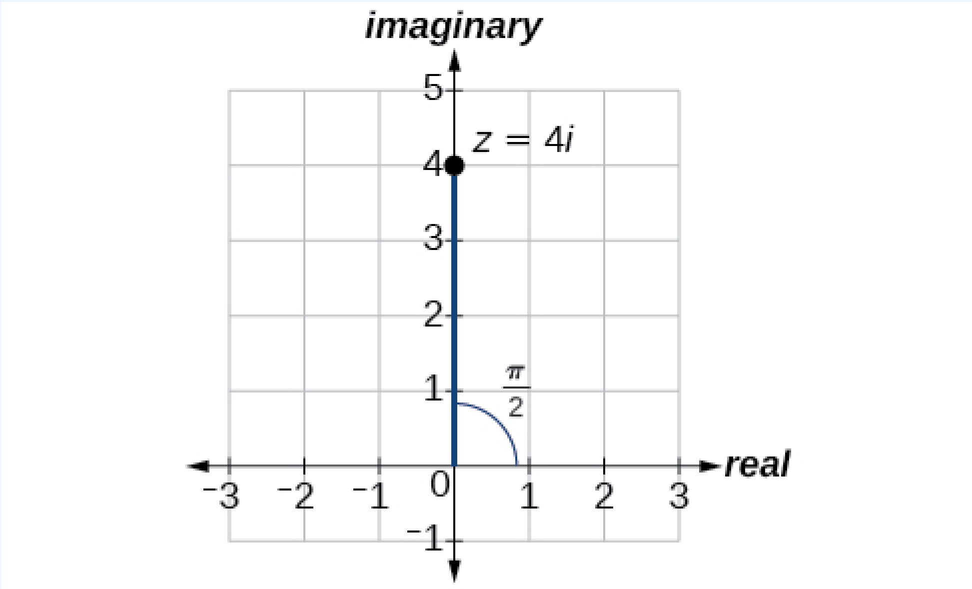


Figure 6.

Try It

Express $z = 3i$ as $r \operatorname{cis} \theta$ in polar form.

Show Solution

$z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

Finding the Polar Form of a Complex Number

Find the polar form of $-4+4i$.

Show Solution

First, find the value of r.

$$r=\sqrt{x^2+y^2}$$
$$r=\sqrt{(-4)^2+(4^2)}$$
$$r=\sqrt{32}$$
$$r=4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta=\frac{x}{r}$$
$$\cos\theta=\frac{-4}{4\sqrt{2}}$$
$$\cos\theta=-\frac{1}{\sqrt{2}}$$
$$\theta=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\frac{3\pi}{4}$$

Thus, the solution is $4\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$.

Try It

Write $z=\sqrt{3}+i$ in polar form.

Show Solution

$$z=2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given $z = r(\cos\theta + i\sin\theta)$, first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$. Then, multiply through by r.

Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:

$z = 12\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

Show Solution

We begin by evaluating the trigonometric expressions.

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

After substitution, the complex number is

$z = 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

We apply the distributive property:

$$\begin{aligned} z &= 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= (12)\frac{\sqrt{3}}{2} + (12)\frac{1}{2}i \\ &= 6\sqrt{3} + 6i \end{aligned}$$

The rectangular form of the given point in complex form is $6\sqrt{3} + 6i$.

Finding the Rectangular Form of a Complex Number

Find the rectangular form of the complex number given $r = 13$ and $\tan\theta = \frac{5}{12}$.

Show Solution

If $\tan\theta = \frac{5}{12}$, and $\tan\theta = \frac{y}{x}$, we first determine $r = \sqrt{x^2+y^2} = \sqrt{12^2+5^2} = 13.$ We then find $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

$$z = 13\left(\cos\theta + i\sin\theta\right)$$
$$= 13\left(\frac{12}{13} + \frac{5}{13}i\right)$$
$$= 12 + 5i$$

The rectangular form of the given number in complex form is $12 + 5i$.

Try It

Convert the complex number to rectangular form:

$$z = 4\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

Show Solution

$$z = 2\sqrt{3} - 2i$$

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham de Moivre (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

Products of Complex Numbers in Polar Form

If $z_1 = r_1 \left(\cos\theta_1 + i\sin\theta_1\right)$ and $z_2 = r_2 \left(\cos\theta_2 + i\sin\theta_2\right)$, then the product of these numbers is given as:

$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]$$
$$z_1 z_2 = r_1 r_2 \text{cis}\left(\theta_1 + \theta_2\right)$$

Notice that the product calls for multiplying the moduli and adding the angles.

Finding the Product of Two Complex Numbers in Polar Form

Find the product of $z_1 z_2$, given $z_1 = 4\left(\cos\left(80\right) + i\sin\left(80\right)\right)$ and $z_2 = 2\left(\cos\left(145\right) + i\sin\left(145\right)\right)$.

Show Solution

Follow the formula

$$z_1 z_2 = 4 \cdot 2\left[\cos\left(80 + 145\right) + i\sin\left(80 + 145\right)\right]$$
$$z_1 z_2 = 8\left[\cos\left(225\right) + i\sin\left(225\right)\right]$$
$$z_1 z_2 = 8\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$z_1 z_2 = 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$z_1 z_2 = -4\sqrt{2} - 4i\sqrt{2}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

Quotients of Complex Numbers in Polar Form

If $z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right],\ \ z_2 \neq 0$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\mathrm{cis}\left(\theta_1 - \theta_2\right),\ \ z_2 \neq 0$$

Notice that the moduli are divided, and the angles are subtracted.

How To

Given two complex numbers in polar form, find the quotient.

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r\left(\cos\theta + i\sin\theta\right)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2\left(\cos\left(213\right) + i\sin\left(213\right)\right)$ and $z_2 = 4\left(\cos\left(33\right) + i\sin\left(33\right)\right)$.

Show Solution

Using the formula, we have

$$\frac{z_1}{z_2} = \frac{2}{4}\left[\cos(213-33) + i\sin(213-33)\right]$$
$$\frac{z_1}{z_2} = \frac{1}{2}\left[\cos(180) + i\sin(180)\right]$$
$$\frac{z_1}{z_2} = \frac{1}{2}\left[-1 + 0i\right]$$
$$\frac{z_1}{z_2} = -\frac{1}{2} + 0i$$
$$\frac{z_1}{z_2} = -\frac{1}{2}$$

Try It

Find the product and the quotient of $z_1 = 2\sqrt{3}\left(\cos(150) + i\sin(150)\right)$ and $z_2 = 2\left(\cos(30) + i\sin(30)\right).$

Show Solution

$z_1 z_2 = -4\sqrt{3};\ \frac{z_1}{z_2} = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using De Moivre's Theorem. It states that, for a positive integer n, z^n is found by raising the modulus to the $n\text{th}$ power and multiplying the argument by n. It is the standard method used in modern mathematics.

De Moivre's Theorem

If $z = r\left(\cos\theta + i\sin\theta\right)$ is a complex number, then

$z^n = r^n\left[\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right]$
$z^n = r^n \text{cis}\left(n\theta\right)$
where n
is a positive integer.

Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

Show Solution

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1+i)$ in polar form. Let us find r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(1)^2 + (1)^2}$$
$$r = \sqrt{2}$$

Then we find θ. Using the formula $\tan\theta = \frac{y}{x}$ gives

$$\tan\theta = \frac{1}{1}$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Use De Moivre's Theorem to evaluate the expression.

$$(a+bi)^n = r^n\left[\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right]$$
$$(1+i)^5 = \left(\sqrt{2}\right)^5\left[\cos\left(5\cdot\frac{\pi}{4}\right) + i\sin\left(5\cdot\frac{\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$(1+i)^5 = -4 - 4i$$

Finding Roots of Complex Numbers in Polar Form

To find the *n*th root of a complex number in polar form, we use the $n\text{th}$ Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding $n\text{th}$ roots of complex numbers in polar form.

The *n*th Root Theorem

To find the $n\text{th}$ root of a complex number in polar form, use the formula given as
$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$

where $k = 0,\ 1,\ 2,\ 3,\ .\ .\ .\ ,\ n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

Finding the *n*th Root of a Complex Number

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$.

Show Solution

We have

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right)\right]$$
$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)\right]$$

There will be three roots: $k = 0,\ 1,\ 2.$ When $k = 0,$ we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)\right)$$

When $k = 1,$ we have

$$\begin{aligned} z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9} + \frac{6\pi}{9}\right) + i\sin\left(\frac{2\pi}{9} + \frac{6\pi}{9}\right)\right] && \text{Add } \frac{2(1)\pi}{3} \text{ to each angle.} \\ z^{\frac{1}{3}} &= 2\left(\cos\left(\frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9}\right)\right) \end{aligned}$$

When $k = 2,$ we have

$$\begin{aligned} z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9} + \frac{12\pi}{9}\right) + i\sin\left(\frac{2\pi}{9} + \frac{12\pi}{9}\right)\right] && \text{Add } \frac{2(2)\pi}{3} \text{ to each angle.} \\ z^{\frac{1}{3}} &= 2\left(\cos\left(\frac{14\pi}{9}\right) + i\sin\left(\frac{14\pi}{9}\right)\right) \end{aligned}$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1,$ the angle simplification is

$$\begin{aligned}\frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3}&=\frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right)\\&=\frac{2\pi}{9}+\frac{6\pi}{9}\\&=\frac{8\pi}{9}\end{aligned}$$

Try It

Find the four fourth roots of $16\left(\cos\left(120\right)+i\sin\left(120\right)\right)$.

Show Solution

$z_0=2\left(\cos\left(30\right)+i\sin\left(30\right)\right)$

$z_1=2\left(\cos\left(120\right)+i\sin\left(120\right)\right)$

$z_2=2\left(\cos\left(210\right)+i\sin\left(210\right)\right)$

$z_3=2\left(\cos\left(300\right)+i\sin\left(300\right)\right)$

Access these online resources for additional instruction and practice with polar forms of complex numbers.

- The Product and Quotient of Complex Numbers in Trigonometric Form
- De Moivre's Theorem

Key Concepts

- Complex numbers in the form $a+bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the *x*-axis as the *real* axis and the *y*-axis as the *imaginary* axis. See (Figure).

- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2 + b^2}$. See (Figure) and (Figure).
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta, y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$. Then, $z = r\left(\cos\theta + i\sin\theta\right)$. See (Figure) and (Figure).
- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See (Figure) and (Figure).
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See (Figure).
- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See (Figure).
- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See (Figure).
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See (Figure).

Section Exercises

Verbal

1. A complex number is $a + bi$. Explain each part.

Show Solution
a is the real part, *b* is the imaginary part, and $i = \sqrt{-1}$

2. What does the absolute value of a complex number represent?

3. How is a complex number converted to polar form?

Show Solution
Polar form converts the real and imaginary part of the complex number in polar form using $x = r\cos\theta$ and $y = r\sin\theta.$

4. How do we find the product of two complex numbers?

5. What is De Moivre's Theorem and what is it used for?

Show Solution
$z^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$ It is used to simplify polar form when a number has been raised to a power.

Algebraic

For the following exercises, find the absolute value of the given complex number.

6. $5 + 3i$

7. $-7 + i$

Show Solution
$5\sqrt{2}$

8. $-3 - 3i$

9. $\sqrt{2} - 6i$

Show Solution

$\sqrt{38}$

10. $2i$

11. $2.2 - 3.1i$

Show Solution
$\sqrt{14.45}$

For the following exercises, write the complex number in polar form.

12. $2 + 2i$

13. $8 - 4i$

Show Solution
$4\sqrt{5}\text{cis}\,(333.4)$

14. $-\frac{1}{2} - \frac{1}{2}i$

15. $\sqrt{3} + i$

Show Solution
$2\text{cis}\left(\frac{\pi}{6}\right)$

16. $3i$

For the following exercises, convert the complex number from polar to rectangular form.

17. $z = 7\text{cis}\left(\frac{\pi}{6}\right)$

Show Solution
$\frac{7\sqrt{3}}{2} + i\frac{7}{2}$

18. $z = 2\text{cis}\left(\frac{\pi}{3}\right)$

19. $z = 4\text{cis}\left(\frac{7\pi}{6}\right)$

Show Solution
$-2\sqrt{3} - 2i$

20. $z = 7\text{cis}\left(25^\circ\right)$

21. $z = 3\text{cis}\left(240^\circ\right)$

Show Solution
$-1.5 - i\frac{3\sqrt{3}}{2}$

22. $z = \sqrt{2}\text{cis}\left(100^\circ\right)$

For the following exercises, find $z_1 z_2$ in polar form.

23. $z_1 = 2\sqrt{3}\text{cis}\left(116^\circ\right); \quad z_2 = 2\text{cis}\left(82^\circ\right)$

Show Solution
$4\sqrt{3}\text{cis}\left(198°\right)$

24. $z_1 = \sqrt{2}\text{cis}\left(205°\right);\ \ z_2 = 2\sqrt{2}\text{cis}\left(118°\right)$
25. $z_1 = 3\text{cis}\left(120°\right);\ \ z_2 = \frac{1}{4}\text{cis}\left(60°\right)$

Show Solution
$\frac{3}{4}\text{cis}\left(180°\right)$

26. $z_1 = 3\text{cis}\left(\frac{\pi}{4}\right);\ \ z_2 = 5\text{cis}\left(\frac{\pi}{6}\right)$
27. $z_1 = \sqrt{5}\text{cis}\left(\frac{5\pi}{8}\right);\ \ z_2 = \sqrt{15}\text{cis}\left(\frac{\pi}{12}\right)$

Show Solution
$5\sqrt{3}\text{cis}\left(\frac{17\pi}{24}\right)$

28. $z_1 = 4\text{cis}\left(\frac{\pi}{2}\right);\ \ z_2 = 2\text{cis}\left(\frac{\pi}{4}\right)$

For the following exercises, find $\frac{z_1}{z_2}$ in polar form.

29. $z_1 = 21\text{cis}\left(135°\right);\ \ z_2 = 3\text{cis}\left(65°\right)$

Show Solution
$7\text{cis}\left(70°\right)$

30. $z_1 = \sqrt{2}\text{cis}\left(90°\right);\ \ z_2 = 2\text{cis}\left(60°\right)$

31. $z_1 = 15\text{cis}\left(120°\right);\ \ z_2 = 3\text{cis}\left(40°\right)$

Show Solution

$5\text{cis}\left(80°\right)$

32. $z_1 = 6\text{cis}\left(\frac{\pi}{3}\right);\ \ z_2 = 2\text{cis}\left(\frac{\pi}{4}\right)$

33. $z_1 = 5\sqrt{2}\text{cis}\left(\pi\right);\ \ z_2 = \sqrt{2}\text{cis}\left(\frac{2\pi}{3}\right)$

Show Solution

$5\text{cis}\left(\frac{\pi}{3}\right)$

34. $z_1 = 2\text{cis}\left(\frac{3\pi}{5}\right);\ \ z_2 = 3\text{cis}\left(\frac{\pi}{4}\right)$

For the following exercises, find the powers of each complex number in polar form.

35. Find z^3 when $z = 5\text{cis}\left(45°\right)$.

Show Solution

$125\text{cis}\left(135°\right)$

36. Find z^4 when $z = 2\text{cis}\left(70°\right)$.

37. Find z^2 when $z = 3\text{cis}\left(120°\right)$.

Show Solution

$9\text{cis}\left(240°\right)$

38. Find z^2 when $z = 4\text{cis}\left(\frac{\pi}{4}\right)$.

39. Find z^4 when $z = \text{cis}\left(\frac{3\pi}{16}\right)$.

Show Solution

$\text{cis}\left(\frac{3\pi}{4}\right)$

40. Find z^3 when $z = 3\text{cis}\left(\frac{5\pi}{3}\right)$.

For the following exercises, evaluate each root.

41. Evaluate the cube root of z when $z = 27\text{cis}\left(240°\right)$.

Show Solution

$3\text{cis}\left(80°\right), 3\text{cis}\left(200°\right), 3\text{cis}\left(320°\right)$

42. Evaluate the square root of z when $z = 16\text{cis}\left(100°\right)$.

43. Evaluate the cube root of z when $z = 32\text{cis}\left(\frac{2\pi}{3}\right)$.

Show Solution

$2\sqrt[3]{4}\text{cis}\left(\frac{2\pi}{9}\right), 2\sqrt[3]{4}\text{cis}\left(\frac{8\pi}{9}\right), 2\sqrt[3]{4}\text{cis}\left(\frac{14\pi}{9}\right)$

44. Evaluate the square root of z when $z = 32\text{cis}\left(\pi\right)$.

45. Evaluate the cube root of z when $z = 8\text{cis}\left(\frac{7\pi}{4}\right)$.

Show Solution
$2\sqrt{2}\text{cis}\left(\frac{7\pi}{8}\right), 2\sqrt{2}\text{cis}\left(\frac{15\pi}{8}\right)$

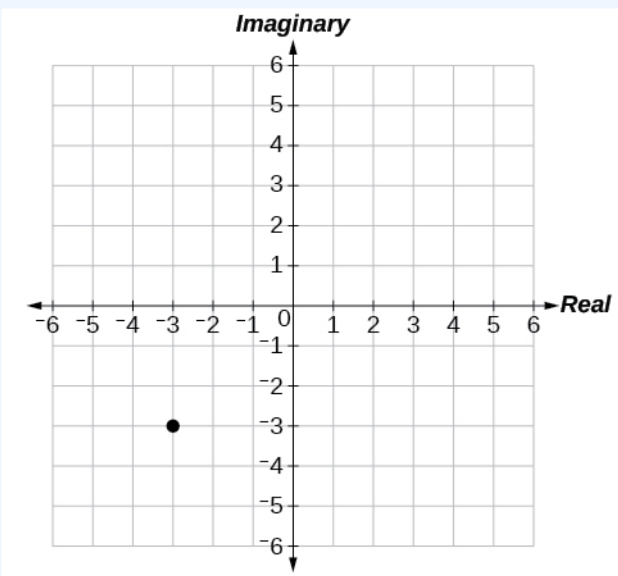
Graphical

For the following exercises, plot the complex number in the complex plane.

46. $2 + 4i$

47. $-3 - 3i$

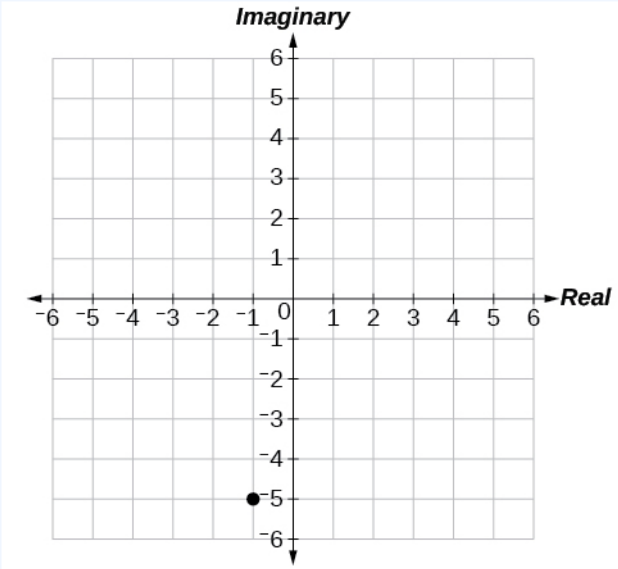
Show Solution



48. $5 - 4i$

49. $-1 - 5i$

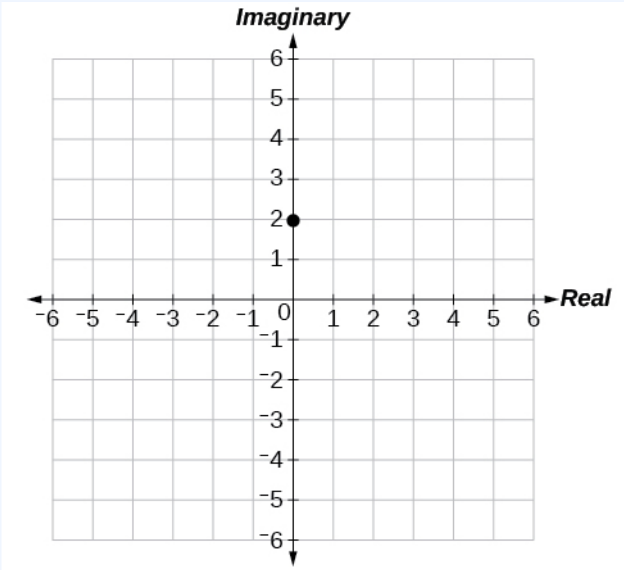
Show Solution



50. $3 + 2i$

51. $2i$

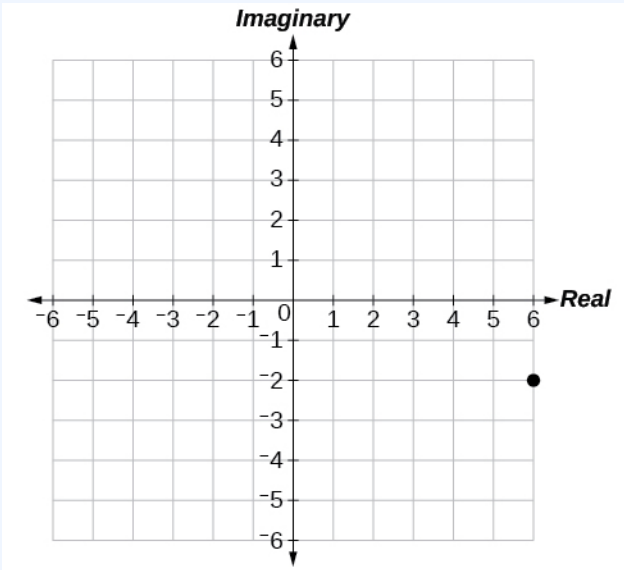
Show Solution



52. -4

53. $6 - 2i$

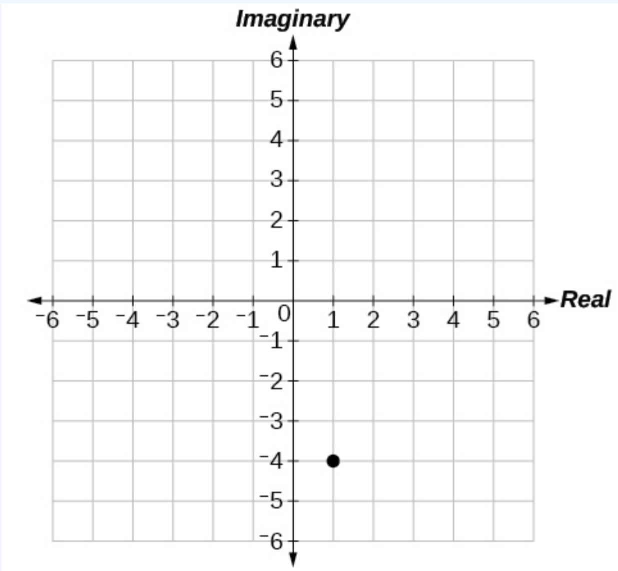
Show Solution



54. $-2+i$

55. $1-4i$

Show Solution



Technology

For the following exercises, find all answers rounded to the nearest hundredth.

56. Use the rectangular to polar feature on the graphing calculator to change $5+5i$ to polar form.

57. Use the rectangular to polar feature on the graphing calculator to change $3-2i$ to polar form.

Show Solution

$3.61e^{-0.59i}$

58. Use the rectangular to polar feature on the graphing calculator to change $-3-8i$ to polar form.

59. Use the polar to rectangular feature on the graphing calculator to change $4\text{cis}\,(120°)$ to rectangular form.

Show Solution
$-2+3.46i$

60. Use the polar to rectangular feature on the graphing calculator to change $2\text{cis}\,(45°)$ to rectangular form.

61. Use the polar to rectangular feature on the graphing calculator to change $5\text{cis}\,(210°)$ to rectangular form.

Show Solution
$-4.33-2.50i$

Glossary

argument
: the angle associated with a complex number; the angle between the line from the origin to the point and the positive real axis

De Moivre's Theorem
: formula used to find the $n\text{th}$ power or *n*th roots of a complex number; states that, for a positive integer $n,\ z^n$ is found by raising the modulus to the $n\text{th}$ power and multiplying the angles by n

modulus
: the absolute value of a complex number, or the distance from the origin to the point (x,y) ; also called the amplitude

polar form of a complex number
: a complex number expressed in terms of an angle θ and its distance from the origin r; can be found by using conversion formulas $x = r\cos\theta,\ y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$

CHAPTER 4.7: PARAMETRIC EQUATIONS

Learning Objectives

In this section, you will:

- Parameterize a curve.
- Eliminate the parameter.
- Find a rectangular equation for a curve defined parametrically.
- Find parametric equations for curves defined by rectangular equations.

Consider the path a moon follows as it orbits a planet, which simultaneously rotates around the sun, as seen in (Figure). At any moment, the moon is located at a particular spot relative to the planet. But how do we write and solve the equation for the position of the moon when the distance from the planet, the speed of the moon's orbit around the planet, and the speed of rotation around the sun are all unknowns? We can solve only for one variable at a time.

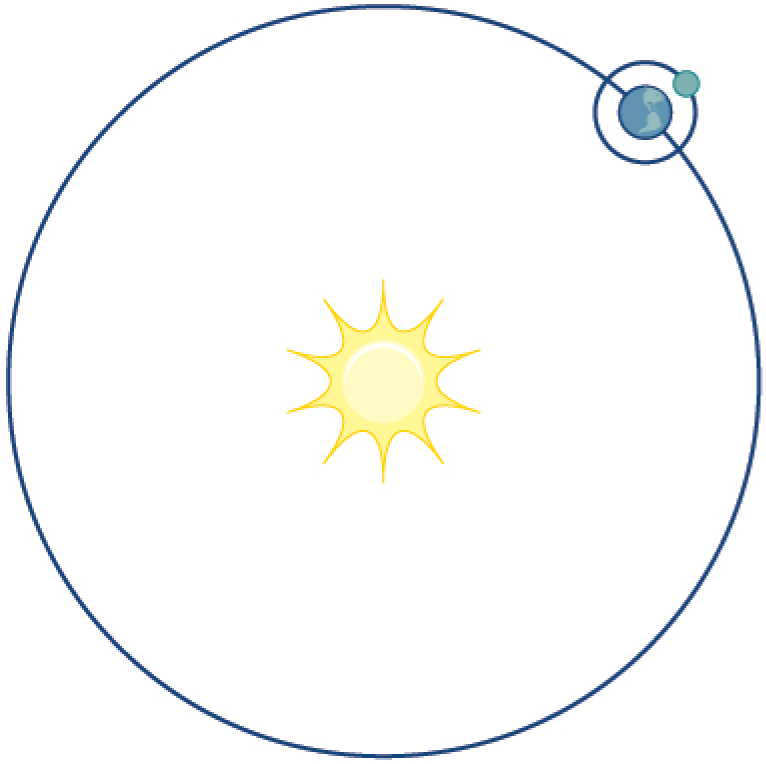


Figure 1.

In this section, we will consider sets of equations given by $x(t)$ and $y(t)$ where t is the independent variable of time. We can use these parametric equations in a number of applications when we are looking for not only a particular position but also the direction of the movement. As we trace out successive values of t, the orientation of the curve becomes clear. This is one of the primary advantages of using parametric equations: we are able to trace the movement of an object along a path according to time. We begin this section with a look at the basic components of parametric equations and what it means to parameterize a curve. Then we will learn how to eliminate the parameter, translate the equations of a curve defined parametrically into rectangular equations, and find the parametric equations for curves defined by rectangular equations.

Parameterizing a Curve

When an object moves along a curve—or curvilinear path—in a given direction and in a given amount of time, the position of the object in the plane is given by the *x*-coordinate and the *y*-coordinate. However, both x and y vary over time and so are functions of time. For this reason, we add another variable, the parameter, upon which both x and y are dependent functions. In the example in the section opener, the parameter is time, t. The x position of the moon at time, t, is represented as the function $x(t)$, and the y position of the moon at time, t, is represented as the function $y(t)$. Together, $x(t)$ and $y(t)$ are called parametric equations, and generate an ordered pair $(x(t),\ y(t))$. Parametric equations primarily describe motion and direction.

When we parameterize a curve, we are translating a single equation in two variables, such as x and y, into an equivalent pair of equations in three variables, x, y, and t. One of the reasons we parameterize a curve is because the parametric equations yield more information: specifically, the direction of the object's motion over time.

When we graph parametric equations, we can observe the individual behaviors of x and of y. There are a number of shapes that cannot be represented in the form $y = f(x)$, meaning that they are not functions. For example, consider the graph of a circle, given as $r^2 = x^2 + y^2$. Solving for y gives $y = \sqrt{r^2 - x^2}$, or two equations: $y_1 = \sqrt{r^2 - x^2}$ and $y_2 = -\sqrt{r^2 - x^2}$. If we graph y_1 and y_2 together, the graph will not pass the vertical line test, as shown in (Figure). Thus, the equation for the graph of a circle is not a function.

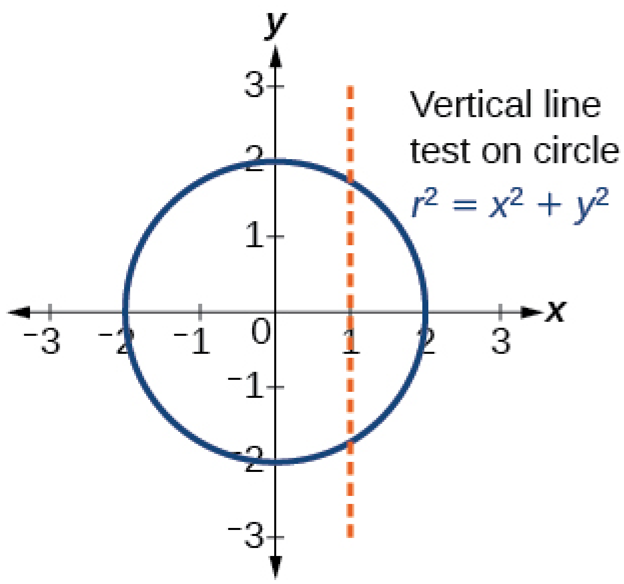


Figure 2.

However, if we were to graph each equation on its own, each one would pass the vertical line test and therefore would represent a function. In some instances, the concept of breaking up the equation for a circle into two functions is similar to the concept of creating parametric equations, as we use two functions to produce a non-function. This will become clearer as we move forward.

Parametric Equations

Suppose t is a number on an interval, I. The set of ordered pairs, $(x(t),\ y(t))$, where

$x = f(t)$ and $y = g(t)$, forms a plane curve based on the parameter t. The equations $x = f(t)$ and $y = g(t)$ are the parametric equations.

Parameterizing a Curve

Parameterize the curve $y = x^2 - 1$ letting $x(t) = t$. Graph both equations.

Show Solution

If $x(t) = t$, then to find $y(t)$ we replace the variable x with the expression given in $x(t)$. In other words, $y(t) = t^2 - 1$. Make a table of values similar to (Figure), and sketch the graph.

t	$x(t)$	$y(t)$
-4	-4	$y(-4) = (-4)^2 - 1 = 15$
-3	-3	$y(-3) = (-3)^2 - 1 = 8$
-2	-2	$y(-2) = (-2)^2 - 1 = 3$
-1	-1	$y(-1) = (-1)^2 - 1 = 0$
0	0	$y(0) = (0)^2 - 1 = -1$
1	1	$y(1) = (1)^2 - 1 = 0$
2	2	$y(2) = (2)^2 - 1 = 3$
3	3	$y(3) = (3)^2 - 1 = 8$
4	4	$y(4) = (4)^2 - 1 = 15$

See the graphs in (Figure). It may be helpful to use the TRACE feature of a graphing calculator to see how the points are generated as t increases.

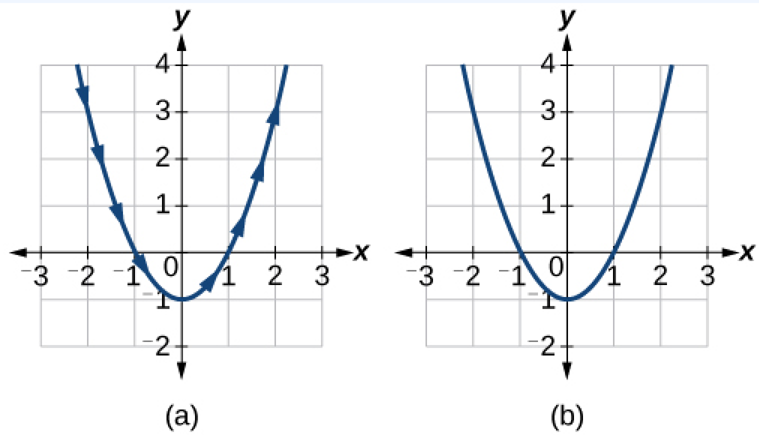


Figure 3. (a) Parametric $y\left(t\right)=t^2-1$ (b) Rectangular $y=x^2-1$

Analysis

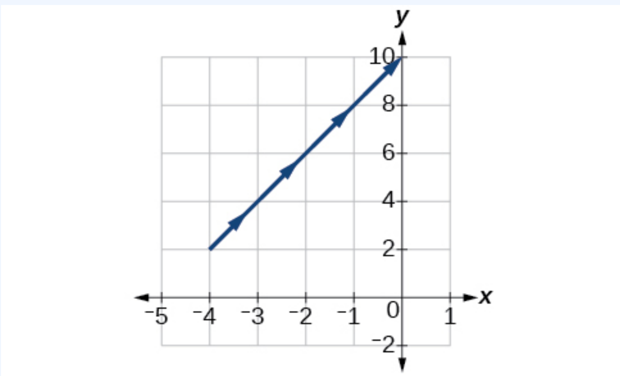
The arrows indicate the direction in which the curve is generated. Notice the curve is identical to the curve of $y=x^2-1$.

Try It

Construct a table of values and plot the parametric equations:
$x\left(t\right)=t-3,\;\;y\left(t\right)=2t+4;\;\;-1\le t\le 2.$

Show Solution

t	$x(t)$	$y(t)$
-1	-4	2
0	-3	4
1	-2	6
2	-1	8



Finding a Pair of Parametric Equations

Find a pair of parametric equations that models the graph of $y = 1 - x^2$, using the parameter $x(t) = t$. Plot some points and sketch the graph.

Show Solution

If $x(t) = t$ and we substitute t for x into the y equation, then $y(t) = 1 - t^2$. Our pair of parametric equations is

$x(t) = t$
$y(t) = 1 - t^2$

To graph the equations, first we construct a table of values like that in (Figure). We can choose values around $t = 0$, from $t = -3$ to $t = 3$. The values in the $x(t)$ column will be the same as those in the t column because $x(t) = t$. Calculate values for the column $y(t)$.

t	$x(t) = t$	$y(t) = 1 - t^2$
-3	-3	$y(-3) = 1 - (-3)^2 = -8$
-2	-2	$y(-2) = 1 - (-2)^2 = -3$
-1	-1	$y(-1) = 1 - (-1)^2 = 0$
0	0	$y(0) = 1 - 0 = 1$
1	1	$y(1) = 1 - (1)^2 = 0$
2	2	$y(2) = 1 - (2)^2 = -3$
3	3	$y(3) = 1 - (3)^2 = -8$

The graph of $y = 1 - t^2$ is a parabola facing downward, as shown in (Figure). We have mapped the curve over the interval $[-3, 3]$, shown as a solid line with arrows indicating the orientation of the curve according to t. Orientation refers to the path traced along the curve in terms of increasing values of t. As this parabola is symmetric with respect to the line $x = 0$, the values of x are reflected across the *y*-axis.

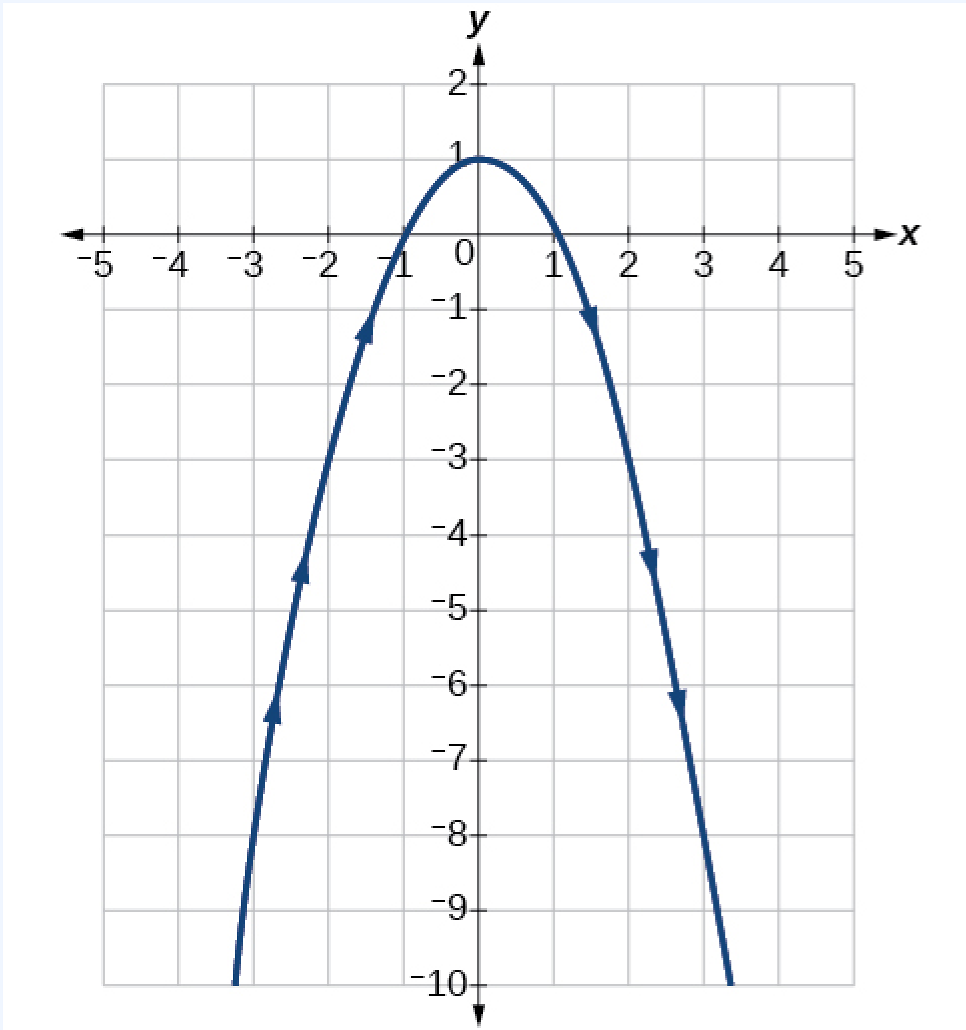


Figure 4.

Try It

Parameterize the curve given by $x = y^3 - 2y.$

Show Solution

$x\left(t\right) = t^3 - 2t$

$y\left(t\right) = t$

Finding Parametric Equations That Model Given Criteria

An object travels at a steady rate along a straight path $\left(-5,\ 3\right)$ to $\left(3,\ -1\right)$ in the same plane in four seconds. The coordinates are measured in meters. Find parametric equations for the position of the object.

Show Solution

The parametric equations are simple linear expressions, but we need to view this problem in a step-by-step fashion. The *x*-value of the object starts at -5 meters and goes to 3 meters. This means the distance *x* has changed by 8 meters in 4 seconds, which is a rate of $\frac{8\ \text{m}}{4\ \text{s}}$, or $2\,\text{m/s}.$ We can write the *x*-coordinate as a linear function with respect to time as $x\left(t\right) = 2t - 5.$ In the linear function template $y = mx + b, 2t = mx$ and $-5 = b.$

Similarly, the *y*-value of the object starts at 3 and goes to $-1,$ which is a change in the distance *y* of -4 meters in 4 seconds, which is a rate of $\frac{-4\ \text{m}}{4\ \text{s}}$, or $-1\text{m/s}.$ We can also write the *y*-coordinate as the linear function $y\left(t\right) = -t + 3.$ Together, these are the parametric equations for the position of the object, where x
and y
are expressed in meters and t
represents time:

$$x(t) = 2t - 5$$
$$y(t) = -t + 3$$

Using these equations, we can build a table of values for t, x, and y (see (Figure)). In this example, we limited values of t to non-negative numbers. In general, any value of t can be used.

t	$x(t) = 2t - 5$	$y(t) = -t + 3$
0	$x = 2(0) - 5 = -5$	$y = -(0) + 3 = 3$
1	$x = 2(1) - 5 = -3$	$y = -(1) + 3 = 2$
2	$x = 2(2) - 5 = -1$	$y = -(2) + 3 = 1$
3	$x = 2(3) - 5 = 1$	$y = -(3) + 3 = 0$
4	$x = 2(4) - 5 = 3$	$y = -(4) + 3 = -1$

From this table, we can create three graphs, as shown in (Figure).

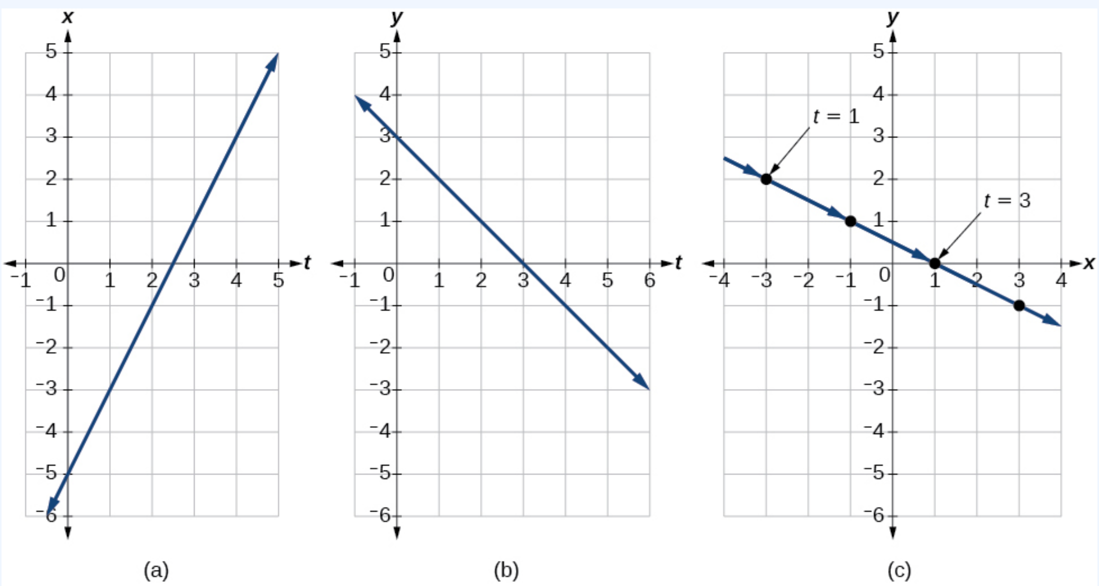


Figure 5. (a) A graph of x vs. t, representing the horizontal position over time. (b) A graph of y vs. t, representing the vertical position over time. (c) A graph of y vs. x, representing the position of the object in the plane at time t.

Analysis

Again, we see that, in (Figure)(c), when the parameter represents time, we can indicate the movement of the object along the path with arrows.

Eliminating the Parameter

In many cases, we may have a pair of parametric equations but find that it is simpler to draw a curve if the equation involves only two variables, such as x and y. Eliminating the parameter is a method that may make graphing some curves easier. However, if we are concerned with the mapping of the equation according to time, then it will be necessary to indicate the orientation of the curve as well. There are various methods for eliminating the parameter t from a set of parametric equations; not every method works for every type of equation. Here we will review the methods for the most common types of equations.

Eliminating the Parameter from Polynomial, Exponential, and Logarithmic Equations

For polynomial, exponential, or logarithmic equations expressed as two parametric equations, we choose the equation that is most easily manipulated and solve for t. We substitute the resulting expression for t into the second equation. This gives one equation in x and y.

Eliminating the Parameter in Polynomials

Given $x\left(t\right)={t}^{2}+1$ and $y\left(t\right)=2+t$, eliminate the parameter, and write the parametric equations as a Cartesian equation.

Show Solution
We will begin with the equation for y because the linear equation is easier to solve for t.

$$\begin{aligned} y &= 2+t \\ y-2 &= t \end{aligned}$$

Next, substitute $y-2$ for t in $x\left(t\right)$.

$$x=t^2+1$$
$$x={\left(y-2\right)}^{2}+1 \qquad \text{Substitute the expression for } t \text{ into } x.$$
$$x=y^2-4y+4+1$$
$$x=y^2-4y+5$$
$$x=y^2-4y+5$$

The Cartesian form is $x=y^2-4y+5.$

Analysis

This is an equation for a parabola in which, in rectangular terms, x is dependent on $y.$ From the curve's vertex at $\left(1,2\right)$, the graph sweeps out to the right. See (Figure). In this section, we consider sets of equations given by the functions $x\left(t\right)$ and $y\left(t\right)$, where t is the independent variable of time. Notice, both x and y are functions of time; so in general y is not a function of x.

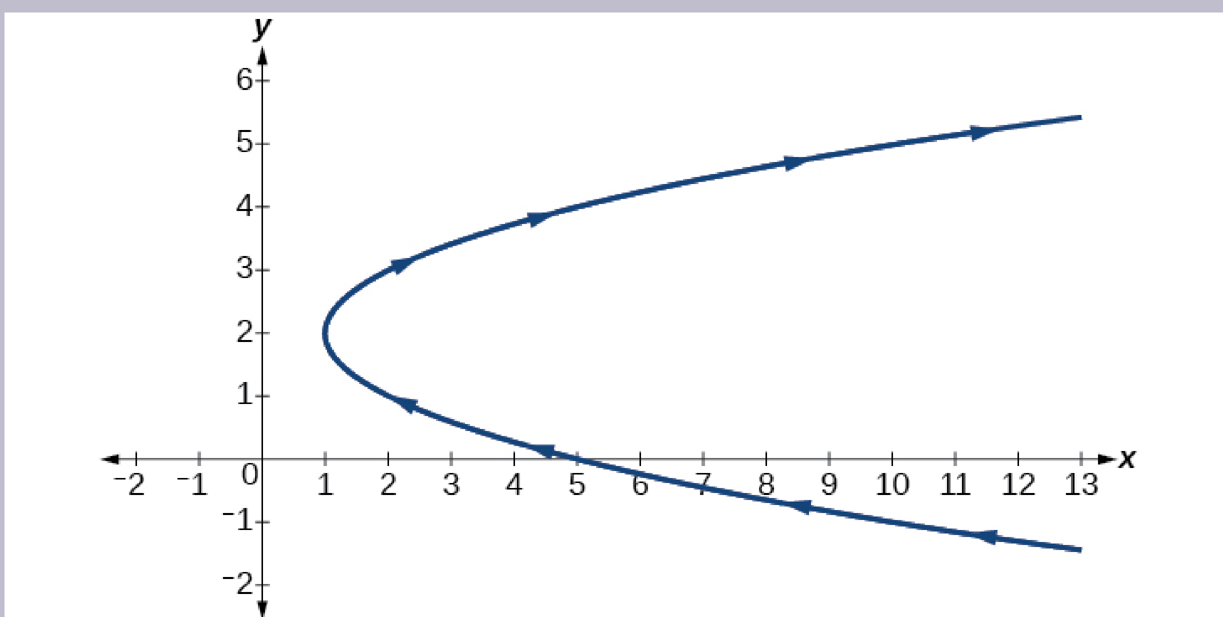


Figure 6.

Try It

Given the equations below, eliminate the parameter and write as a rectangular equation for y as a function

$x(t) = 2t^2 + 6$
$y(t) = 5 - t$

Show Solution

$y = 5 - \sqrt{\frac{1}{2}x - 3}$

Eliminating the Parameter in Exponential Equations

Eliminate the parameter and write as a Cartesian equation: $x(t) = e^{-t}$ and $y(t) = 3e^t,\ t > 0.$

Show Solution

Isolate e^t.

$x = e^{-t}$
$e^t = \frac{1}{x}$

Substitute the expression into $y(t)$.

$y = 3e^t$
$y = 3\left(\frac{1}{x}\right)$
$y = \frac{3}{x}$

The Cartesian form is $y = \frac{3}{x}$.

Analysis

The graph of the parametric equation is shown in (Figure)**(a)**. The domain is restricted to $t > 0$. The Cartesian equation, $y = \frac{3}{x}$ is shown in (Figure)**(b)** and has only one restriction on the domain, $x \neq 0$.

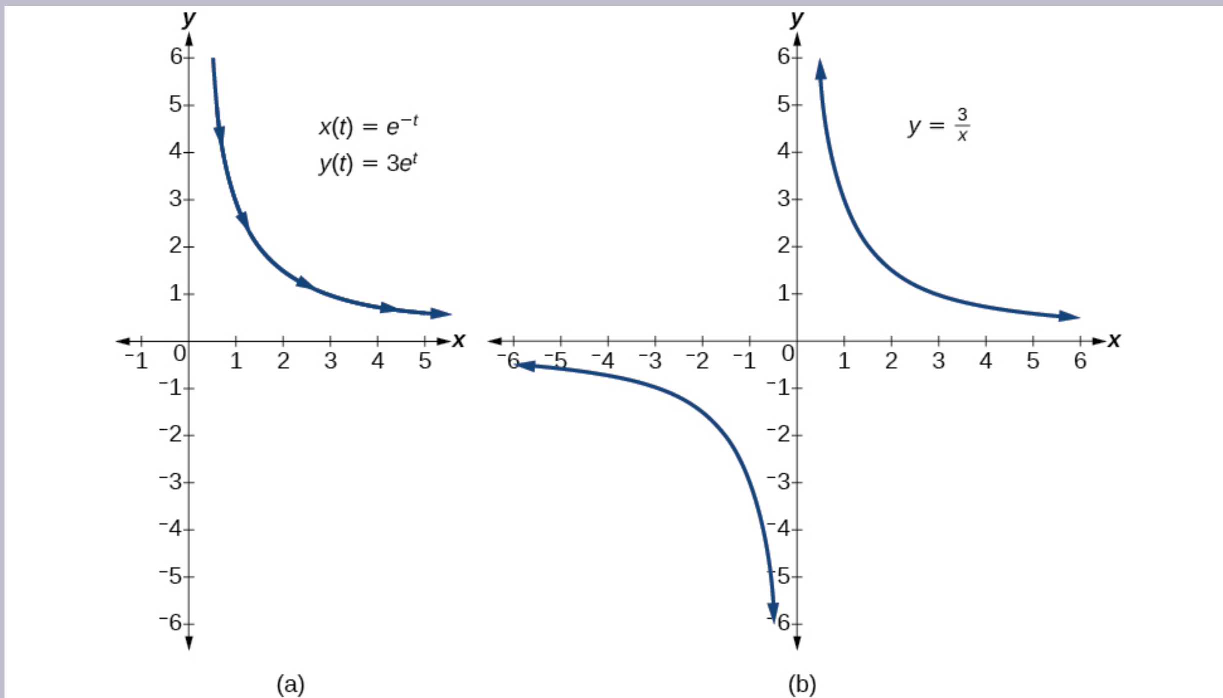


Figure 7.

Eliminating the Parameter in Logarithmic Equations

Eliminate the parameter and write as a Cartesian equation: $x(t) = \sqrt{t} + 2$ and $y(t) = \log(t)$.

Show Solution
Solve the first equation for t.

$x = \sqrt{t} + 2$
$x - 2 = \sqrt{t}$
$(x - 2)^2 = t \quad \text{Square both sides.}$

Then, substitute the expression for t into the y equation.

$y = \log(t)$
$y = \log(x - 2)^2$

The Cartesian form is $y = \log(x - 2)^2$.

Analysis

To be sure that the parametric equations are equivalent to the Cartesian equation, check the domains. The parametric equations restrict the domain on $x = \sqrt{t} + 2$ to $t > 0$; we restrict the domain on x to $x > 2$. The domain for the parametric equation $y = \log(t)$ is restricted to $t > 0$; we limit the domain on $y = \log(x - 2)^2$ to $x > 2$.

Try It

Eliminate the parameter and write as a rectangular equation.

$x(t) = t^2$
$y(t) = \ln t \qquad t > 0$

Show Solution

$y = \ln\sqrt{x}$

Eliminating the Parameter from Trigonometric Equations

Eliminating the parameter from trigonometric equations is a straightforward substitution. We can use a few of the familiar trigonometric identities and the Pythagorean Theorem.

First, we use the identities:

$$x(t) = a\cos t$$
$$y(t) = b\sin t$$

Solving for $\cos t$ and $\sin t$, we have

$$\frac{x}{a} = \cos t$$
$$\frac{y}{b} = \sin t$$

Then, use the Pythagorean Theorem:

$$\cos^2 t + \sin^2 t = 1$$

Substituting gives

$$\cos^2 t + \sin^2 t = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Eliminating the Parameter from a Pair of Trigonometric Parametric Equations

Eliminate the parameter from the given pair of trigonometric equations where $0 \leq t \leq 2\pi$ and sketch the graph.

$$x(t) = 4\cos t$$
$$y(t) = 3\sin t$$

Show Solution

Solving for $\cos t$ and $\sin t$, we have

$$x = 4\cos t$$
$$\frac{x}{4} = \cos t$$
$$y = 3\sin t$$
$$\frac{y}{3} = \sin t$$

Next, use the Pythagorean identity and make the substitutions.

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

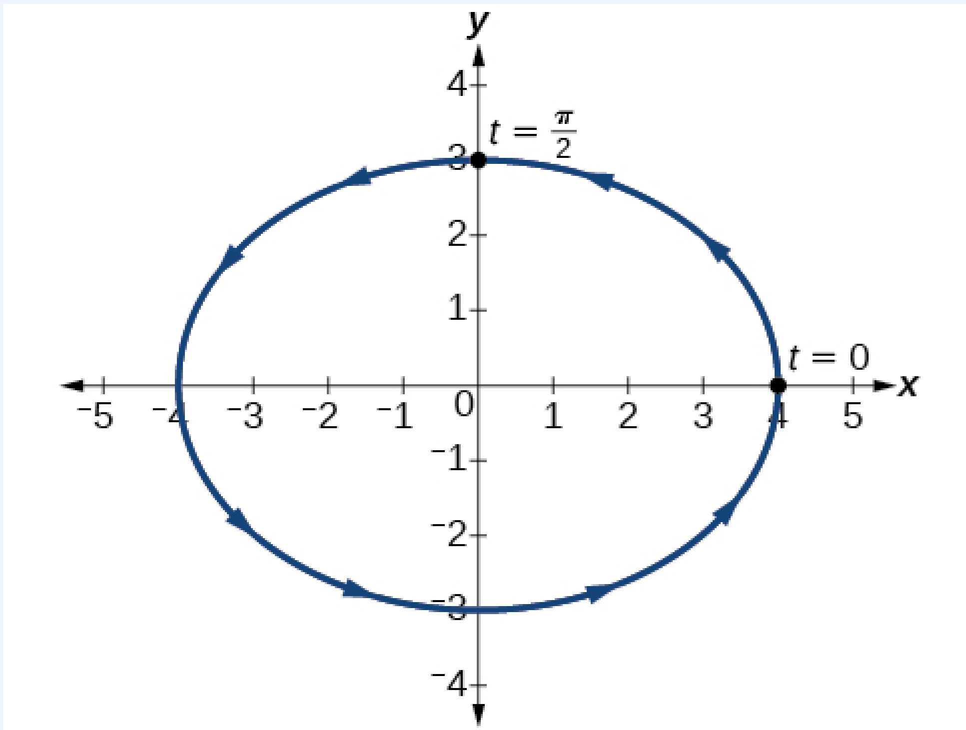


Figure 8.

The graph for the equation is shown in (Figure).

Analysis

Applying the general equations for conic sections (introduced in Analytic Geometry, we can identify $\frac{x^2}{16} + \frac{y^2}{9} = 1$ as an ellipse centered at $(0, 0)$. Notice that when $t = 0$ the coordinates are $(4, 0)$, and when $t = \frac{\pi}{2}$ the coordinates are $(0, 3)$. This shows the orientation of the curve with increasing values of t.

Try It

Eliminate the parameter from the given pair of parametric equations and write as a Cartesian equation: $x\left(t\right)=2\cos t$ and $y\left(t\right)=3\sin t$.

Show Solution
$\frac{x^2}{4}+\frac{y^2}{9}=1$

Finding Cartesian Equations from Curves Defined Parametrically

When we are given a set of parametric equations and need to find an equivalent Cartesian equation, we are essentially "eliminating the parameter." However, there are various methods we can use to rewrite a set of parametric equations as a Cartesian equation. The simplest method is to set one equation equal to the parameter, such as $x\left(t\right)=t$. In this case, $y\left(t\right)$ can be any expression. For example, consider the following pair of equations.

$x\left(t\right)=t$
$y\left(t\right)=t^2-3$

Rewriting this set of parametric equations is a matter of substituting x for t. Thus, the Cartesian equation is $y=x^2-3$.

Finding a Cartesian Equation Using Alternate Methods

Use two different methods to find the Cartesian equation equivalent to the given set of parametric equations.

$x\left(t\right)=3t-2$
$y\left(t\right)=t+1$

Show Solution

Method 1. First, let's solve the x equation for $t.$ Then we can substitute the result into the y equation.

$x = 3t - 2$
$x + 2 = 3t$
$\frac{x+2}{3} = t$

Now substitute the expression for t into the y equation.

$y = t + 1$
$y = \left(\frac{x+2}{3}\right) + 1$
$y = \frac{x}{3} + \frac{2}{3} + 1$
$y = \frac{1}{3}x + \frac{5}{3}$

Method 2. Solve the y equation for t and substitute this expression in the x equation.

$y = t + 1$
$y - 1 = t$

Make the substitution and then solve for $y.$

$x = 3\left(y - 1\right) - 2$
$x = 3y - 3 - 2$
$x = 3y - 5$
$x + 5 = 3y$
$\frac{x+5}{3} = y$
$y = \frac{1}{3}x + \frac{5}{3}$

Try It

Write the given parametric equations as a Cartesian equation: $x\left(t\right) = t^3$ and $y\left(t\right) = t^6.$

Show Solution

$y = x^2$

Finding Parametric Equations for Curves Defined by Rectangular

Equations

Although we have just shown that there is only one way to interpret a set of parametric equations as a rectangular equation, there are multiple ways to interpret a rectangular equation as a set of parametric equations. Any strategy we may use to find the parametric equations is valid if it produces equivalency. In other words, if we choose an expression to represent x, and then substitute it into the y equation, and it produces the same graph over the same domain as the rectangular equation, then the set of parametric equations is valid. If the domain becomes restricted in the set of parametric equations, and the function does not allow the same values for x as the domain of the rectangular equation, then the graphs will be different.

Finding a Set of Parametric Equations for Curves Defined by Rectangular Equations

Find a set of equivalent parametric equations for $y=(x+3)^2+1$.

Show Solution

An obvious choice would be to let $x(t)=t$. Then $y(t)=(t+3)^2+1$. But let's try something more interesting. What if we let $x=t+3$? Then we have

$$y=(x+3)^2+1$$
$$y=((t+3)+3)^2+1$$
$$y=(t+6)^2+1$$

The set of parametric equations is

$$x(t)=t+3$$
$$y(t)=(t+6)^2+1$$

See (Figure).

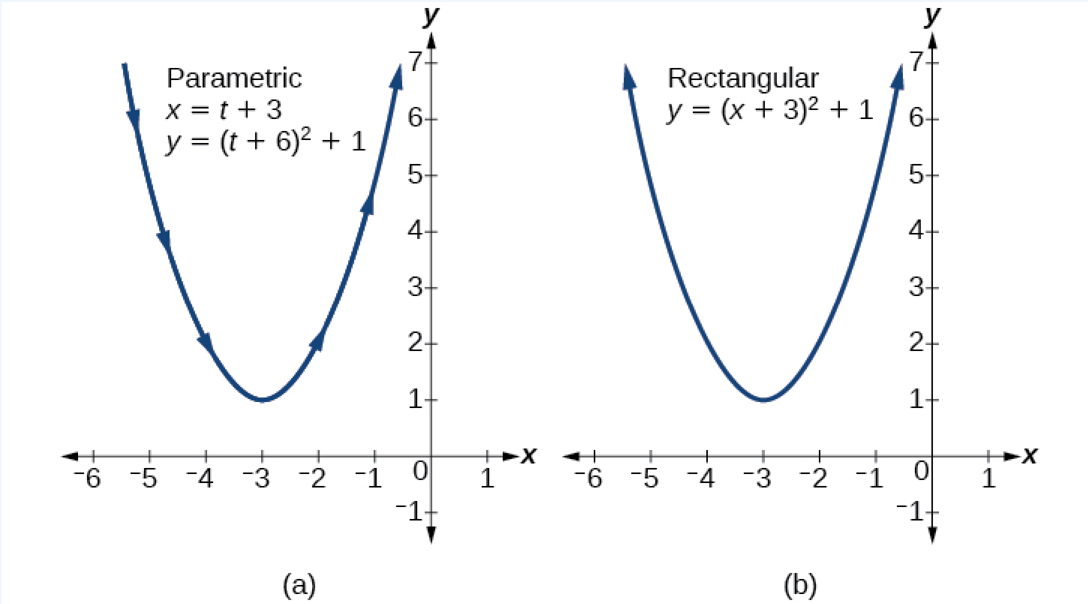


Figure 6.

Access these online resources for additional instruction and practice with parametric equations.

- Introduction to Parametric Equations
- Converting Parametric Equations to Rectangular Form

Key Concepts

- Parameterizing a curve involves translating a rectangular equation in two variables, x and y, into two equations in three variables, *x*, *y*, and *t*. Often, more information is obtained from a set of parametric equations. See (Figure), (Figure), and (Figure).
- Sometimes equations are simpler to graph when written in rectangular form. By eliminating t, an equation in x and y is the result.

- To eliminate t, solve one of the equations for t, and substitute the expression into the second equation. See (Figure), (Figure), (Figure), and (Figure).
- Finding the rectangular equation for a curve defined parametrically is basically the same as eliminating the parameter. Solve for t in one of the equations, and substitute the expression into the second equation. See (Figure).
- There are an infinite number of ways to choose a set of parametric equations for a curve defined as a rectangular equation.
- Find an expression for x such that the domain of the set of parametric equations remains the same as the original rectangular equation. See (Figure).

Section Exercises

Verbal

1. What is a system of parametric equations?

Show Solution
A pair of functions that is dependent on an external factor. The two functions are written in terms of the same parameter. For example, $x = f\left(t\right)$ and $y = f\left(t\right).$

2. Some examples of a third parameter are time, length, speed, and scale. Explain when time is used as a parameter.

3. Explain how to eliminate a parameter given a set of parametric equations.

Show Solution
Choose one equation to solve for $t,$ substitute into the other equation and simplify.

4. What is a benefit of writing a system of parametric equations as a Cartesian equation?

5. What is a benefit of using parametric equations?

Show Solution

Some equations cannot be written as functions, like a circle. However, when written as two parametric equations, separately the equations are functions.

6. Why are there many sets of parametric equations to represent on Cartesian function?

Algebraic

For the following exercises, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

7. $\begin{cases} x(t) = 5 - t \\ y(t) = 8 - 2t \end{cases}$

Show Solution

$y = -2 + 2x$

8. $\begin{cases} x(t) = 6 - 3t \\ y(t) = 10 - t \end{cases}$

---

9. $\begin{cases} x(t) = 2t + 1 \\ y(t) = 3\sqrt{t} \end{cases}$

Show Solution

$y = 3\sqrt{\frac{x-1}{2}}$

10. $\begin{cases} x(t) = 3t - 1 \\ y(t) = 2t^2 \end{cases}$

---

11. $\begin{cases} x(t) = 2e^t \\ y(t) = 1 - 5t \end{cases}$

Show Solution

$x = 2e^{\frac{1-y}{5}}$ or $y = 1 - 5ln\left(\frac{x}{2}\right)$

12. $\begin{cases} x(t) = e^{-2t} \\ y(t) = 2e^{-t} \end{cases}$

---

13. $\begin{cases} x(t) = 4\log(t) \\ y(t) = 3 + 2t \end{cases}$

Show Solution

$x = 4\log\left(\frac{y-3}{2}\right)$

14. $\begin{cases} x(t) = \log(2t) \\ y(t) = \sqrt{t-1} \end{cases}$

---

15. $\begin{cases} x(t) = t^3 - t \\ y(t) = 2t \end{cases}$

Show Solution

$x=\left(\frac{y}{2}\right)^3-\frac{y}{2}$

16. $\begin{cases} x(t)=t-t^4 \\ y(t)=t+2 \end{cases}$

---

17. $\begin{cases} x(t)=e^{2t} \\ y(t)=e^{6t} \end{cases}$

Show Solution

$y=x^3$

18. $\begin{cases} x(t)=t^5 \\ y(t)=t^{10} \end{cases}$

---

19. $\begin{cases} x(t)=4\cos t \\ y(t)=5\sin t \end{cases}$

Show Solution

$\left(\frac{x}{4}\right)^2+\left(\frac{y}{5}\right)^2=1$

20. $\begin{cases} x(t)=3\sin t \\ y(t)=6\cos t \end{cases}$

---

21. $\begin{cases} x(t) = 2\cos^2 t \\ y(t) = -\sin t \end{cases}$

Show Solution

$y^2 = 1 - \frac{1}{2}x$

22. $\begin{cases} x(t) = \cos t + 4 \\ y(t) = 2\sin^2 t \end{cases}$

---

23. $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$

Show Solution

$y = x^2 + 2x + 1$

24. $\begin{cases} x(t) = -t \\ y(t) = t^3 + 1 \end{cases}$

---

25. $\begin{cases} x(t) = 2t - 1 \\ y(t) = t^3 - 2 \end{cases}$

Show Solution

$y = \left(\frac{x+1}{2}\right)^3 - 2$

For the following exercises, rewrite the parametric equation as a Cartesian equation by building an x-y table.

25. $\begin{cases} x(t) = 2t - 1 \\ y(t) = t + 4 \end{cases}$

---

26. $\begin{cases} x(t) = 4 - t \\ y(t) = 3t + 2 \end{cases}$

Show Solution

$y = -3x + 14$

27. $\begin{cases} x(t) = 2t - 1 \\ y(t) = 5t \end{cases}$

---

28. $\begin{cases} x(t) = 4t - 1 \\ y(t) = 4t + 2 \end{cases}$

Show Solution

$y = x + 3$

For the following exercises, parameterize (write parametric equations for) each Cartesian equation by setting $x\left(t\right) = t$ or by setting $y\left(t\right) = t$.

29. $y\left(x\right) = 3x^2 + 3$

30. $y\left(x\right) = 2\sin x + 1$

Show Solution

$\begin{cases} x(t) = t \\ y(t) = 2\text{sin}t + 1 \end{cases}$

40. $x\left(y\right)=3\log\left(y\right)+y$

41. $x\left(y\right)=\sqrt{y}+2y$

Show Solution
$\begin{cases} x(t)=\sqrt{t}+2t \\ y(t)=t \end{cases}$

For the following exercises, parameterize (write parametric equations for) each Cartesian equation by using $x\left(t\right)=a\cos t$ and $y\left(t\right)=b\sin t.$ Identify the curve.

42. $\frac{x^2}{4}+\frac{y^2}{9}=1$

---

43. $\frac{x^2}{16}+\frac{y^2}{36}=1$

Show Solution
$\begin{cases} x(t)=4\cos t \\ y(t)=6\sin t \end{cases}$; Ellipse

44. $x^2+y^2=16$

45. $x^2+y^2=10$

Show Solution
$\begin{cases} x(t)=\sqrt{10}\text{cos}t \\ y(t)=\sqrt{10}\text{sin}t \end{cases}$;
Circle

46. Parameterize the line from $(3, 0)$ to $(-2, -5)$ so that the line is at $(3, 0)$ at $t = 0$, and at $(-2, -5)$ at $t = 1$.

47. Parameterize the line from $(-1, 0)$ to $(3, -2)$ so that the line is at $(-1, 0)$ at $t = 0$, and at $(3, -2)$ at $t = 1$.

Show Solution

$$\begin{cases} x(t) = -1 + 4t \\ y(t) = -2t \end{cases}$$

48. Parameterize the line from $(-1, 5)$ to $(2, 3)$ so that the line is at $(-1, 5)$ at $t = 0$, and at $(2, 3)$ at $t = 1$.

49. Parameterize the line from $(4, 1)$ to $(6, -2)$ so that the line is at $(4, 1)$ at $t = 0$, and at $(6, -2)$ at $t = 1$.

Show Solution

$$\begin{cases} x(t) = 4 + 2t \\ y(t) = 1 - 3t \end{cases}$$

Technology

For the following exercises, use the table feature in the graphing calculator to determine whether the graphs intersect.

50. $\begin{cases} x_1(t) = 3t \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = t + 3 \\ y_2(t) = 4t - 4 \end{cases}$

---

51. $\begin{cases} x_1(t) = t^2 \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = -t + 6 \\ y_2(t) = t + 1 \end{cases}$

Show Solution

yes, at $t = 2$

For the following exercises, use a graphing calculator to complete the table of values for each set of parametric equations.

52. $\begin{cases} x_1(t) = 3t^2 - 3t + 7 \\ y_1(t) = 2t + 3 \end{cases}$

t	x	y
–1		
0		
1		

53. $\begin{cases} x_1(t) = t^2 - 4 \\ y_1(t) = 2t^2 - 1 \end{cases}$

t	x	y
1		
2		
3		

Show Solution

t	x	y
1	-3	1
2	0	7
3	5	17

54. $\begin{cases} x_1(t) = t^4 \\ y_1(t) = t^3 + 4 \end{cases}$

t	x	y
-1		
0		
1		
2		

Extensions

55. Find two different sets of parametric equations for $y = (x+1)^2$.

Show Solution

answers may vary: $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$ and $\begin{cases} x(t) = t + 1 \\ y(t) = (t+2)^2 \end{cases}$

56. Find two different sets of parametric equations for $y = 3x - 2$.

57. Find two different sets of parametric equations for $y = x^2 - 4x + 4$.

Show Solution

answers may vary: , $\begin{cases} x(t) = t \\ y(t) = t^2 - 4t + 4 \end{cases}$ and $\begin{cases} x(t) = t + 2 \\ y(t) = t^2 \end{cases}$

Glossary

parameter
: a variable, often representing time, upon which x and y are both dependent

CHAPTER 4.8: PARAMETRIC EQUATIONS: GRAPHS

Learning Objectives

In this section you will:

- Graph plane curves described by parametric equations by plotting points.
- Graph parametric equations.

It is the bottom of the ninth inning, with two outs and two men on base. The home team is losing by two runs. The batter swings and hits the baseball at 140 feet per second and at an angle of approximately 45 to the horizontal. How far will the ball travel? Will it clear the fence for a game-winning home run? The outcome may depend partly on other factors (for example, the wind), but mathematicians can model the path of a projectile and predict approximately how far it will travel using parametric equations. In this section, we'll discuss parametric equations and some common applications, such as projectile motion problems.

Figure 1. Parametric equations can model the path of a projectile. (credit: Paul Kreher, Flickr)

Graphing Parametric Equations by Plotting Points

In lieu of a graphing calculator or a computer graphing program, plotting points to represent the graph of an equation is the standard method. As long as we are careful in calculating the values, point-plotting is highly dependable.

How To

Given a pair of parametric equations, sketch a graph by plotting points.

1. Construct a table with three columns: $t, x\left(t\right), \text{and } y\left(t\right)$.
2. Evaluate x and y for values of t over the interval for which the functions are defined.
3. Plot the resulting pairs $\left(x, y\right)$.

Sketching the Graph of a Pair of Parametric Equations by Plotting Points

Sketch the graph of the parametric equations $x\left(t\right)={t}^{2}+1,\;y\left(t\right)=2+t.$

Show Solution

Construct a table of values for $t,x\left(t\right)$, and $y\left(t\right)$, as in (Figure), and plot the points in a plane.

t	$x\left(t\right)={t}^{2}+1$	$y\left(t\right)=2+t$
-5	26	-3
-4	17	-2
-3	10	-1
-2	5	0
-1	2	1
0	1	2
1	2	3
2	5	4
3	10	5
4	17	6
5	26	7

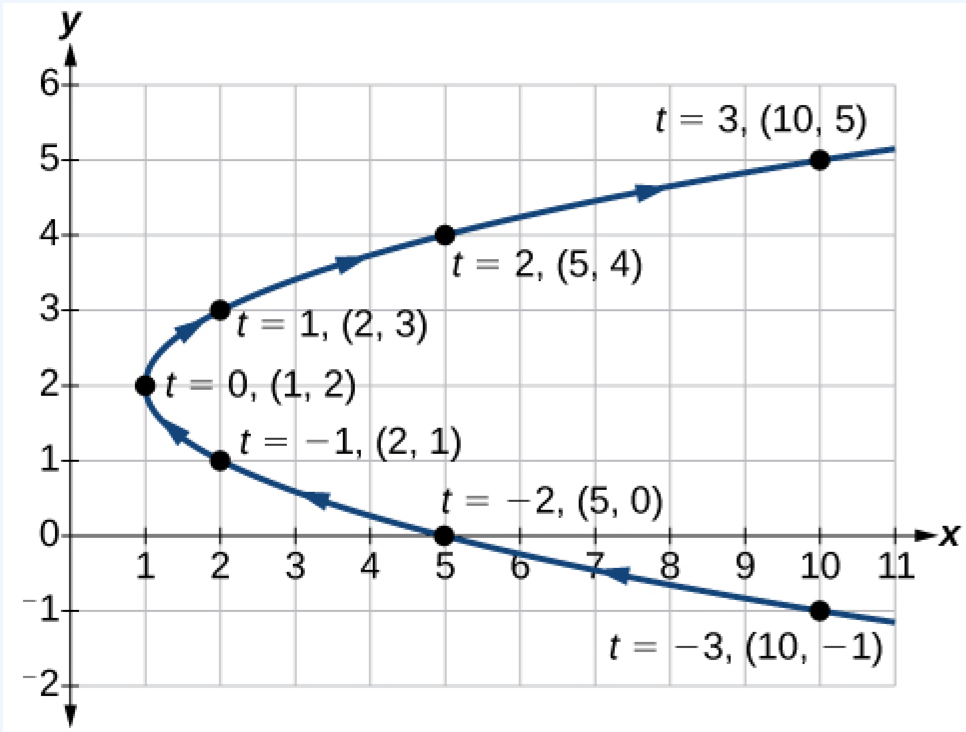


Figure 2.

The graph is a parabola with vertex at the point $(1,2)$, opening to the right. See (Figure).

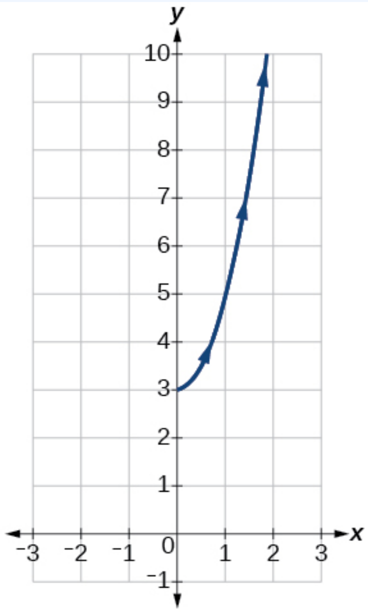
Analysis

As values for t progress in a positive direction from 0 to 5, the plotted points trace out the top half of the parabola. As values of t become negative, they trace out the lower half of the parabola. There are no restrictions on the domain. The arrows indicate direction according to increasing values of t. The graph does not represent a function, as it will fail the vertical line test. The graph is drawn in two parts: the positive values for t, and the negative values for t.

Try It

Sketch the graph of the parametric equations $x = \sqrt{t},\ y = 2t + 3,\ \ 0 \leq t \leq 3.$

Show Solution



Sketching the Graph of Trigonometric Parametric Equations

Construct a table of values for the given parametric equations and sketch the graph:

$x = 2\cos t$
$y = 4\sin t$

Show Solution

Construct a table like that in (Figure) using angle measure in radians as inputs for t, and evaluating x and y. Using angles with known sine and cosine values for t makes calculations easier.

t	$x = 2\cos t$	$y = 4\sin t$
0	$x = 2\cos(0) = 2$	$y = 4\sin(0) = 0$
$\frac{\pi}{6}$	$x = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$	$y = 4\sin\left(\frac{\pi}{6}\right) = 2$
$\frac{\pi}{3}$	$x = 2\cos\left(\frac{\pi}{3}\right) = 1$	$y = 4\sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}$
$\frac{\pi}{2}$	$x = 2\cos\left(\frac{\pi}{2}\right) = 0$	$y = 4\sin\left(\frac{\pi}{2}\right) = 4$
$\frac{2\pi}{3}$	$x = 2\cos\left(\frac{2\pi}{3}\right) = -1$	$y = 4\sin\left(\frac{2\pi}{3}\right) = 2\sqrt{3}$
$\frac{5\pi}{6}$	$x = 2\cos\left(\frac{5\pi}{6}\right) = -\sqrt{3}$	$y = 4\sin\left(\frac{5\pi}{6}\right) = 2$
π	$x = 2\cos(\pi) = -2$	$y = 4\sin(\pi) = 0$
$\frac{7\pi}{6}$	$x = 2\cos\left(\frac{7\pi}{6}\right) = -\sqrt{3}$	$y = 4\sin\left(\frac{7\pi}{6}\right) = -2$
$\frac{4\pi}{3}$	$x = 2\cos\left(\frac{4\pi}{3}\right) = -1$	$y = 4\sin\left(\frac{4\pi}{3}\right) = -2\sqrt{3}$
$\frac{3\pi}{2}$	$x = 2\cos\left(\frac{3\pi}{2}\right) = 0$	$y = 4\sin\left(\frac{3\pi}{2}\right) = -4$
$\frac{5\pi}{3}$	$x = 2\cos\left(\frac{5\pi}{3}\right) = 1$	$y = 4\sin\left(\frac{5\pi}{3}\right) = -2\sqrt{3}$
$\frac{11\pi}{6}$	$x = 2\cos\left(\frac{11\pi}{6}\right) = \sqrt{3}$	$y = 4\sin\left(\frac{11\pi}{6}\right) = -2$
2π	$x = 2\cos(2\pi) = 2$	$y = 4\sin(2\pi) = 0$

(Figure) shows the graph.

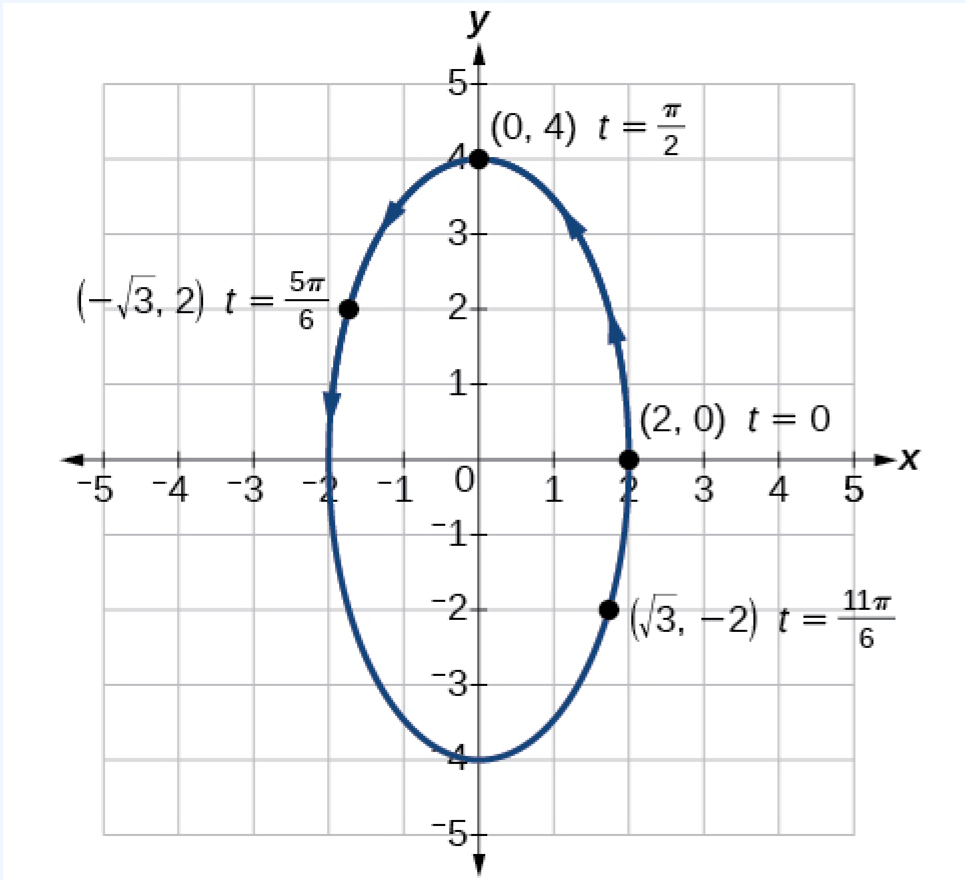


Figure 3.

By the symmetry shown in the values of x and y, we see that the parametric equations represent an ellipse. The ellipse is mapped in a counterclockwise direction as shown by the arrows indicating increasing t values.

Analysis

We have seen that parametric equations can be graphed by plotting points. However, a graphing calculator will save some time and reveal nuances in a graph that may be too tedious to discover using only hand calculations.

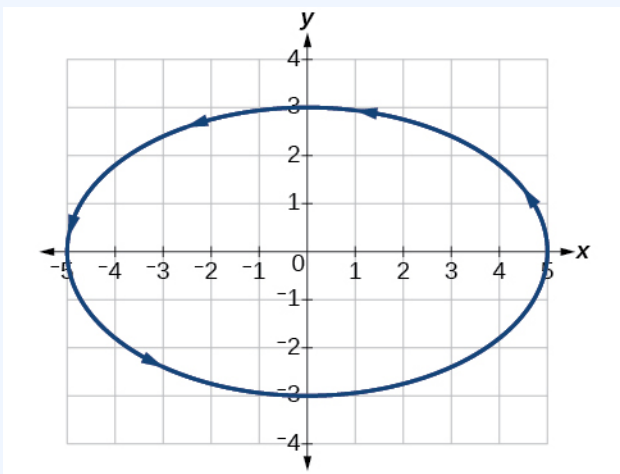
Make sure to change the mode on the calculator to parametric (PAR). To confirm, the $Y =$ window should show

$X_{1T} =$
$Y_{1T} =$
instead of $Y_1 =$.

Try It

Graph the parametric equations: $x = 5\cos t, \;\; y = 3\sin t.$

Show Solution



Graphing Parametric Equations and Rectangular Form Together

Graph the parametric equations $x = 5\cos t$ and $y = 2\sin t.$ First, construct the graph using data points generated from the parametric form. Then graph the rectangular form of the equation. Compare the two graphs.

Show Solution

Construct a table of values like that in (Figure).

t	$x = 5\cos t$	$y = 2\sin t$
0	$x = 5\cos(0) = 5$	$y = 2\sin(0) = 0$
1	$x = 5\cos(1) \approx 2.7$	$y = 2\sin(1) \approx 1.7$
2	$x = 5\cos(2) \approx -2.1$	$y = 2\sin(2) \approx 1.8$
3	$x = 5\cos(3) \approx -4.95$	$y = 2\sin(3) \approx 0.28$
4	$x = 5\cos(4) \approx -3.3$	$y = 2\sin(4) \approx -1.5$
5	$x = 5\cos(5) \approx 1.4$	$y = 2\sin(5) \approx -1.9$
-1	$x = 5\cos(-1) \approx 2.7$	$y = 2\sin(-1) \approx -1.7$
-2	$x = 5\cos(-2) \approx -2.1$	$y = 2\sin(-2) \approx -1.8$
-3	$x = 5\cos(-3) \approx -4.95$	$y = 2\sin(-3) \approx -0.28$
-4	$x = 5\cos(-4) \approx -3.3$	$y = 2\sin(-4) \approx 1.5$
-5	$x = 5\cos(-5) \approx 1.4$	$y = 2\sin(-5) \approx 1.9$

Plot the (x, y) values from the table. See (Figure).

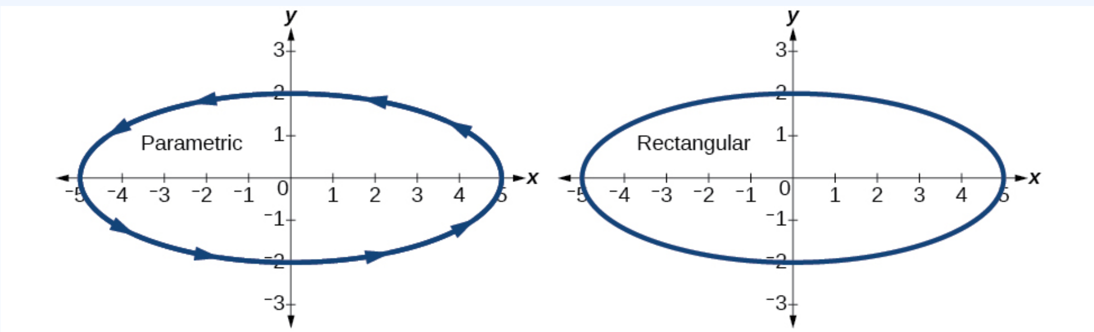


Figure 4.

Next, translate the parametric equations to rectangular form. To do this, we solve for t in either $x(t)$ or $y(t)$, and then substitute the expression for t in the other equation. The result will be a function $y(x)$ if solving for t as a function of x, or $x(y)$ if solving for t as a function of y.

$$\begin{aligned} x &= 5\cos t \\ \tfrac{x}{5} &= \cos t && \text{Solve for } \cos t. \\ y &= 2\sin t && \text{Solve for } \sin t. \\ \tfrac{y}{2} &= \sin t \end{aligned}$$

Then, use the Pythagorean Theorem.

$$\begin{aligned} \cos^2 t + \sin^2 t &= 1 \\ \left(\tfrac{x}{5}\right)^2 + \left(\tfrac{y}{2}\right)^2 &= 1 \\ \tfrac{x^2}{25} + \tfrac{y^2}{4} &= 1 \end{aligned}$$

Analysis

In (Figure), the data from the parametric equations and the rectangular equation are plotted together. The parametric equations are plotted in blue; the graph for the rectangular equation is drawn on top of the parametric in a dashed style colored red. Clearly, both forms produce the same graph.

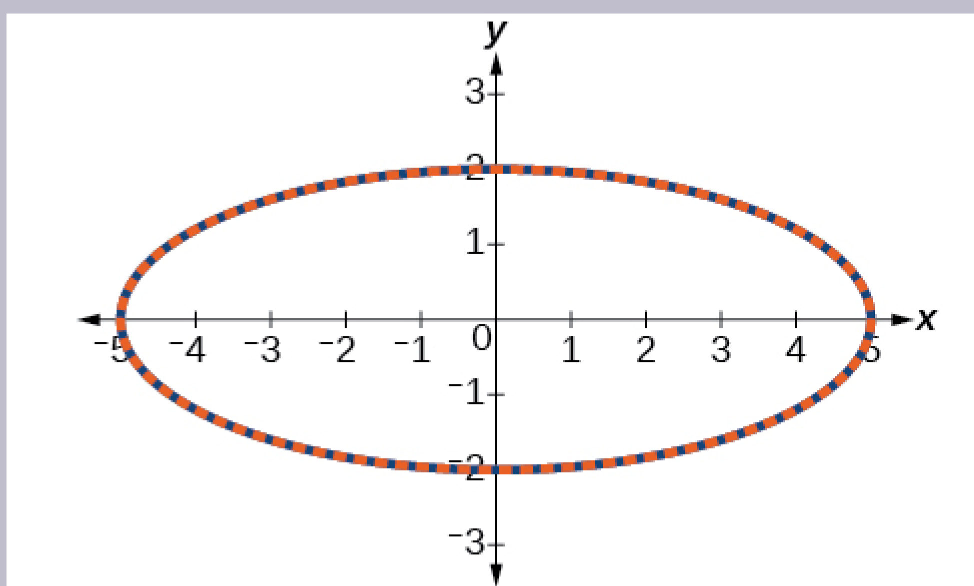


Figure 5.

Graphing Parametric Equations and Rectangular Equations on the Coordinate System

Graph the parametric equations $x = t + 1$ and $y = \sqrt{t},\ t \geq 0,$ and the rectangular equivalent $y = \sqrt{x - 1}$ on the same coordinate system.

Show Solution

Construct a table of values for the parametric equations, as we did in the previous example, and graph $y = \sqrt{t},\ t \geq 0$ on the same grid, as in (Figure).

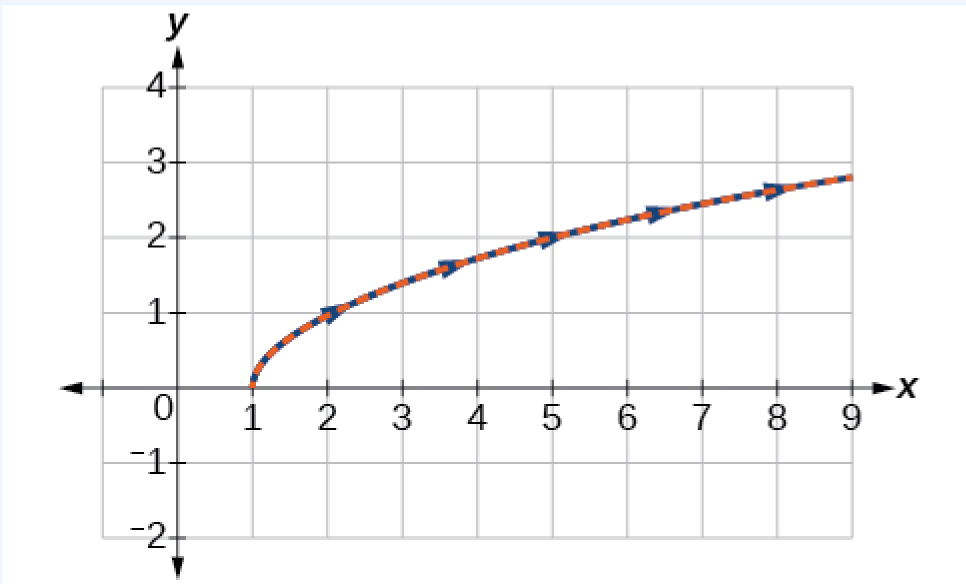


Figure 6.

Analysis

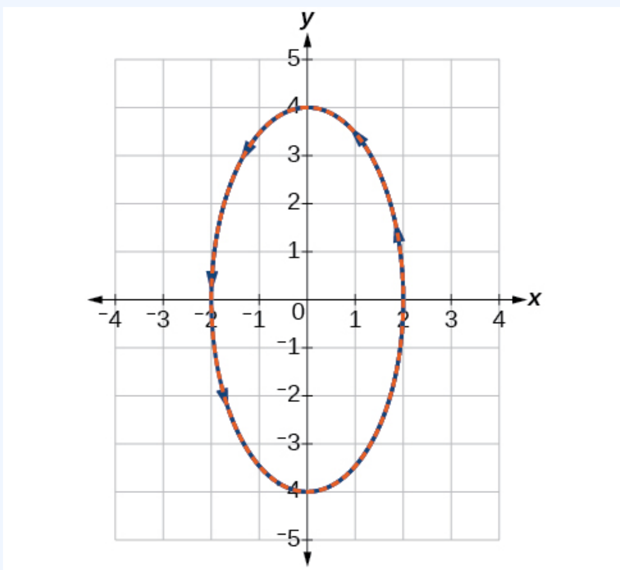
With the domain on t restricted, we only plot positive values of t. The parametric data is graphed in blue and the graph of the rectangular equation is dashed in red. Once again, we see that the two forms overlap.

Try It

Sketch the graph of the parametric equations $x = 2\cos\theta \text{ and } y = 4\sin\theta$, along with the rectangular equation on the same grid.

Show Solution

The graph of the parametric equations is in red and the graph of the rectangular equation is drawn in blue dots on top of the parametric equations.



Applications of Parametric Equations

Many of the advantages of parametric equations become obvious when applied to solving real-world problems. Although rectangular equations in *x* and *y* give an overall picture of an object's path, they do not reveal the position of an object at a specific time. Parametric equations, however, illustrate how the values of *x* and *y* change depending on *t*, as the location of a moving object at a particular time.

A common application of parametric equations is solving problems involving projectile motion. In this type of motion, an object is propelled forward in an upward direction forming an angle of θ to the horizontal, with an initial speed of v_0, and at a height h above the horizontal.

The path of an object propelled at an inclination of θ to the horizontal, with initial speed v_0, and at a height h above the horizontal, is given by
$x = (v_0 \cos\theta)\, t$
$y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)\, t + h$

where g accounts for the effects of gravity and h is the initial height of the object. Depending on the units involved in the problem, use $g = 32\,\mathrm{ft}\,/\,\mathrm{s}^2$ or $g = 9.8\,\mathrm{m}\,/\,\mathrm{s}^2$. The equation for x gives horizontal distance, and the equation for y gives the vertical distance.

How To

Given a projectile motion problem, use parametric equations to solve.

1. The horizontal distance is given by $x = (v_0 \cos\theta)\, t$. Substitute the initial speed of the object for v_0.
2. The expression $\cos\theta$ indicates the angle at which the object is propelled. Substitute that angle in degrees for $\cos\theta$.
3. The vertical distance is given by the formula $y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)\, t + h$. The term $-\frac{1}{2}gt^2$ represents the effect of gravity. Depending on units involved, use $g = 32\,\mathrm{ft/s}^2$ or $g = 9.8\,\mathrm{m/s}^2$. Again, substitute the initial speed for v_0, and the height at which the object was propelled for h.
4. Proceed by calculating each term to solve for t.

Finding the Parametric Equations to Describe the Motion of a Baseball

Solve the problem presented at the beginning of this section. Does the batter hit the game-winning home run? Assume that the ball is hit with an initial velocity of 140 feet per second at an angle of 45 to the horizontal, making contact 3 feet above the ground.

a. Find the parametric equations to model the path of the baseball.
b. Where is the ball after 2 seconds?
c. How long is the ball in the air?
d. Is it a home run?

Show Solution

a. Use the formulas to set up the equations. The horizontal position is found using the parametric equation for x. Thus,

$$\begin{array}{l} x = (v_0 \cos\theta)\, t \\ x = (140 \cos(45))\, t \end{array} \quad [/latex < /div > Theverticalpositionisfoundusingtheparametricequationfor[latex]\, y.$$

Thus,

$$\begin{array}{l} y = -16t^2 + (v_0 \sin\theta)\, t + h \\ y = -16t^2 + (140 \sin(45))\, t + 3 \end{array}$$

b. Substitute 2 into the equations to find the horizontal and vertical positions of the ball.

$$\begin{array}{l} x = (140 \cos(45))\,(2) \\ x = 198 \text{ feet} \end{array}$$

$$\begin{array}{l} y = -16(2)^2 + (140 \sin(45))\,(2) + 3 \\ y = 137 \text{ feet} \end{array}$$

After 2 seconds, the ball is 198 feet away from the batter's box and 137 feet above the ground.

c. To calculate how long the ball is in the air, we have to find out when it will hit ground, or when $y = 0$. Thus,

$$\begin{array}{ll} y = -16t^2 + (140 \sin(45^\circ))\, t + 3 & \\ y = 0 & \text{Set } y(t) = 0 \text{ and solve the quadratic.} \\ t = 6.2173 & \end{array}$$

When $t = 6.2173$ seconds, the ball has hit the ground. (The quadratic equation can be solved in various ways, but this problem was solved using a computer math program.)

d. We cannot confirm that the hit was a home run without considering the size of the outfield, which varies from field to field. However, for simplicity's sake, let's assume that the outfield wall is 400 feet from home plate in the deepest part of the park. Let's also assume that the wall is 10 feet high. In order to determine whether the ball clears the wall, we need to calculate how high the ball is when *x* = 400 feet. So we will set *x* = 400, solve for t, and input t into y.

$$x = (140\cos(45))\,t$$
$$400 = (140\cos(45))\,t$$
$$t = 4.04$$

$$y = -16(4.04)^2 + (140\sin(45))\,(4.04) + 3$$
$$y = 141.8$$

The ball is 141.8 feet in the air when it soars out of the ballpark. It was indeed a home run. See (Figure).

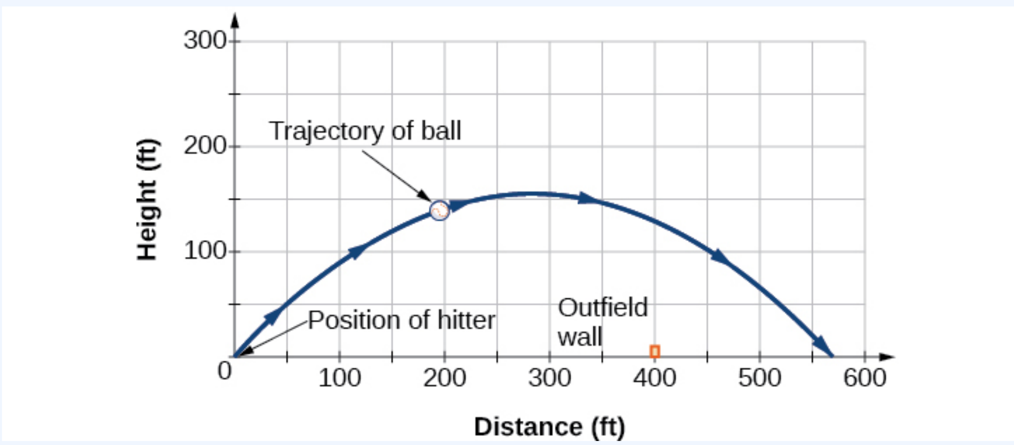


Figure 7.

Access the following online resource for additional instruction and practice with graphs of parametric equations.

- Graphing Parametric Equations on the TI-84

Key Concepts

- When there is a third variable, a third parameter on which x and y depend, parametric equations can be used.
- To graph parametric equations by plotting points, make a table with three columns labeled $t, x\left(t\right)$, and $y\left(t\right)$. Choose values for t in increasing order. Plot the last two columns for x and y. See (Figure) and (Figure).
- When graphing a parametric curve by plotting points, note the associated *t*-values and show arrows on the graph indicating the orientation of the curve. See (Figure) and (Figure).
- Parametric equations allow the direction or the orientation of the curve to be shown on the graph. Equations that are not functions can be graphed and used in many applications involving motion. See (Figure).
- Projectile motion depends on two parametric equations: $x=\left({v}_{0}\mathrm{cos}\,\theta\right)t$ and $y=-16{t}^{2}+\left({v}_{0}\mathrm{sin}\,\theta\right)t+h.$ Initial velocity is symbolized as ${v}_{0}.$ θ represents the initial angle of the object when thrown, and h represents the height at which the object is propelled.

Section Exercises

Verbal

1. What are two methods used to graph parametric equations?

Show Solution
plotting points with the orientation arrow and a graphing calculator

2. What is one difference in point-plotting parametric equations compared to Cartesian equations?

3. Why are some graphs drawn with arrows?

Show Solution

The arrows show the orientation, the direction of motion according to increasing values of t.

4. Name a few common types of graphs of parametric equations.

5. Why are parametric graphs important in understanding projectile motion?

Show Solution

The parametric equations show the different vertical and horizontal motions over time.

Graphical

For the following exercises, graph each set of parametric equations by making a table of values. Include the orientation on the graph.

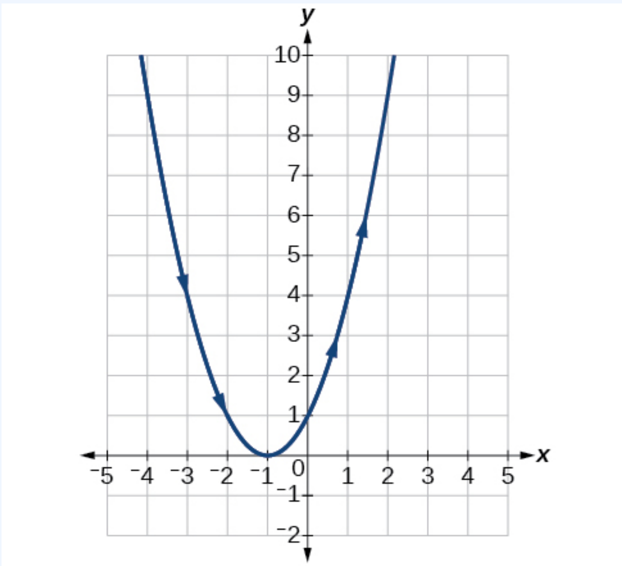
6. $\begin{cases} x(t) = t \\ y(t) = t^2 - 1 \end{cases}$

t	x	y
-3		
-2		
-1		
0		
1		
2		
3		

7. $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$

t	-3	-2	-1	0	1	2
x						
y						

Show Solution



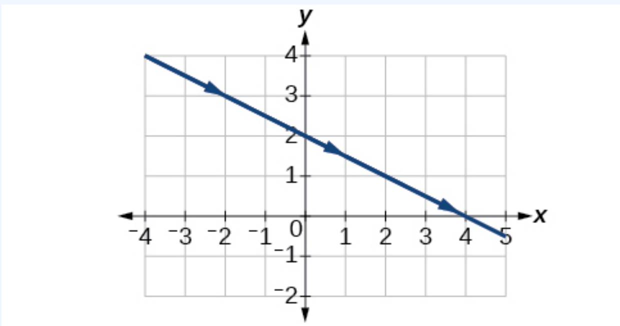
8. $\begin{cases} x(t) = 2 + t \\ y(t) = 3 - 2t \end{cases}$

t	-2	-1	0	1	2	3
x						
y						

9. $\begin{cases} x(t) = -2 - 2t \\ y(t) = 3 + t \end{cases}$

t	-3	-2	-1	0	1
x					
y					

Show Solution



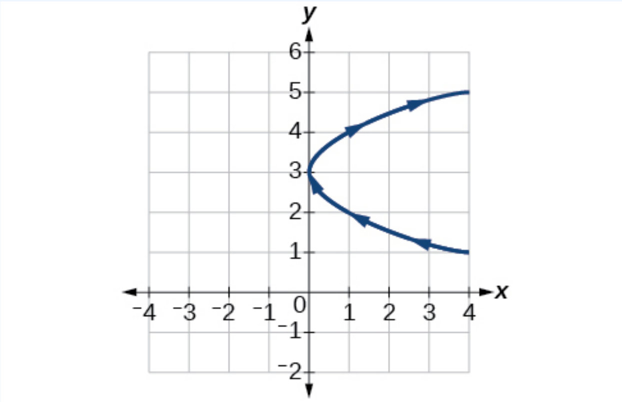
10. $\begin{cases} x(t) = t^3 \\ y(t) = t + 2 \end{cases}$

t	-2	-1	0	1	2
x					
y					

11. $\begin{cases} x(t) = t^2 \\ y(t) = t + 3 \end{cases}$

t	-2	-1	0	1	2
x					
y					

Show Solution

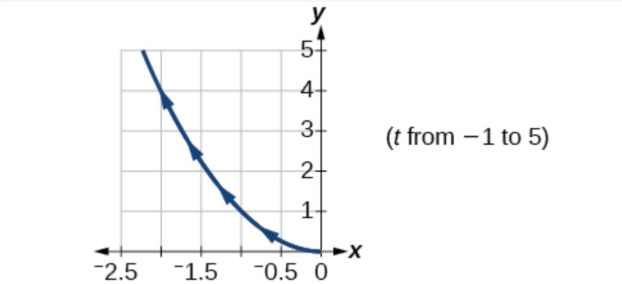


For the following exercises, sketch the curve and include the orientation.

12. $\begin{cases} x(t) = t \\ y(t) = \sqrt{t} \end{cases}$

13. $\begin{cases} x(t) = -\sqrt{t} \\ y(t) = t \end{cases}$

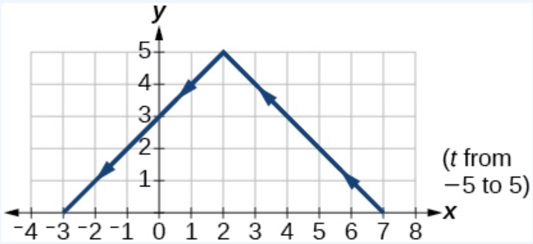
Show Solution



14. $\begin{cases} x(t) = 5 - |t| \\ y(t) = t + 2 \end{cases}$

15. $\begin{cases} x(t) = -t + 2 \\ y(t) = 5 - |t| \end{cases}$

Show Solution

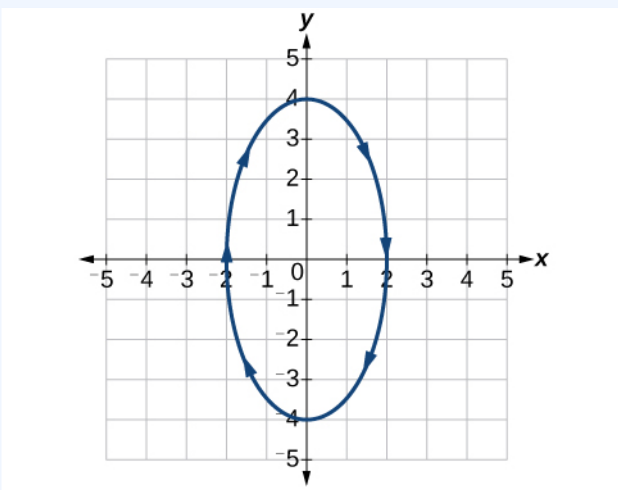


16. $\begin{cases} x(t) = 4\sin t \\ y(t) = 2\cos t \end{cases}$

---

17. $\begin{cases} x(t) = 2\sin t \\ y(t) = 4\cos t \end{cases}$

Show Solution

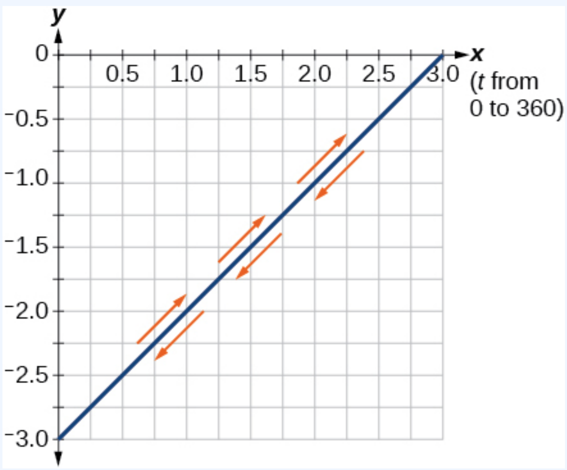


18. $\begin{cases} x(t) = 3\cos^2 t \\ y(t) = -3\sin t \end{cases}$

---

19. $\begin{cases} x(t) = 3\cos^2 t \\ y(t) = -3\sin^2 t \end{cases}$

Show Solution

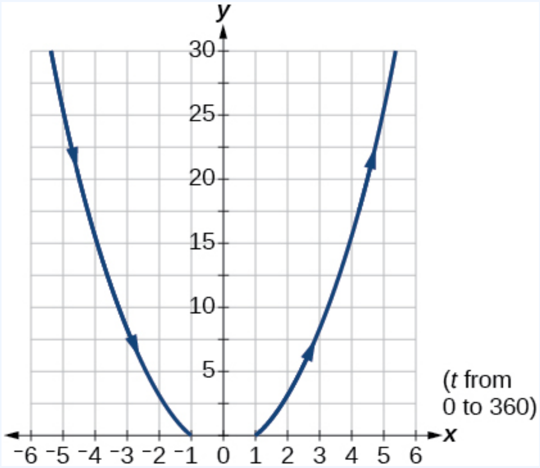


20. $\begin{cases} x(t) = \sec t \\ y(t) = \tan t \end{cases}$

---

21. $\begin{cases} x(t) = \sec t \\ y(t) = \tan^2 t \end{cases}$

Show Solution

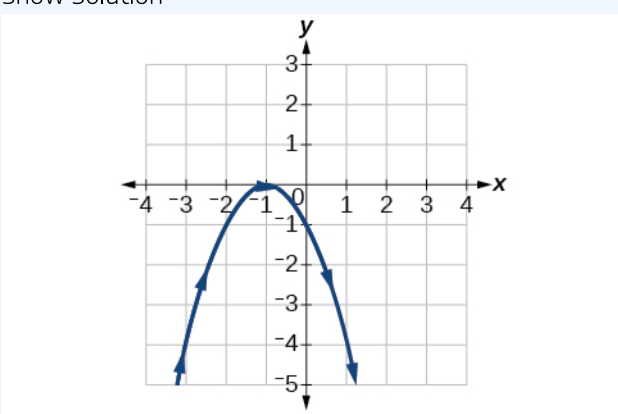


22. $\begin{cases} x(t) = \frac{1}{e^{2t}} \\ y(t) = e^{-t} \end{cases}$

For the following exercises, graph the equation and include the orientation. Then, write the Cartesian equation.

23. $\begin{cases} x(t) = t - 1 \\ y(t) = -t^2 \end{cases}$

Show Solution

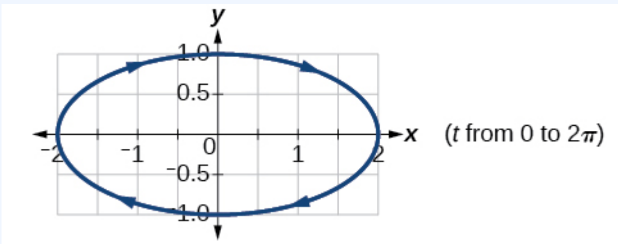


24. $\begin{cases} x(t) = t^3 \\ y(t) = t + 3 \end{cases}$

---

25. $\begin{cases} x(t) = 2\cos t \\ y(t) = -\sin t \end{cases}$

Show Solution

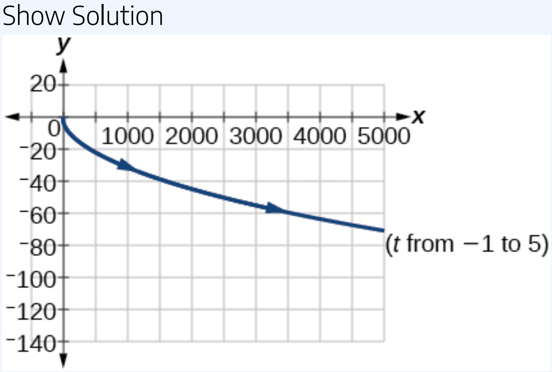


26. $\begin{cases} x(t) = 7\cos t \\ y(t) = 7\sin t \end{cases}$

---

27. $\begin{cases} x(t) = e^{2t} \\ y(t) = -e^{t} \end{cases}$

Show Solution

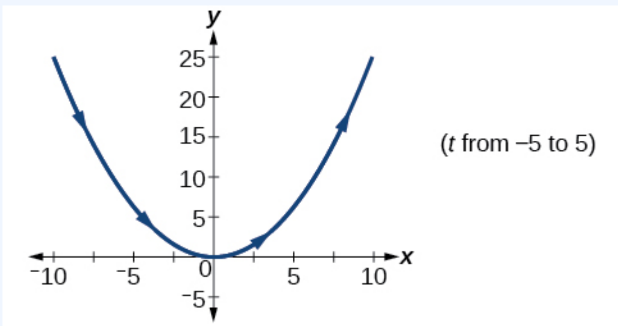


For the following exercises, graph the equation and include the orientation.

28. $x = t^2,\ y = 3t,\ 0 \leq t \leq 5$

29. $x = 2t,\ y = t^2,\ -5 \leq t \leq 5$

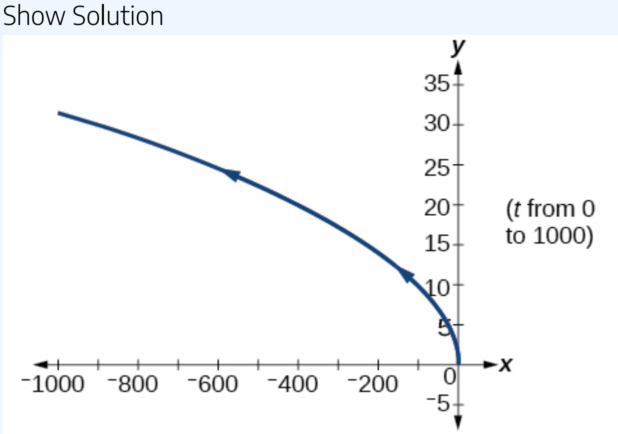
Show Solution



30. $x = t,\ y = \sqrt{25 - t^2},\ 0 < t \leq 5$

31. $x\left(t\right) = -t, y\left(t\right) = \sqrt{t},\ t \geq 0$

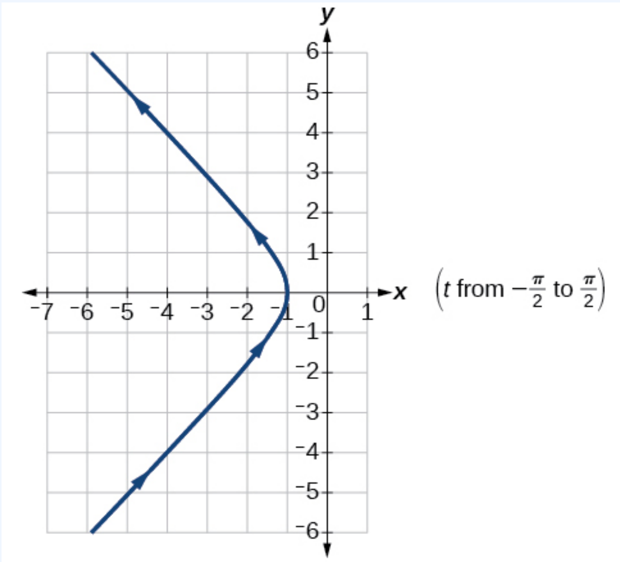
Show Solution



32. $x = -2\cos t,\ y = 6\sin t,\ 0 \leq t \leq \pi$

33. $x = -\sec t,\ y = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$

Show Solution



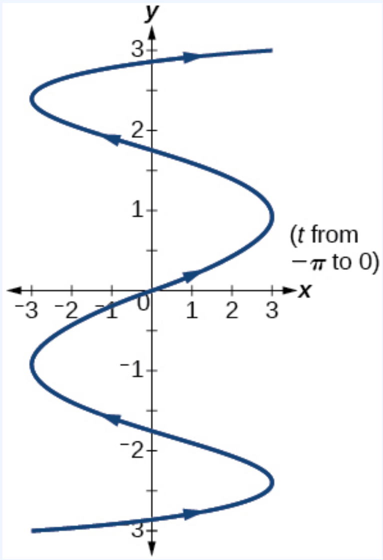
For the following exercises, use the parametric equations for integers *a* and *b*:

$$x(t) = a\cos((a+b)t)$$
$$y(t) = a\cos((a-b)t)$$

34. Graph on the domain $[-\pi, 0]$, where $a = 2$ and $b = 1$, and include the orientation.

35. Graph on the domain $[-\pi, 0]$, where $a = 3$ and $b = 2$, and include the orientation.

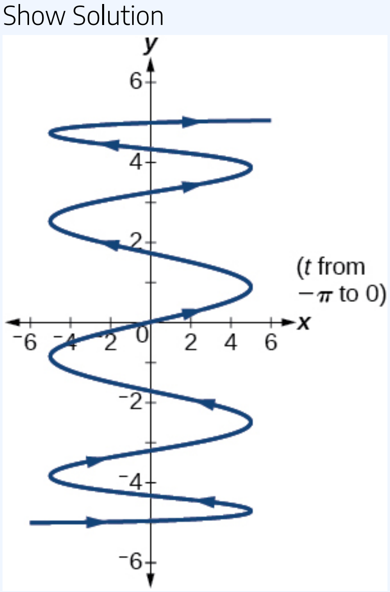
Show Solution



36. Graph on the domain $[-\pi, 0]$, where $a = 4$ and $b = 3$, and include the orientation.

37. Graph on the domain $[-\pi, 0]$, where $a = 5$ and $b = 4$, and include the orientation.

Show Solution



38. If a is 1 more than b, describe the effect the values of a and b have on the graph of the parametric equations.

39. Describe the graph if $a = 100$ and $b = 99$.

Show Solution
There will be 100 back-and-forth motions.

40. What happens if b is 1 more than a? Describe the graph.

41. If the parametric equations $x\left(t\right) = t^2$ and $y\left(t\right) = 6 - 3t$ have the graph of a horizontal parabola opening to the right, what would change the direction of the curve?

Show Solution
Take the opposite of the $x\left(t\right)$ equation.

For the following exercises, describe the graph of the set of parametric equations.

42. $x\left(t\right) = -t^2$ and $y\left(t\right)$ is linear

43. $y\left(t\right) = t^2$ and $x\left(t\right)$ is linear

Show Solution
The parabola opens up.

44. $y\left(t\right) = -t^2$ and $x\left(t\right)$ is linear

45. Write the parametric equations of a circle with center $\left(0, 0\right)$, radius 5, and a counterclockwise orientation.

Show Solution

$$\begin{cases} x(t) = 5\text{cos}t \\ y(t) = 5\text{sin}t \end{cases}$$

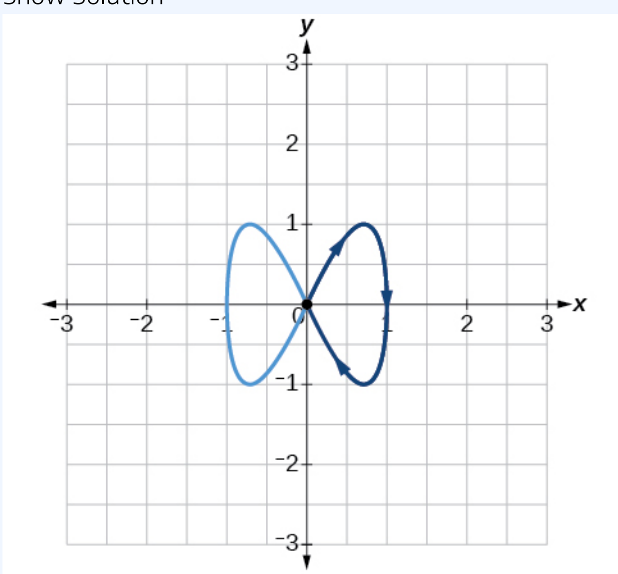
46. Write the parametric equations of an ellipse with center $(0, 0)$, major axis of length 10, minor axis of length 6, and a counterclockwise orientation.

For the following exercises, use a graphing utility to graph on the window $[-3, 3]$ by $[-3, 3]$ on the domain $[0, 2\pi)$ for the following values of a and b , and include the orientation.

$$\begin{cases} x(t) = \sin(at) \\ y(t) = \sin(bt) \end{cases}$$

47. $a = 1, b = 2$

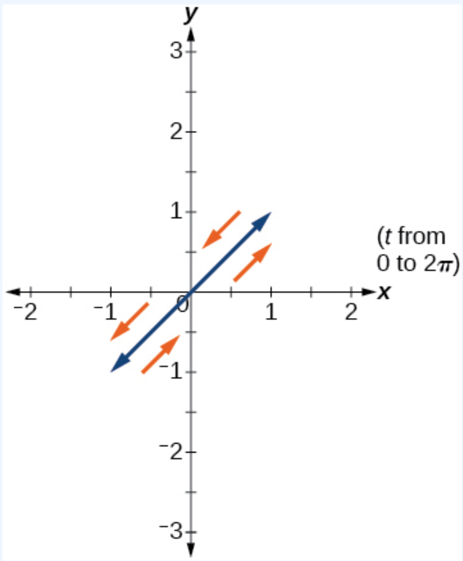
Show Solution



48. $a = 2, b = 1$

49. $a = 3, b = 3$

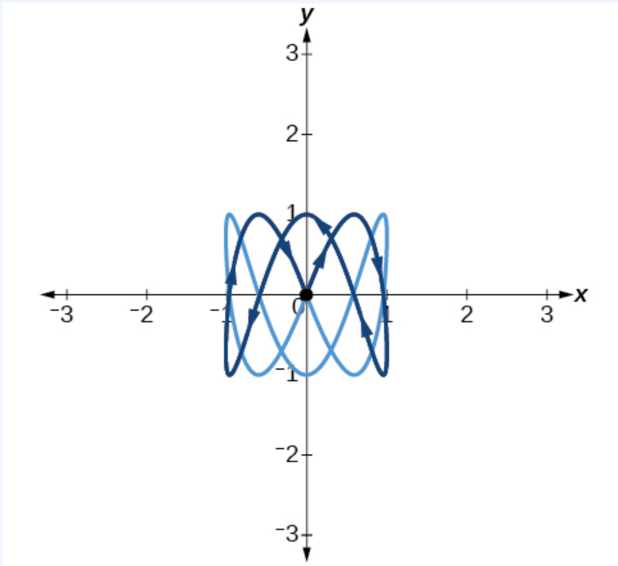
Show Solution



50. $a = 5, b = 5$

51. $a = 2, b = 5$

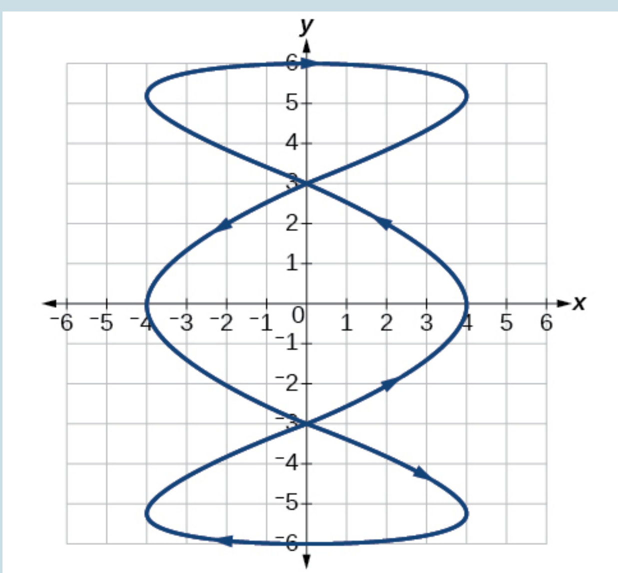
Show Solution



52. $a = 5, b = 2$

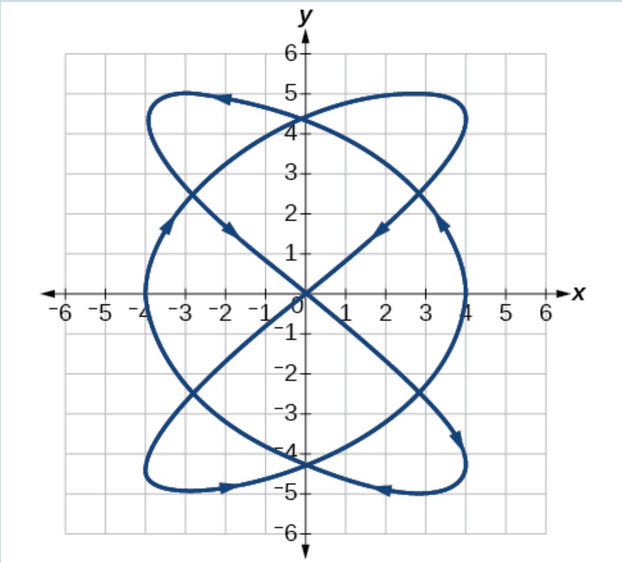
Technology

For the following exercises, look at the graphs that were created by parametric equations of the form $\begin{cases} x(t) = a\cos(bt) \\ y(t) = c\sin(dt) \end{cases}$. Use the parametric mode on the graphing calculator to find the values of $a, b, c,$ and d to achieve each graph.

53. 

Show Solution

$a = 4,\ b = 3,\ c = 6,\ d = 1$

54. 

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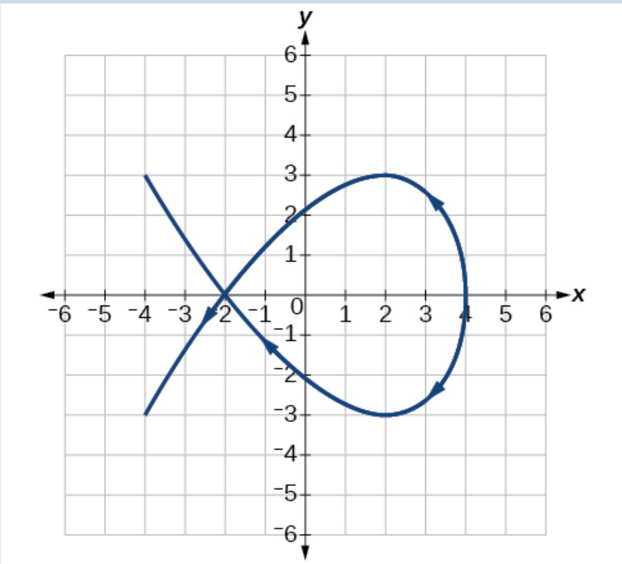
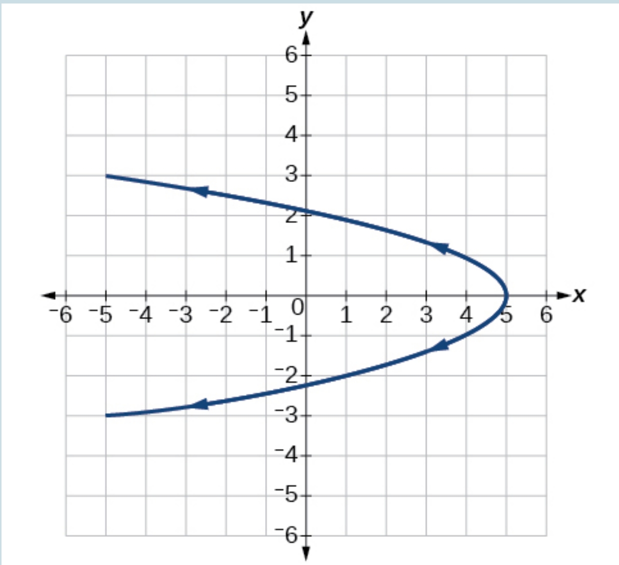
55. 

Show Solution

$a = 4,\ b = 2,\ c = 3,\ d = 3$

56. 

For the following exercises, use a graphing utility to graph the given parametric equations.

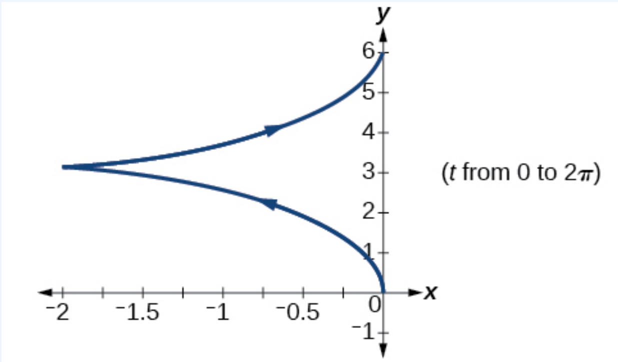
a. $\begin{cases} x(t) = \cos t - 1 \\ y(t) = \sin t + t \end{cases}$

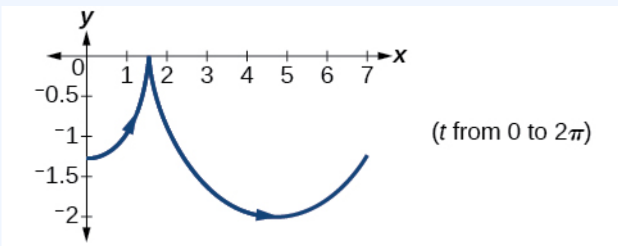
b. $\begin{cases} x(t) = \cos t + t \\ y(t) = \sin t - 1 \end{cases}$

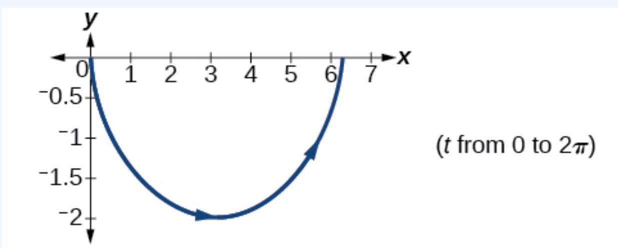
c. $\begin{cases} x(t) = t - \sin t \\ y(t) = \cos t - 1 \end{cases}$

57. Graph all three sets of parametric equations on the domain $[0,\ 2\pi]$.

Show Solution



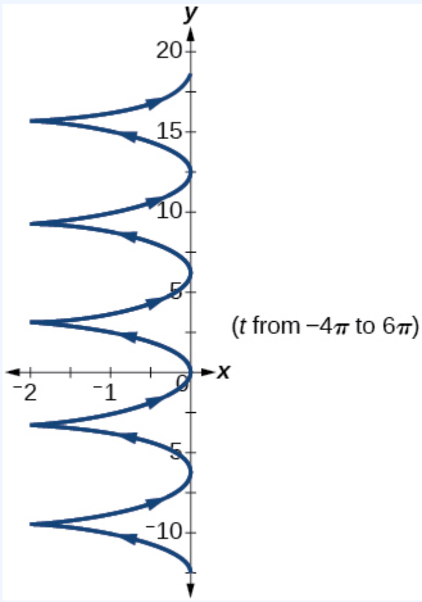


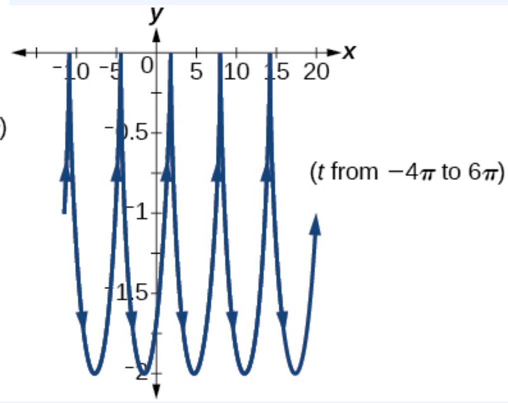


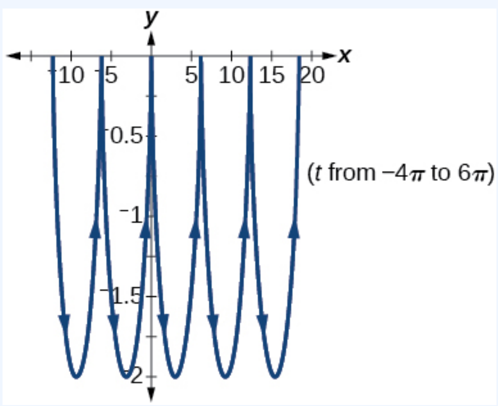
58. Graph all three sets of parametric equations on the domain $[0, 4\pi]$.

59. Graph all three sets of parametric equations on the domain $[-4\pi, 6\pi]$.

Show Solution







60. The graph of each set of parametric equations appears to “creep” along one of the axes. What controls which axis the graph creeps along?

61. Explain the effect on the graph of the parametric equation when we switched $\sin t$ and $\cos t$.

Show Solution
The y -intercept changes.

62. Explain the effect on the graph of the parametric equation when we changed the domain.

Extensions

63. An object is thrown in the air with vertical velocity of 20 ft/s and horizontal velocity of 15 ft/s. The object's height can be described by the equation $y\left(t\right)=-16t^2+20t$, while the object moves horizontally with constant velocity 15 ft/s. Write parametric equations for the object's position, and then eliminate time to write height as a function of horizontal position.

Show Solution

$y\left(x\right)=-16{\left(\frac{x}{15}\right)}^2+20\left(\frac{x}{15}\right)$

64. A skateboarder riding on a level surface at a constant speed of 9 ft/s throws a ball in the air, the height of which can be described by the equation $y\left(t\right)=-16t^2+10t+5.$ Write parametric equations for the ball's position, and then eliminate time to write height as a function of horizontal position.

For the following exercises, use this scenario: A dart is thrown upward with an initial velocity of 65 ft/s at an angle of elevation of 52°. Consider the position of the dart at any time $t.$ Neglect air resistance.

65. Find parametric equations that model the problem situation.

Show Solution

$$\left\{\begin{array}{l}x(t)=64t\cos(52)\\ y(t)=-16t^2+64t\sin(52)\end{array}\right.$$

66. Find all possible values of x that represent the situation.

67. When will the dart hit the ground?

Show Solution

approximately 3.2 seconds

68. Find the maximum height of the dart.

69. At what time will the dart reach maximum height?

Show Solution

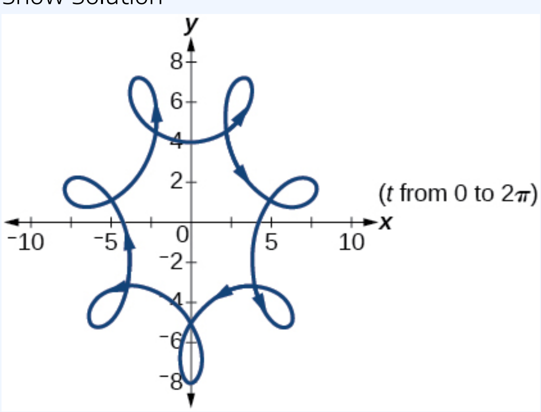
1.6 seconds

For the following exercises, look at the graphs of each of the four parametric equations. Although they look unusual and beautiful, they are so common that they have names, as indicated in each exercise. Use a graphing utility to graph each on the indicated domain.

70. An epicycloid: $\begin{cases} x(t) = 14\cos t - \cos(14t) \\ y(t) = 14\sin t + \sin(14t) \end{cases}$ on the domain $[0, 2\pi]$.

71. A hypocycloid: $\begin{cases} x(t) = 6\sin t + 2\sin(6t) \\ y(t) = 6\cos t - 2\cos(6t) \end{cases}$ on the domain $[0, 2\pi]$.

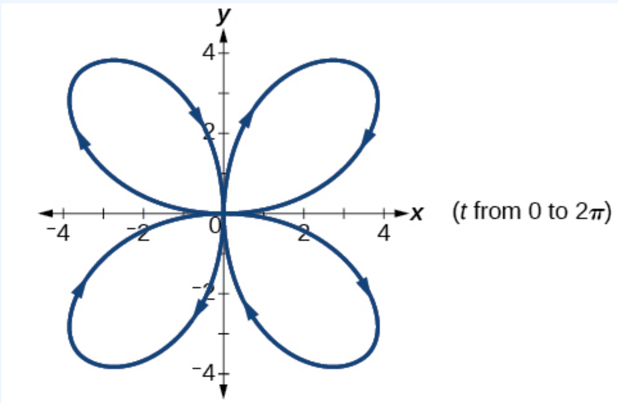
Show Solution



72. A hypotrochoid: $\begin{cases} x(t) = 2\sin t + 5\cos(6t) \\ y(t) = 5\cos t - 2\sin(6t) \end{cases}$ on the domain $[0, 2\pi]$.

73. A rose: $\begin{cases} x(t) = 5\sin(2t)\sin t \\ y(t) = 5\sin(2t)\cos t \end{cases}$ on the domain $[0, 2\pi]$.

Show Solution



CHAPTER 4.9: VECTORS

Learning Objectives

In this section you will:

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of v .
- Perform operations with vectors in terms of i and j .
- Find the dot product of two vectors.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in (Figure). What are the ground speed and actual bearing of the plane?

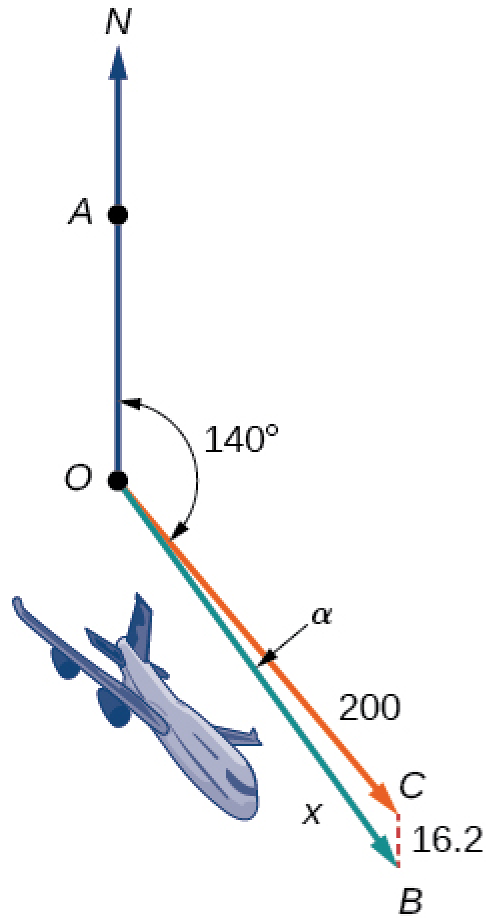


Figure 1.

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane's groundspeed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

A vector is a specific quantity drawn as a line segment with an arrowhead at one end. It has an initial point, where it begins, and a terminal point, where it ends. A vector is defined by its magnitude, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as $v,\ u,\ w,\ \vec{v},\ \vec{u},\ \vec{w}$.
- Given initial point P and terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrowhead on top

is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of $(0,0)$ and terminal point (a,b) , a vector may be represented as a,b.

This last symbol a,b has special significance. It is called the standard position. The position vector has an initial point $(0,0)$ and a terminal point a,b. To change any vector into the position vector, we think about the change in the *x*-coordinates and the change in the *y*-coordinates. Thus, if the initial point of a vector $\vec{CD}$ is $C\ (x_1,y_1)$ and the terminal point is $D\ (x_2,y_2)$, then the position vector is found by calculating

$$\begin{aligned}\vec{AB} &= \ x_2-x_1,y_2-y_1\\ &= \ a,b\end{aligned}$$

In (Figure), we see the original vector $\vec{CD}$ and the position vector $\vec{AB}$.

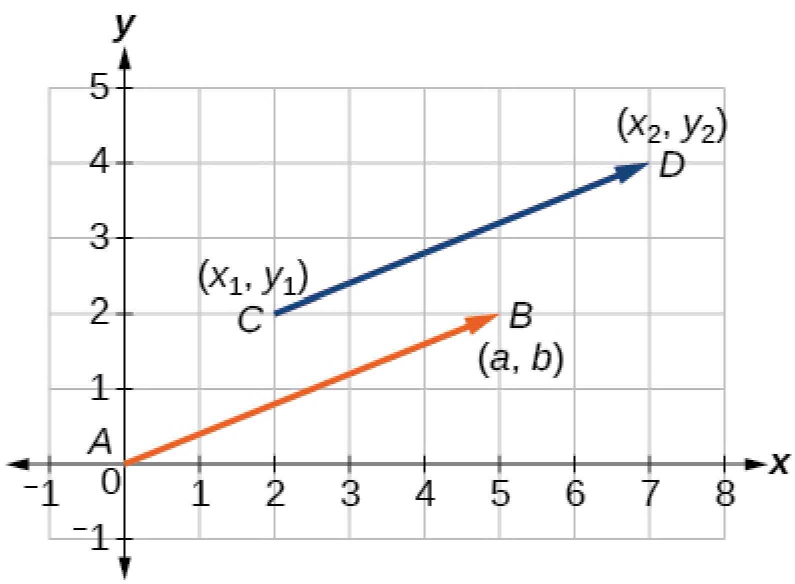


Figure 2.

Properties of Vectors

A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at $(0,0)$ and is identified by its terminal point a,b.

Find the Position Vector

Consider the vector whose initial point is $P\ (2,3)$ and terminal point is $Q\ (6,4)$. Find the position vector.

Show Solution

The position vector is found by subtracting one *x*-coordinate from the other *x*-coordinate, and one *y*-coordinate from the other *y*-coordinate. Thus

$$v = 6 - 2, 4 - 3$$
$$= 4, 1$$

The position vector begins at $(0,0)$ and terminates at $(4,1)$. The graphs of both vectors are shown in (Figure).

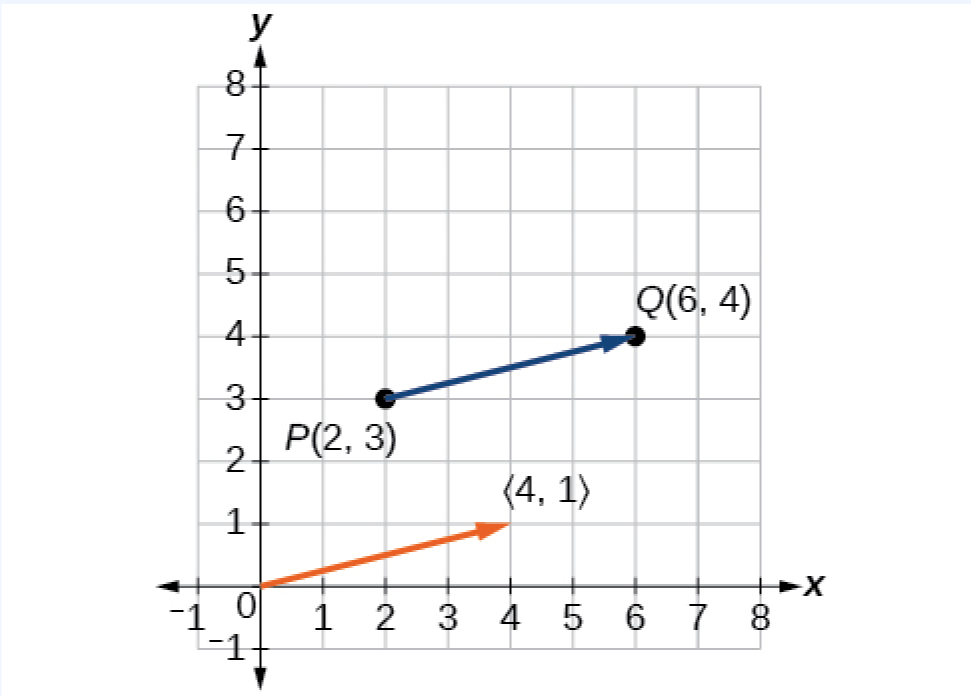


Figure 3.

We see that the position vector is $4, 1$.

Drawing a Vector with the Given Criteria and Its Equivalent Position Vector

Find the position vector given that vector v has an initial point at $(-3, 2)$ and a terminal point at $(4, 5)$, then graph both vectors in the same plane.

Show Solution

The position vector is found using the following calculation:

$$v = 4 - (-3), 5 - 2$$
$$= 7, 3$$

Thus, the position vector begins at $(0, 0)$ and terminates at $(7, 3)$. See (Figure).

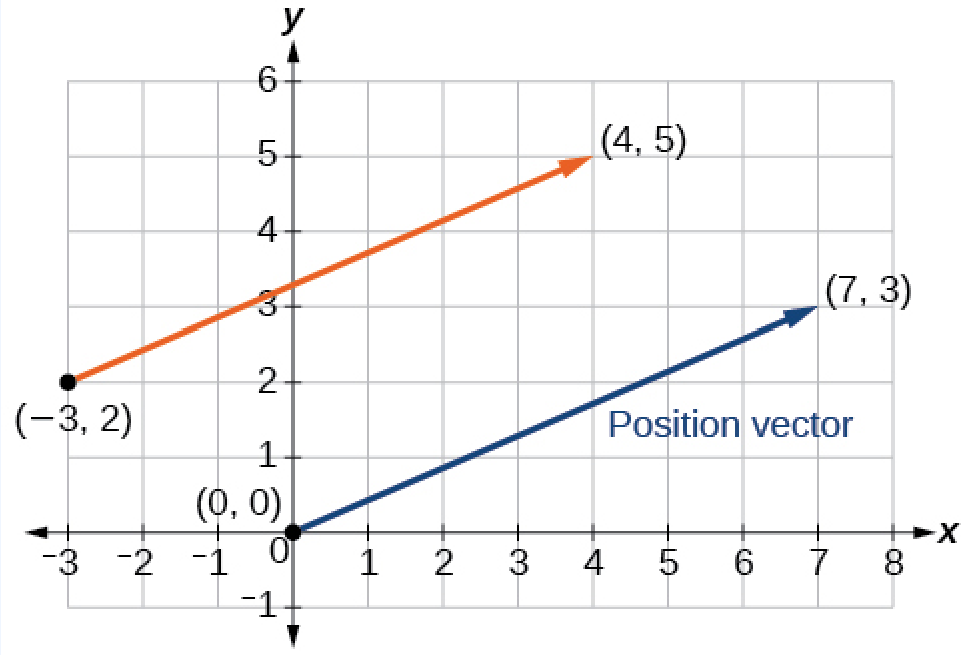
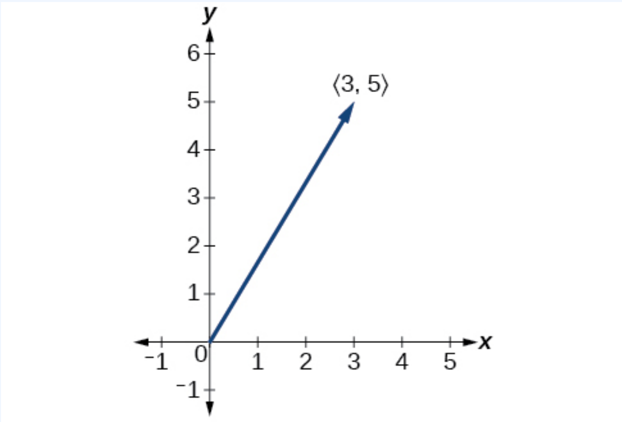


Figure 4.

Try It

Draw a vector v that connects from the origin to the point $(3, 5)$.

Show Solution



Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

Magnitude and Direction of a Vector

Given a position vector $v = a, b$, the magnitude is found by $|v| = \sqrt{a^2 + b^2}$. The direction is equal to the angle formed with the *x*-axis, or with the *y*-axis, depending on the application. For a position vector, the direction is found by $\tan\theta = \left(\frac{b}{a}\right) \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in (Figure).

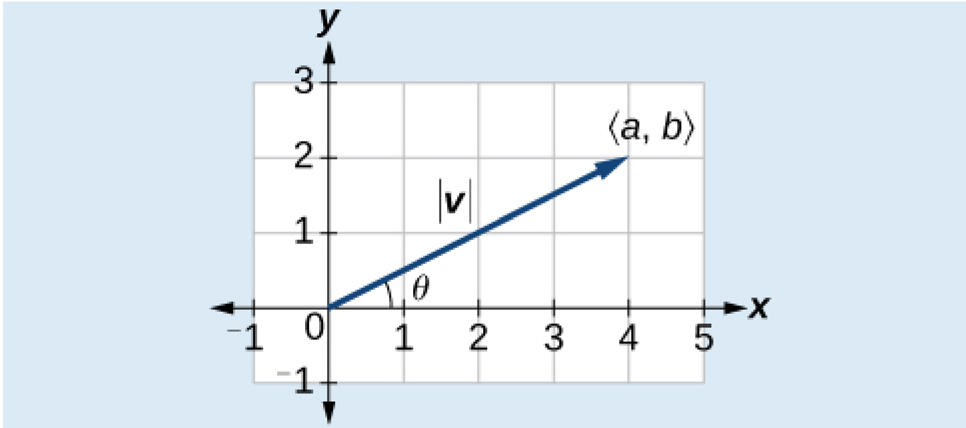


Figure 5.

Two vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point $P\left(-8,1\right)$ and terminal point $Q\left(-2,-5\right)$. Draw the vector.

Show Solution

First, find the position vector.

$$u = -2, -\left(-8\right), -5 - 1$$
$$= 6, -6$$

We use the Pythagorean Theorem to find the magnitude.

$$|u| = \sqrt{\left(6\right)^2 + \left(-6\right)^2}$$
$$= \sqrt{72}$$
$$= 6\sqrt{2}$$

The direction is given as

$$\tan\theta = \frac{-6}{6} = -1\theta = \tan^{-1}\left(-1\right)$$
$$= -45°$$

However, the angle terminates in the fourth quadrant, so we add 360° to obtain a positive angle. Thus, $-45 + 360 = 315$. See (Figure).

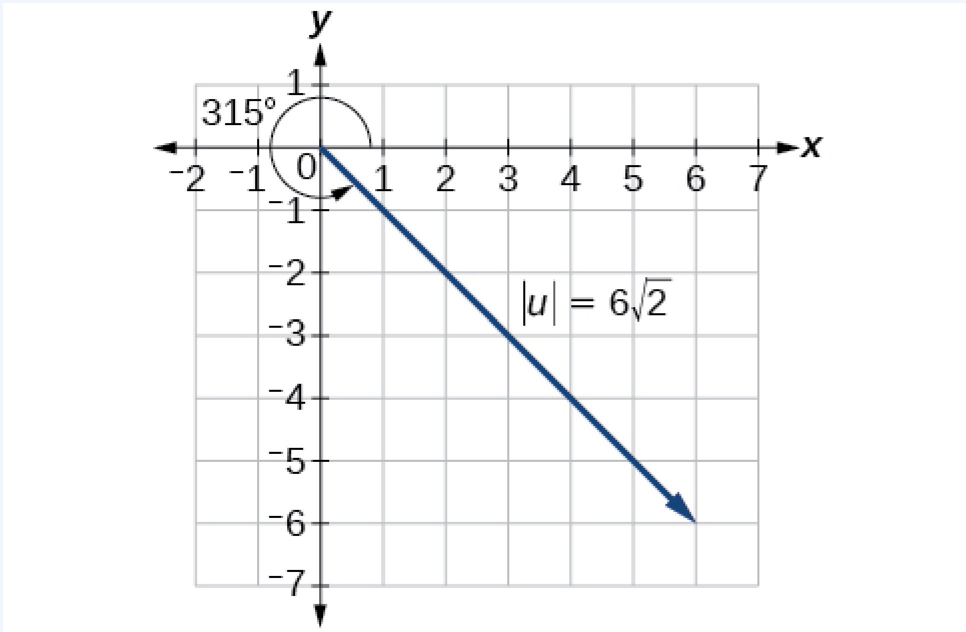


Figure 6.

Showing That Two Vectors Are Equal

Show that vector ***v*** with initial point at $(5,-3)$ and terminal point at $(-1,2)$ is equal to vector ***u*** with initial point at $(-1,-3)$ and terminal point at $(-7,2)$. Draw the position vector on the same grid as ***v*** and ***u***. Next, find the magnitude and direction of each vector.

Show Solution

As shown in (Figure), draw the vector v starting at initial $(5,-3)$ and terminal point $(-1,2)$. Draw the vector u with initial point $(-1,-3)$ and terminal point $(-7,2)$. Find the standard position for each.

Next, find and sketch the position vector for ***v*** and ***u***. We have

$$v = -1 - 5, 2 - (-3)$$
$$= -6, 5$$

$$u = -7 - (-1), 2 - (-3)$$
$$= -6, 5$$

Since the position vectors are the same, ***v*** and ***u*** are the same.

An alternative way to check for vector equality is to show that the magnitude and direction are the same for both vectors. To show that the magnitudes are equal, use the Pythagorean Theorem.

$$|v| = \sqrt{(-1-5)^2 + (2-(-3))^2}$$
$$= \sqrt{(-6)^2 + (5)^2}$$
$$= \sqrt{36+25}$$
$$= \sqrt{61}$$
$$|u| = \sqrt{(-7-(-1))^2 + (2-(-3))^2}$$
$$= \sqrt{(-6)^2 + (5)^2}$$
$$= \sqrt{36+25}$$
$$= \sqrt{61}$$

As the magnitudes are equal, we now need to verify the direction. Using the tangent function with the position vector gives

$$\tan\theta = -\tfrac{5}{6}\theta = \tan^{-1}\left(-\tfrac{5}{6}\right)$$
$$= -39.8^\circ$$

However, we can see that the position vector terminates in the second quadrant, so we add 180°. Thus, the direction is $-39.8^\circ + 180^\circ = 140.2^\circ$.

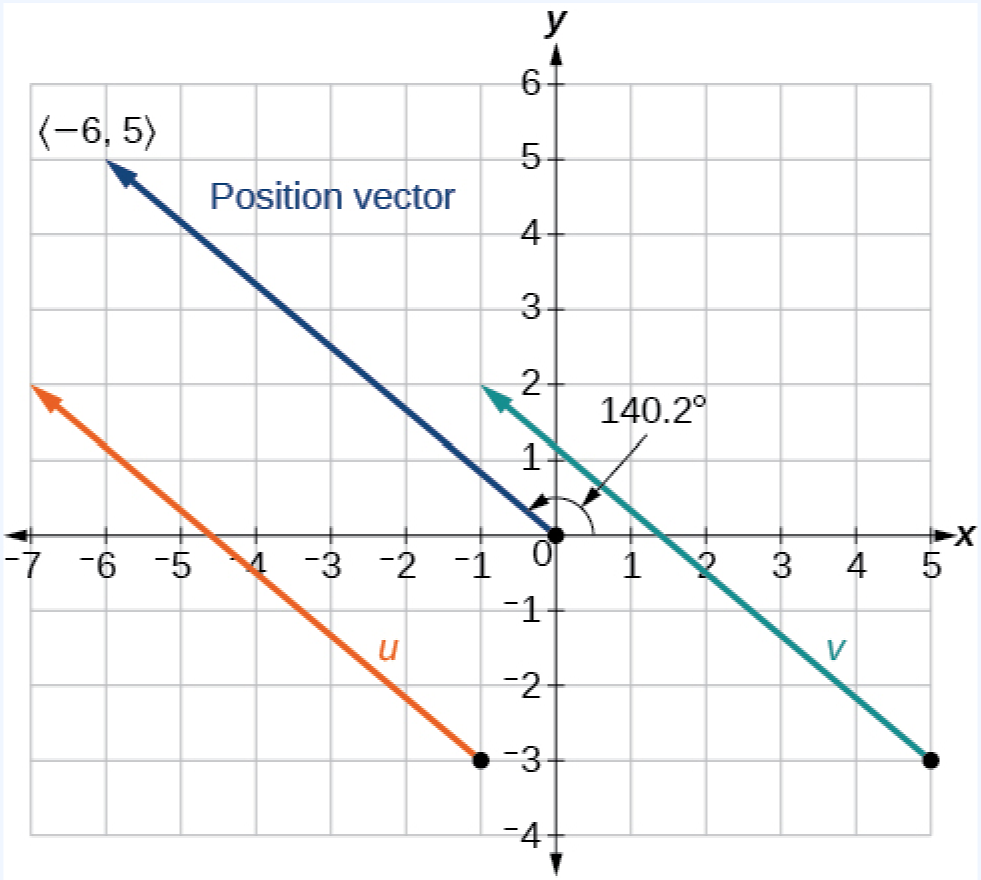


Figure 7.

Performing Vector Addition and Scalar Multiplication

Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector $u = x, y$ as an arrow or directed line segment from the origin to the point (x, y) , vectors can be situated anywhere in the plane. The sum of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, or vector addition, produces a third vector $\boldsymbol{u}$+ $\boldsymbol{v}$, the resultant vector.

To find $\boldsymbol{u} + \boldsymbol{v}$, we first draw the vector $\boldsymbol{u}$, and from the terminal end of $\boldsymbol{u}$, we drawn the vector $\boldsymbol{v}$. In other words, we have the initial point of $\boldsymbol{v}$ meet the terminal end of $\boldsymbol{u}$. This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum $\boldsymbol{u}$

+ ***v*** is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of ***u*** to the end of ***v*** in a straight path, as shown in (Figure).

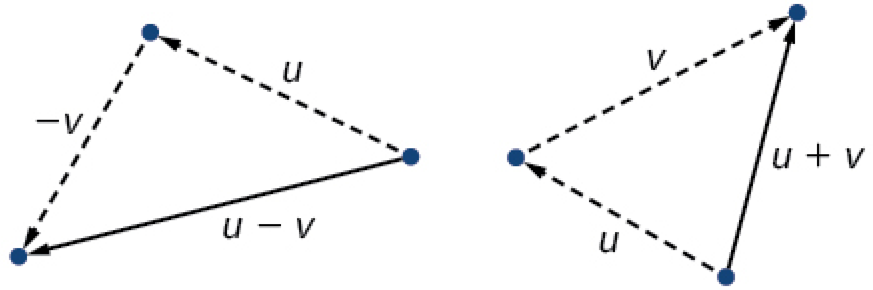


Figure 8.

Vector subtraction is similar to vector addition. To find ***u*** − ***v***, view it as ***u*** + (−***v***). Adding −***v*** is reversing direction of ***v*** and adding it to the end of ***u***. The new vector begins at the start of ***u*** and stops at the end point of −***v***. See (Figure) for a visual that compares vector addition and vector subtraction using parallelograms.

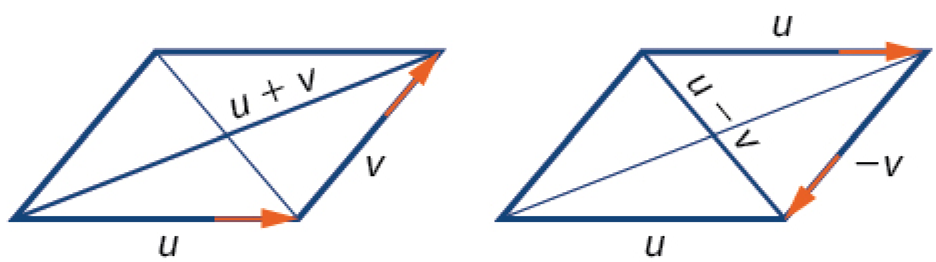


Figure 9.

Adding and Subtracting Vectors

Given $u = 3, -2$ and $v = -1, 4$, find two new vectors ***u*** + ***v***, and ***u*** − ***v***.

Show Solution

To find the sum of two vectors, we add the components. Thus,

$$\begin{aligned} u + v &= 3, -2 + -1, 4 \\ &= 3 + (-1), -2 + 4 \\ &= 2, 2 \end{aligned}$$

See (Figure)**(a)**.

To find the difference of two vectors, add the negative components of v to u. Thus,

$$\begin{aligned} u + (-v) &= 3, -2 + 1, -4 \\ &= 3 + 1, -2 + (-4) \\ &= 4, -6 \end{aligned}$$

See (Figure)**(b).**

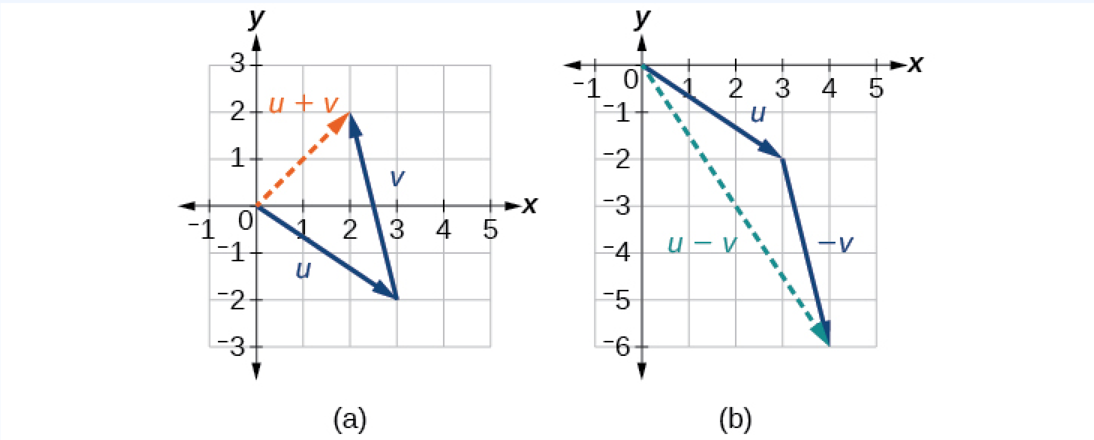


Figure 10. (a) Sum of two vectors (b) Difference of two vectors

Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a scalar, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

Scalar Multiplication

Scalar multiplication involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply $v = a, b$ by k , we have

$$kv = ka, kb$$

Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Performing Scalar Multiplication

Given vector $v = 3, 1$, find 3***v***, $\frac{1}{2}v$, and -***v***.

Show Solution

See (Figure) for a geometric interpretation. If $v = 3, 1$, then

$$\begin{aligned} 3v &= 3 \cdot 3, 3 \cdot 1 \\ &= 9, 3 \\ \tfrac{1}{2}v &= \tfrac{1}{2} \cdot 3, \tfrac{1}{2} \cdot 1 \\ &= \tfrac{3}{2}, \tfrac{1}{2} \\ -v &= -3, -1 \end{aligned}$$

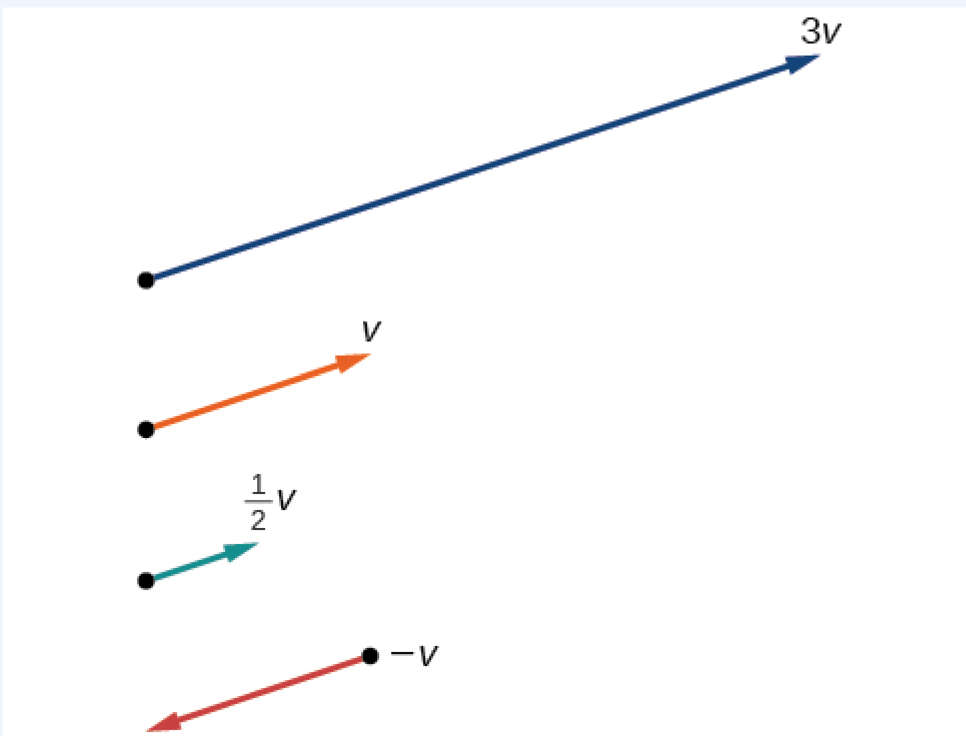


Figure 11.

Analysis

Notice that the vector 3***v*** is three times the length of ***v***, $\frac{1}{2}v$ is half the length of ***v***, and –***v*** is the same length of ***v***, but in the opposite direction.

Try It

Find the scalar multiple 3 u given $u = 5, 4.$

Show Solution
$3u = 15, 12$

Using Vector Addition and Scalar Multiplication to Find a New Vector

Given $u = 3, -2$ and $v = -1, 4,$ find a new vector ***w*** = 3***u*** + 2***v***.

Show Solution
First, we must multiply each vector by the scalar.

$$3u = 33, -2$$
$$= 9, -6$$
$$2v = 2 - 1, 4$$
$$= -2, 8$$

Then, add the two together.

$$\begin{aligned} w &= 3u + 2v \\ &= 9, -6 + -2, 8 \\ &= 9 - 2, -6 + 8 \\ &= 7, 2 \end{aligned}$$

So, $w = 7, 2.$

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in (Figure) that the position vector $2, 3$ comes from adding the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. We have $\boldsymbol{v}_1$ with initial point $(0, 0)$ and terminal point $(2, 0)$.

$$\begin{aligned} v_1 &= 2 - 0, 0 - 0 \\ &= 2, 0 \end{aligned}$$

We also have $\boldsymbol{v}_2$ with initial point $(0, 0)$ and terminal point $(0,\ 3)$.

$$\begin{aligned} v_2 &= 0 - 0, 3 - 0 \\ &= 0, 3 \end{aligned}$$

Therefore, the position vector is

$$\begin{aligned} v &= 2 + 0, 3 + 0 \\ &= 2, 3 \end{aligned}$$

Using the Pythagorean Theorem, the magnitude of $\boldsymbol{v}_1$ is 2, and the magnitude of $\boldsymbol{v}_2$ is 3. To find the magnitude of $\boldsymbol{v}$, use the formula with the position vector.

$$\begin{aligned} |v| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

The magnitude of $\boldsymbol{v}$ is $\sqrt{13}$. To find the direction, we use the tangent function $\tan\theta = \frac{y}{x}$.

$$\tan\theta = \frac{v_2}{v_1}$$
$$\tan\theta = \frac{3}{2}$$
$$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.3°$$

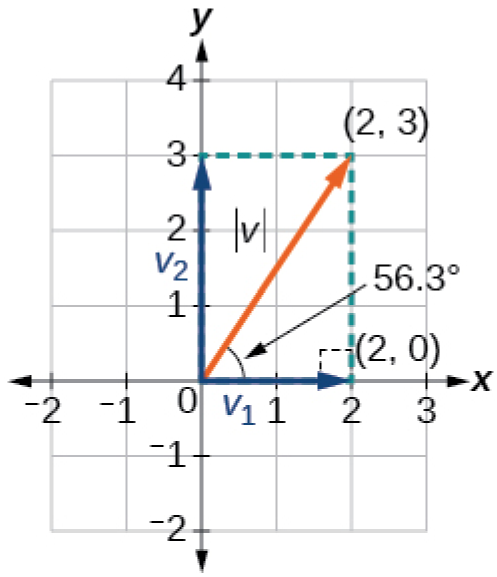


Figure 12.

Thus, the magnitude of v is $\sqrt{13}$ and the direction is $56.3°$ off the horizontal.

Finding the Components of the Vector

Find the components of the vector v with initial point $(3, 2)$ and terminal point $(7, 4)$.

Show Solution

First find the standard position.

$$v = 7 - 3, 4 - 2$$
$$= 4, 2$$

See the illustration in (Figure).

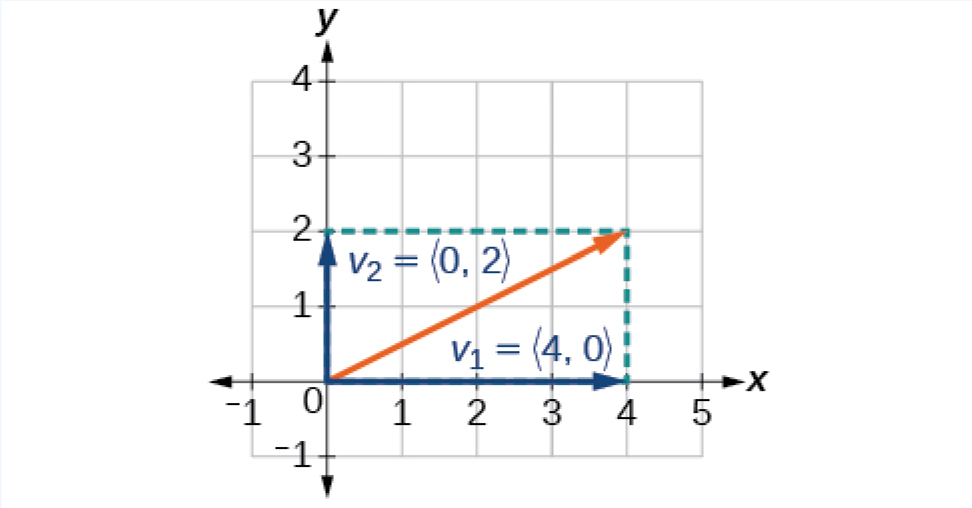


Figure 13.

The horizontal component is $v_1 = 4, 0$ and the vertical component is $v_2 = 0, 2$.

Finding the Unit Vector in the Direction of *v*

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a unit vector. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as $i = 1, 0$ and is directed along the positive horizontal axis. The vertical unit vector is written as $j = 0, 1$ and is directed along the positive vertical axis. See (Figure).

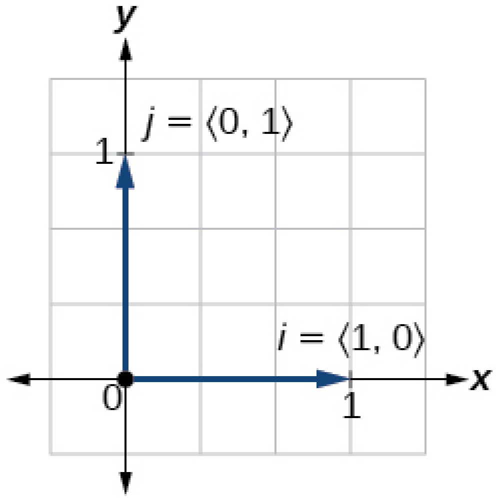


Figure 14.

The Unit Vectors

If v is a nonzero vector, then $\frac{v}{|v|}$ is a unit vector in the direction of v. Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Finding the Unit Vector in the Direction of *v*

Find a unit vector in the same direction as $v = -5, 12$.

Show Solution

First, we will find the magnitude.

$$\begin{aligned}|v| &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13\end{aligned}$$

Then we divide each component by $|v|$, which gives a unit vector in the same direction as ***v***.

$$\frac{v}{|v|} = -\frac{5}{13}i + \frac{12}{13}j$$

or, in component form

$\frac{v}{|v|} = -\frac{5}{13}, \frac{12}{13}$

See (Figure).

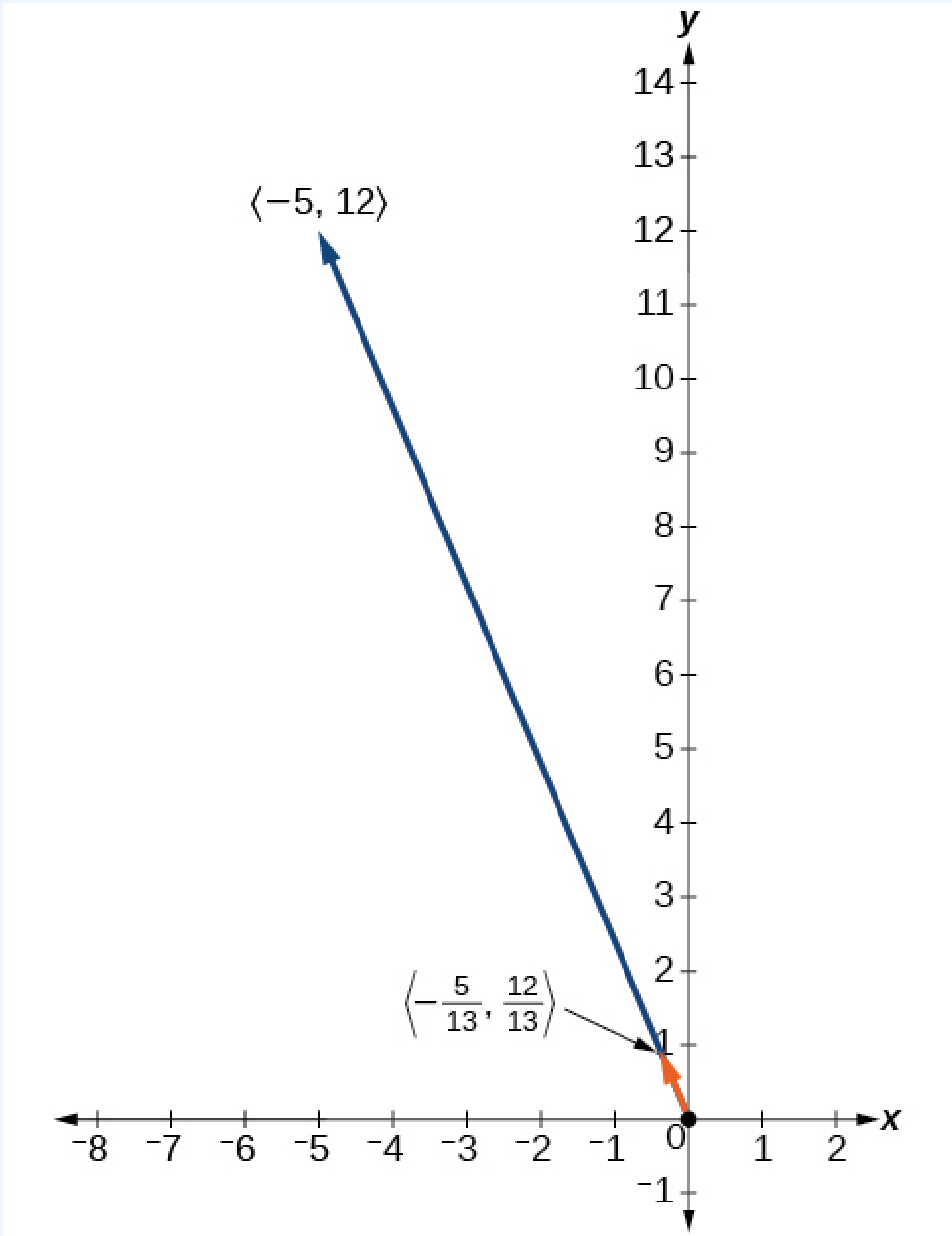


Figure 15.

Verify that the magnitude of the unit vector equals 1. The magnitude of $-\frac{5}{13}i + \frac{12}{13}j$ is given as

$$\sqrt{\left(-\frac{5}{13}\right)^2+\left(\frac{12}{13}\right)^2}=\sqrt{\frac{25}{169}+\frac{144}{169}}$$
$$=\sqrt{\frac{169}{169}}=1$$

The vector ***u*** $=\frac{5}{13}$***i*** $+\frac{12}{13}$***j*** is the unit vector in the same direction as ***v*** $=-5, 12.$

Performing Operations with Vectors in Terms of *i* and *j*

So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of ***i*** and ***j***.

Vectors in the Rectangular Plane

Given a vector v with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2)$, ***v*** is written as
$v=(x_2-x_1)\,i+(y_2-y_1)\,j$

The position vector from $(0,0)$ to (a,b) , where $(x_2-x_1)=a$ and $(y_2-y_1)=b$, is written as ***v*** = *a**i*** + *b**j***. This vector sum is called a linear combination of the vectors ***i*** and ***j***.

The magnitude of ***v*** = *a**i*** + *b**j*** is given as $|v|=\sqrt{a^2+b^2}$. See (Figure).

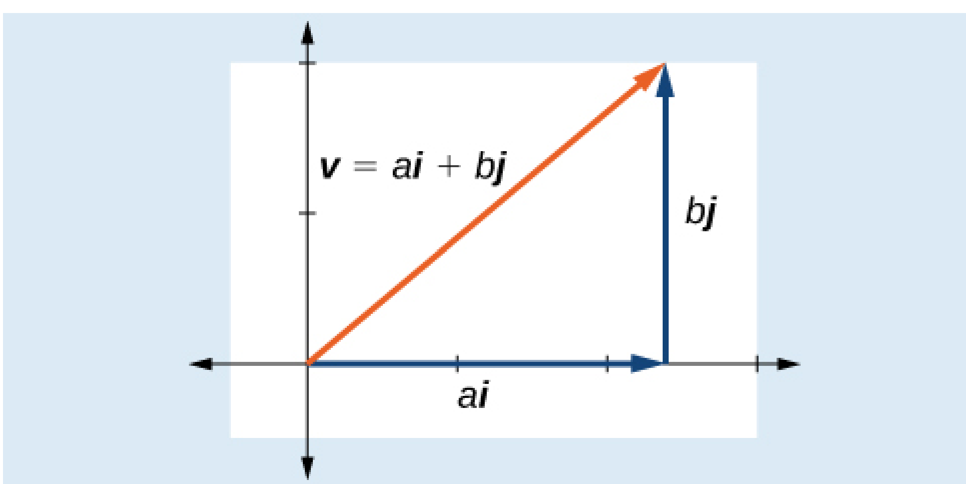


Figure 16.

Writing a Vector in Terms of *i* and *j*

Given a vector v with initial point $P = (2, -6)$ and terminal point $Q = (-6, 6)$, write the vector in terms of i and j.

Show Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$v = (x_2 - x_1)\,i + (y_2 - y_1)\,j$$
$$= (-6 - 2)\,i + (6 - (-6))\,j$$
$$= -8i + 12j$$

Writing a Vector in Terms of *i* and *j* Using Initial and Terminal Points

Given initial point $P_1 = (-1, 3)$ and terminal point $P_2 = (2, 7)$, write the vector v in terms of i and j.

Show Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$v = (x_2 - x_1)\,i + (y_2 - y_1)\,j$$
$$v = (2 - (-1))\,i + (7 - 3)\,j$$
$$= 3i + 4j$$

Try It

Write the vector u with initial point $P=(-1,6)$ and terminal point $Q=(7,-5)$ in terms of i and j.

Show Solution

$u=8i-11j$

Performing Operations on Vectors in Terms of *i* and *j*

When vectors are written in terms of i and j, we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

Adding and Subtracting Vectors in Rectangular Coordinates

Given $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ and $\boldsymbol{u} = c\boldsymbol{i} + d\boldsymbol{j}$, then

$$v+u=(a+c)\,i+(b+d)\,j$$
$$v-u=(a-c)\,i+(b-d)\,j$$

Finding the Sum of the Vectors

Find the sum of $v_1=2i-3j$ and $v_2=4i+5j$.

Show Solution

According to the formula, we have

$$\begin{aligned} v_1 + v_2 &= (2+4)\,i + (-3+5)\,j \\ &= 6i + 2j \end{aligned}$$

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using $i \text{ and } j$. For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with $|v|$ replacing r.

Vector Components in Terms of Magnitude and Direction

Given a position vector $v = x, y$ and a direction angle θ,

$$\cos\theta = \frac{x}{|v|} \quad \text{and} \quad \sin\theta = \frac{y}{|v|}$$

$$x = |v|\cos\theta \qquad\qquad y = |v|\sin\theta$$

Thus, $v = xi + yj = |v|\cos\theta i + |v|\sin\theta j$, and magnitude is expressed as $|v| = \sqrt{x^2 + y^2}$.

Writing a Vector in Terms of Magnitude and Direction

Write a vector with length 7 at an angle of 135° to the positive *x*-axis in terms of magnitude and direction.

Show Solution

Using the conversion formulas $x = |v|\cos\theta i$ and $y = |v|\sin\theta j$, we find that

$$x = 7\cos(135°)\,i$$
$$= -\frac{7\sqrt{2}}{2}$$
$$y = 7\sin(135°)\,j$$
$$= \frac{7\sqrt{2}}{2}$$

This vector can be written as $v = 7\cos(135°)\,i + 7\sin(135)\,j$ or simplified as

$$v = -\frac{7\sqrt{2}}{2}i + \frac{7\sqrt{2}}{2}j$$

A vector travels from the origin to the point $(3, 5)$. Write the vector in terms of magnitude and direction.

Show Solution

$$v = \sqrt{34}\cos(59°)\,i + \sqrt{34}\sin(59°)\,j$$

Magnitude = $\sqrt{34}$

$$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.04°$$

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the *dot product* and the *cross product*. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

Dot Product

The dot product of two vectors $v = a, b$ and $u = c, d$ is the sum of the product of the horizontal components and the product of the vertical components.
$v \cdot u = ac + bd$

To find the angle between the two vectors, use the formula below.
$\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$

Finding the Dot Product of Two Vectors

Find the dot product of $v = 5, 12$ and $u = -3, 4$.

Show Solution
Using the formula, we have

$$\begin{aligned} v \cdot u &= 5, 12 \cdot -3, 4 \\ &= 5 \cdot (-3) + 12 \cdot 4 \\ &= -15 + 48 \\ &= 33 \end{aligned}$$

Finding the Dot Product of Two Vectors and the Angle between Them

Find the dot product of $\boldsymbol{v}_1 = 5\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{v}_2 = 3\boldsymbol{i} + 7\boldsymbol{j}$. Then, find the angle between the two vectors.

Show Solution
Finding the dot product, we multiply corresponding components.

$$\begin{aligned} v_1 \cdot v_2 &= 5, 2 \cdot 3, 7 \\ &= 5 \cdot 3 + 2 \cdot 7 \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

To find the angle between them, we use the formula $\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$.

$$\begin{aligned} \frac{v}{|v|} \cdot \frac{u}{|u|} &= \frac{5}{\sqrt{29}} + \frac{2}{\sqrt{29}} \cdot \frac{3}{\sqrt{58}} + \frac{7}{\sqrt{58}} \\ &= \frac{5}{\sqrt{29}} \cdot \frac{3}{\sqrt{58}} + \frac{2}{\sqrt{29}} \cdot \frac{7}{\sqrt{58}} \\ &= \frac{15}{\sqrt{1682}} + \frac{14}{\sqrt{1682}} = \frac{29}{\sqrt{1682}} \\ &= 0.707107 \end{aligned}$$

$$\cos^{-1}(0.707107) = 45°$$

See (Figure).

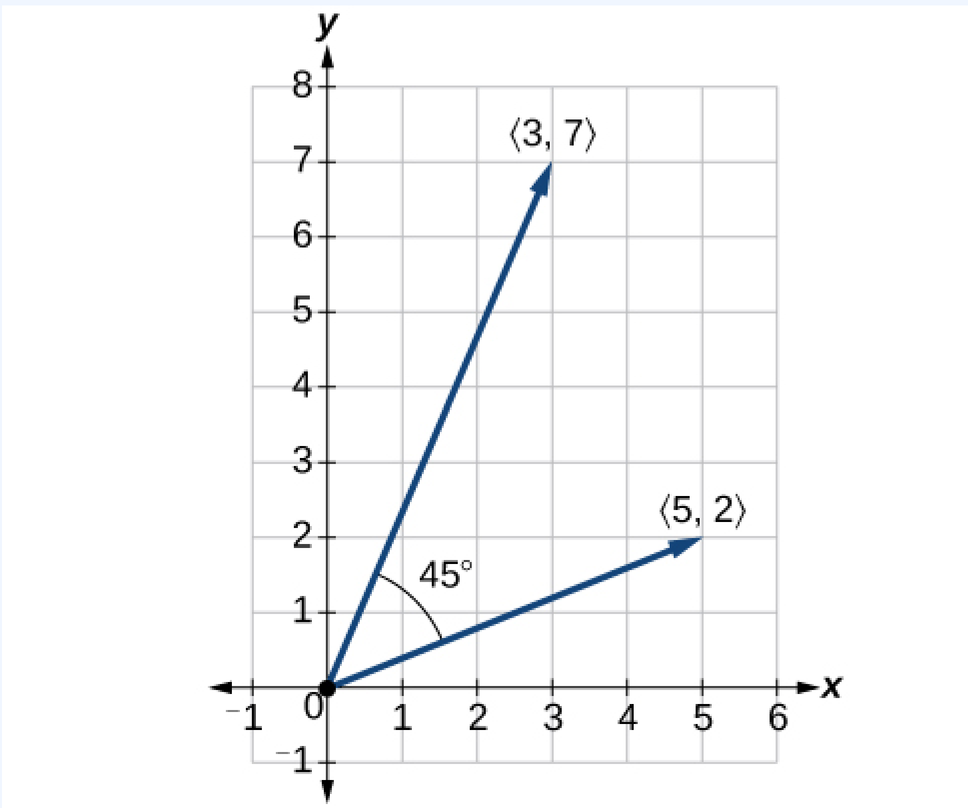


Figure 17.

Finding the Angle between Two Vectors

Find the angle between $u = -3, 4$ and $v = 5, 12$.

Show Solution

Using the formula, we have

$$\theta = \cos^{-1}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right)$$

$$\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) = \frac{-3i+4j}{5} \cdot \frac{5i+12j}{13}$$

$$= \left(-\frac{3}{5} \cdot \frac{5}{13}\right) + \left(\frac{4}{5} \cdot \frac{12}{13}\right)$$

$$= -\frac{15}{65} + \frac{48}{65}$$

$$= \frac{33}{65}$$

$$\theta = \cos^{-1}\left(\frac{33}{65}\right)$$

$$= 59.5°$$

See (Figure).

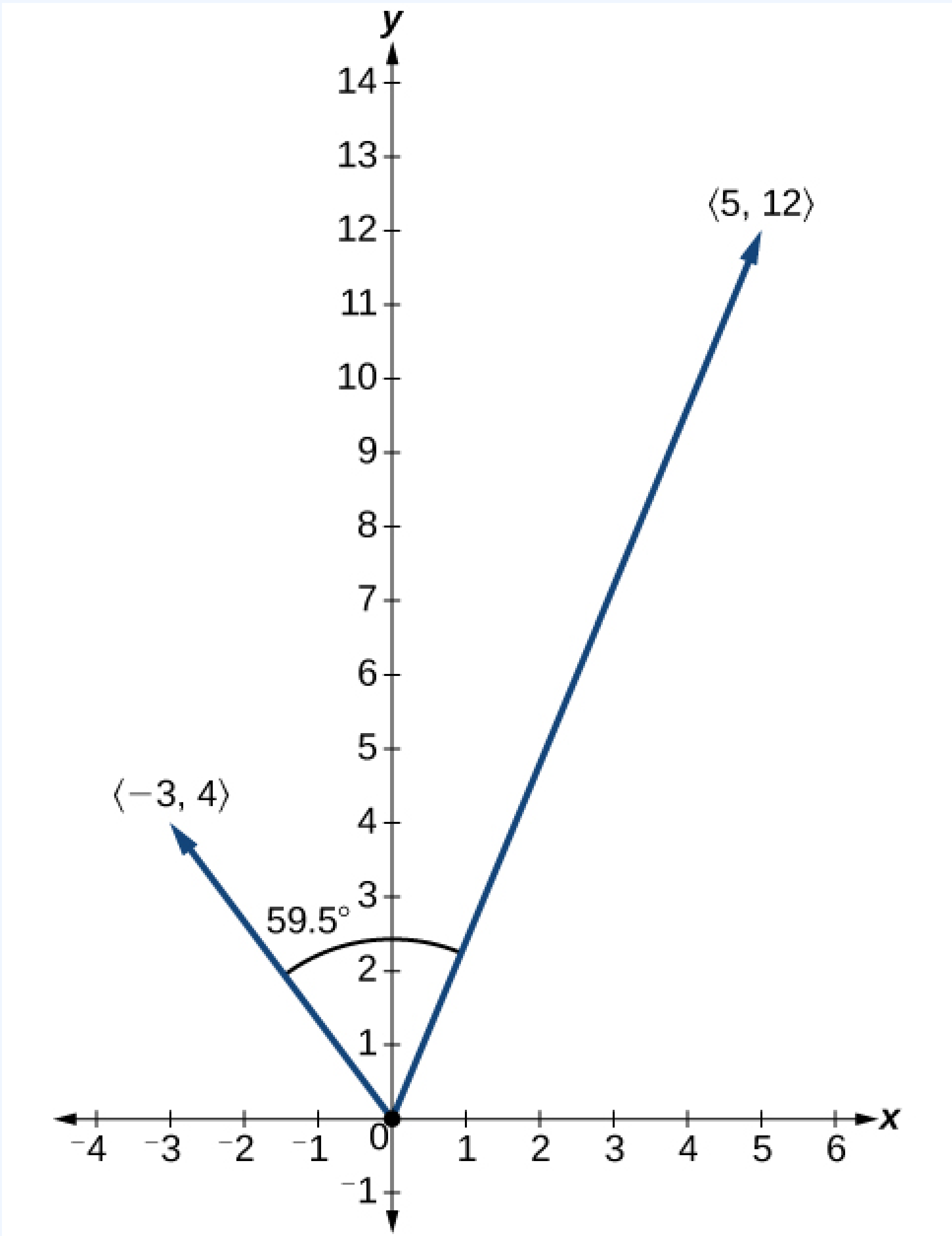


Figure 18.

Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See (Figure).

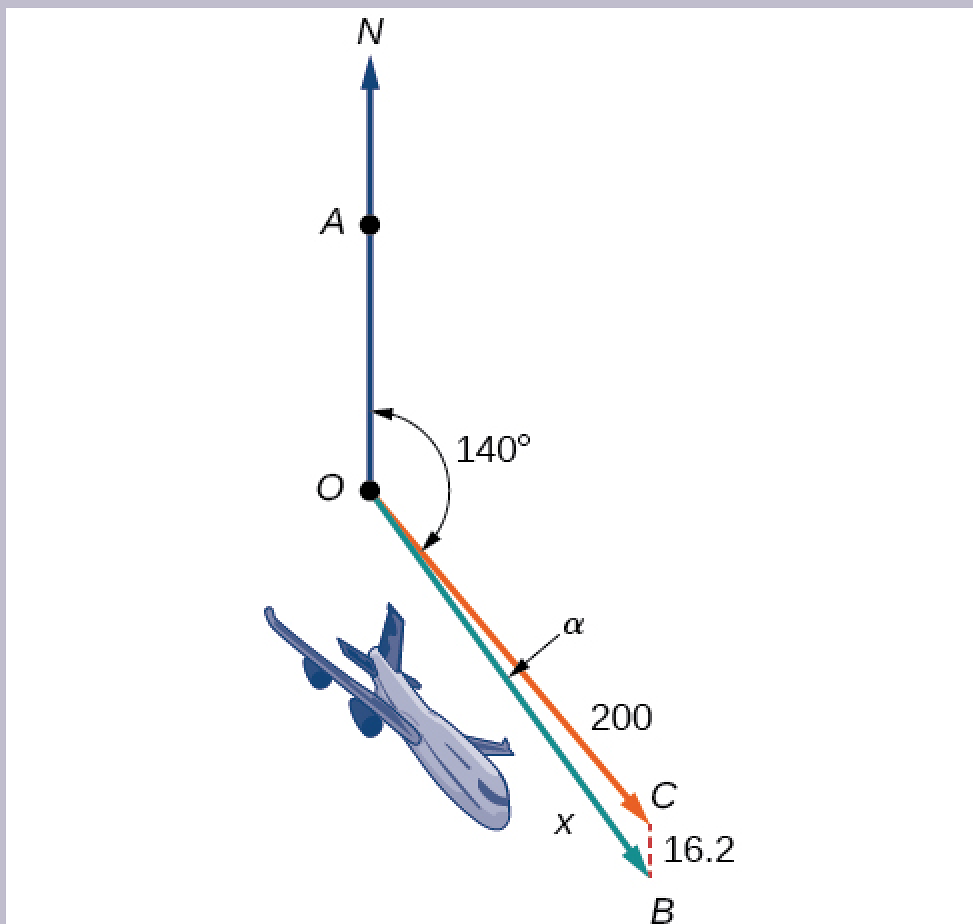


Figure 19.

Show Solution

The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140^\circ + \alpha$.

Notice in (Figure), that angle BCO must be equal to angle AOC by the rule of alternating interior angles, so angle BCO is 140°. We can find x by the Law of Cosines:

$$x^2 = (16.2)^2 + (200)^2 - 2\,(16.2)\,(200)\cos(140°)$$
$$x^2 = 45,226.41$$
$$x = \sqrt{45,226.41}$$
$$x = 212.7$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$$\frac{\sin\alpha}{16.2} = \frac{\sin(140°)}{212.7}$$
$$\sin\alpha = \frac{16.2\sin(140°)}{212.7}$$
$$= 0.04896$$
$$\sin^{-1}(0.04896) = 2.8°$$

Therefore, the plane has a SE bearing of 140°+2.8°=142.8°. The ground speed is 212.7 miles per hour.

Access these online resources for additional instruction and practice with vectors.

- Introduction to Vectors
- Vector Operations
- The Unit Vector

Key Concepts

- The position vector has its initial point at the origin. See (Figure).
- If the position vector is the same for two vectors, they are equal. See (Figure).
- Vectors are defined by their magnitude and direction. See (Figure).
- If two vectors have the same magnitude and direction, they are equal. See (Figure).
- Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See (Figure).

- Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See (Figure) and (Figure).
- Vectors are comprised of two components: the horizontal component along the positive *x*-axis, and the vertical component along the positive *y*-axis. See (Figure).
- The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.
- The magnitude of a vector in the rectangular coordinate system is $|v| = \sqrt{a^2 + b^2}$. See (Figure)**.**
- In the rectangular coordinate system, unit vectors may be represented in terms of i and j where i represents the horizontal component and j represents the vertical component. Then, ***v*** = a***i*** + b***j*** is a scalar multiple of v by real numbers $a \text{ and } b.$ See (Figure) and (Figure).
- Adding and subtracting vectors in terms of *i* and *j* consists of adding or subtracting corresponding coefficients of *i* and corresponding coefficients of *j*. See (Figure).
- A vector *v* = *a****i*** + *b****j*** is written in terms of magnitude and direction as $v = |v| \cos \theta i + |v| \sin \theta j.$ See (Figure).
- The dot product of two vectors is the product of the i terms plus the product of the j terms. See (Figure).
- We can use the dot product to find the angle between two vectors. (Figure) and (Figure).
- Dot products are useful for many types of physics applications. See (Figure).

Section Exercises

Verbal

1. What are the characteristics of the letters that are commonly used to represent vectors?

Show Solution
lowercase, bold letter, usually u, v, w

2. How is a vector more specific than a line segment?

3. What are i and j, and what do they represent?

Show Solution
They are unit vectors. They are used to represent the horizontal and vertical components of a vector. They each have a magnitude of 1.

4. What is component form?

5. When a unit vector is expressed as a, b, which letter is the coefficient of the i and which the j?

Show Solution
The first number always represents the coefficient of the i, and the second represents the j.

Algebraic

6. Given a vector with initial point $(5, 2)$ and terminal point $(-1, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form a, b.

7. Given a vector with initial point $(-4, 2)$ and terminal point $(3, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form a, b.

Show Solution
$7, -5$

8. Given a vector with initial point $(7, -1)$ and terminal point $(-1, -7)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form a, b.

For the following exercises, determine whether the two vectors u and v are equal, where u has an initial point P_1 and a terminal point P_2 and v has an initial point P_3 and a terminal point P_4 .

9. $P_1 = (5, 1)$, $P_2 = (3, -2)$, $P_3 = (-1, 3)$, and $P_4 = (9, -4)$

Show Solution
not equal

10. $P_1 = (2, -3)$, $P_2 = (5, 1)$, $P_3 = (6, -1)$, and $P_4 = (9, 3)$

11. $P_1 = (-1, -1)$, $P_2 = (-4, 5)$, $P_3 = (-10, 6)$, and $P_4 = (-13, 12)$

Show Solution
equal

12. $P_1 = (3, 7)$, $P_2 = (2, 1)$, $P_3 = (1, 2)$, and $P_4 = (-1, -4)$

13. $P_1 = (8, 3)$, $P_2 = (6, 5)$, $P_3 = (11, 8)$, and $P_4 = (9, 10)$

Show Solution
equal

14. Given initial point $P_1 = (-3, 1)$ and terminal point $P_2 = (5, 2)$, write the vector v in terms of i and j.

15. Given initial point $P_1 = (6, 0)$ and terminal point $P_2 = (-1, -3)$, write the vector v in terms of i and j.

Show Solution
$7i - 3j$

For the following exercises, use the vectors ***u*** = ***i*** + 5***j***, ***v*** = −2***i***− 3***j***, and ***w*** = 4***i*** − ***j***.

16. Find ***u*** + (***v*** − ***w***)

17. Find 4***v*** + 2***u***

Show Solution

$-6i - 2j$

For the following exercises, use the given vectors to compute ***u*** + ***v***, ***u*** − ***v***, and 2***u*** − 3***v***.

18. $u = 2, -3, v = 1, 5$

19. $u = -3, 4, v = -2, 1$

Show Solution

$u + v = -5, 5, u - v = -1, 3, 2u - 3v = 0, 5$

20. Let ***v*** = −4***i*** + 3***j***. Find a vector that is half the length and points in the same direction as v.

21. Let ***v*** = 5***i*** + 2***j***. Find a vector that is twice the length and points in the opposite direction as v.

Show Solution

$-10i - 4j$

For the following exercises, find a unit vector in the same direction as the given vector.

22.***a*** = 3***i*** + 4***j***

23.***b*** = −2***i*** + 5***j***

Show Solution
$-\frac{2\sqrt{29}}{29}i+\frac{5\sqrt{29}}{29}j$

24. ***c*** = 10***i*** – ***j***

25. $d=-\frac{1}{3}i+\frac{5}{2}j$

Show Solution
$-\frac{2\sqrt{229}}{229}i+\frac{15\sqrt{229}}{229}j$

26.***u*** = 100***i*** + 200***j***

27. ***u*** = -14***i*** + 2***j***

Show Solution
$-\frac{7\sqrt{2}}{10}i+\frac{\sqrt{2}}{10}j$

For the following exercises, find the magnitude and direction of the vector, $0\leq\theta<2\pi$.

28. $0,4$

29. $6,5$

Show Solution
$|v|=7.810,\theta=39.806°$

30. $2,-5$

31. $-4, -6$

Show Solution
$|v| = 7.211, \theta = 236.310°$

32. Given ***u*** = 3***i*** − 4***j*** and ***v*** = −2***i*** + 3***j***, calculate $u \cdot v$.

33. Given ***u*** = −***i*** − ***j*** and ***v*** = ***i*** + 5***j***, calculate $u \cdot v$.

Show Solution
-6

34. Given $u = -2, 4$ and $v = -3, 1$, calculate $u \cdot v$.

35. Given ***u*** $= -1, 6$ and ***v*** $= 6, -1$, calculate $u \cdot v$.

Show Solution
-12

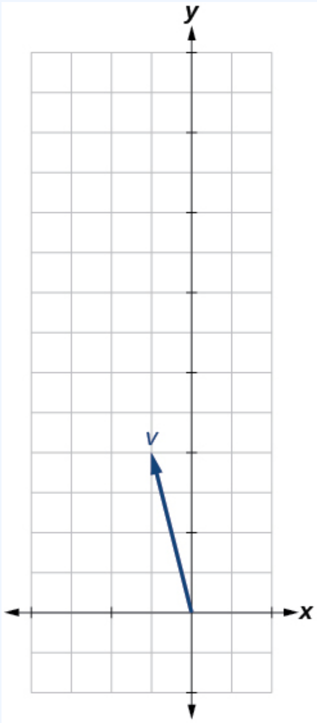
Graphical

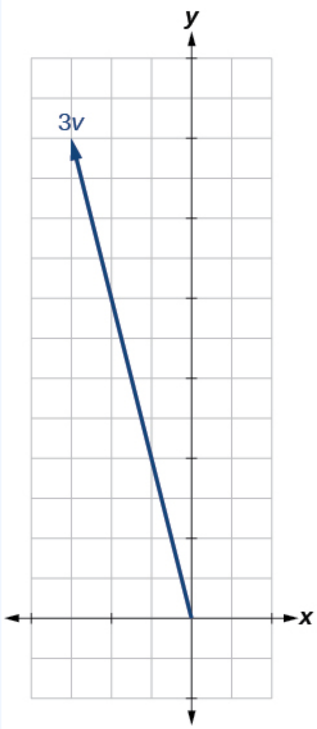
For the following exercises, given v, draw v, 3***v*** and $\frac{1}{2}v$.

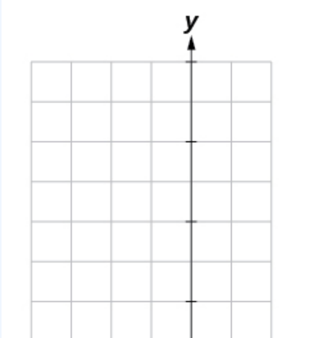
36. $2, -1$

37. $-1, 4$

Show Solution


y
v
x


y
3v
x


y

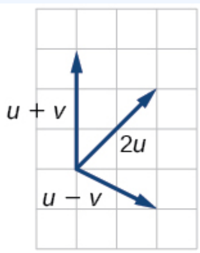
38. $-3, -2$

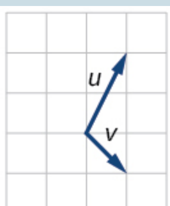
For the following exercises, use the vectors shown to sketch

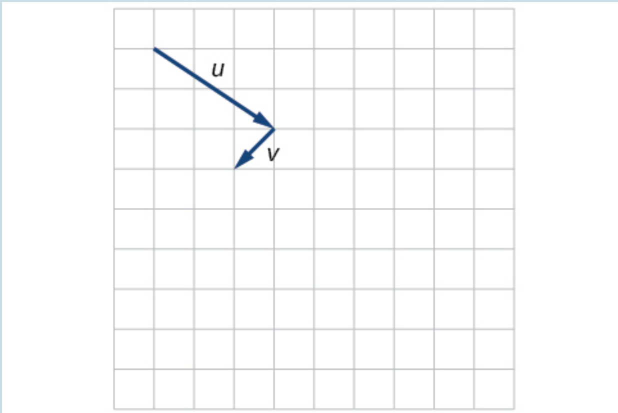
u + ***v***, ***u*** − ***v***, and 2***u***.

39. 

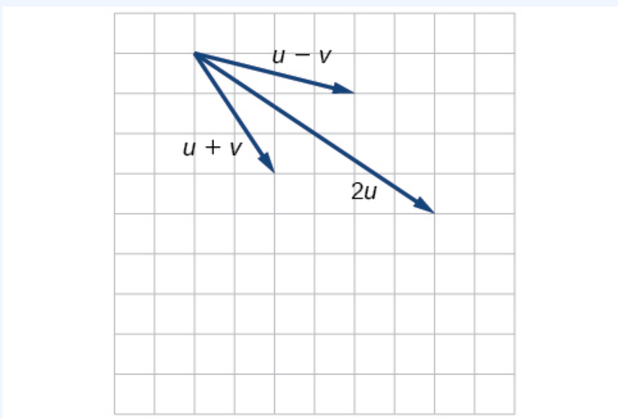
Show Solution



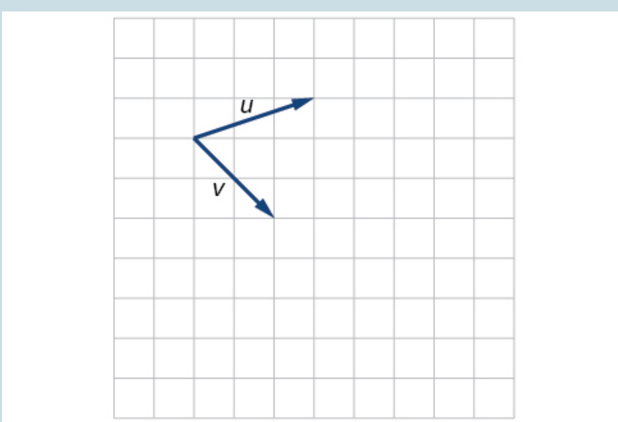
40. 

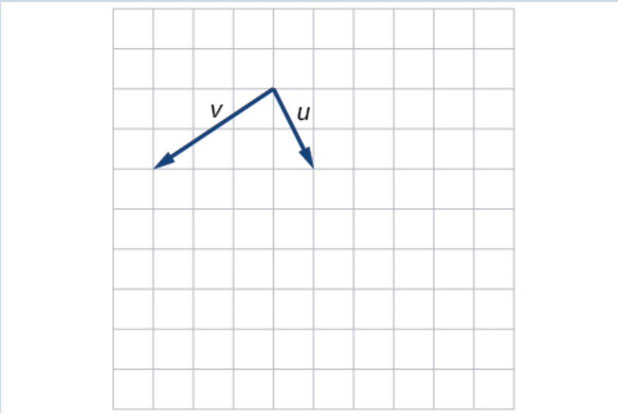
41. 

Show Solution



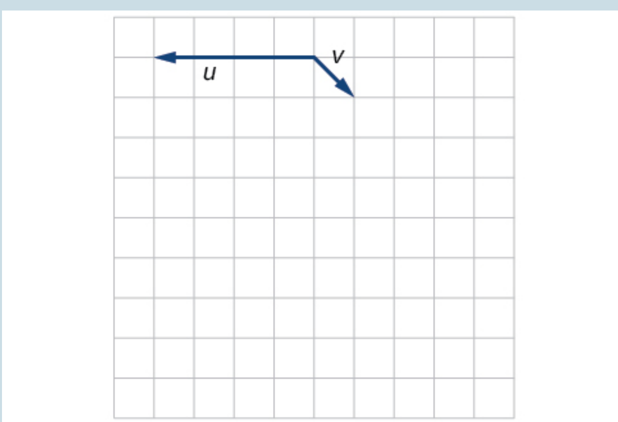
For the following exercises, use the vectors shown to sketch 2***u*** + ***v***.

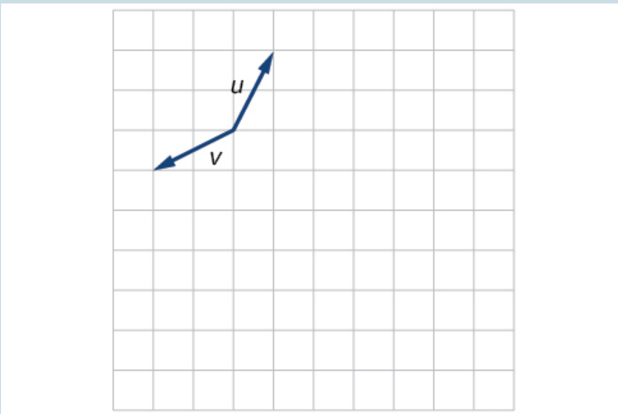
42. 

43. 

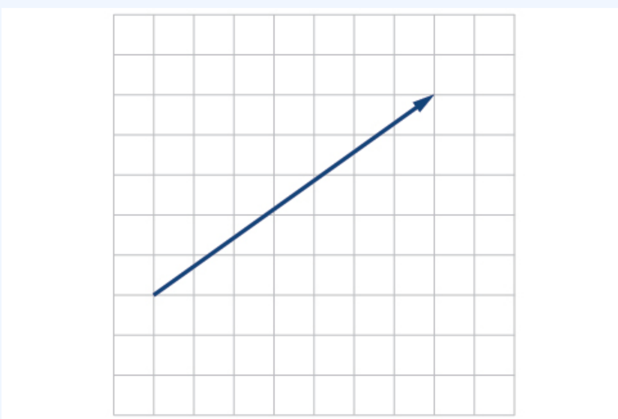
Show Solution

For the following exercises, use the vectors shown to sketch ***u*** – 3***v***.

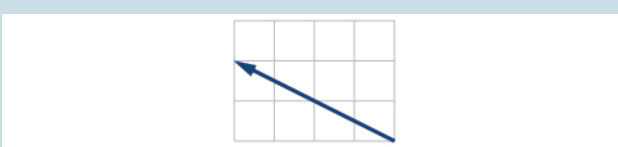
44. 

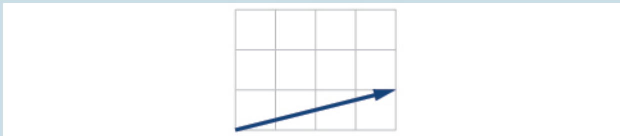
45. 

Show Solution



For the following exercises, write the vector shown in component form.

46. 

47. 

Show Solution

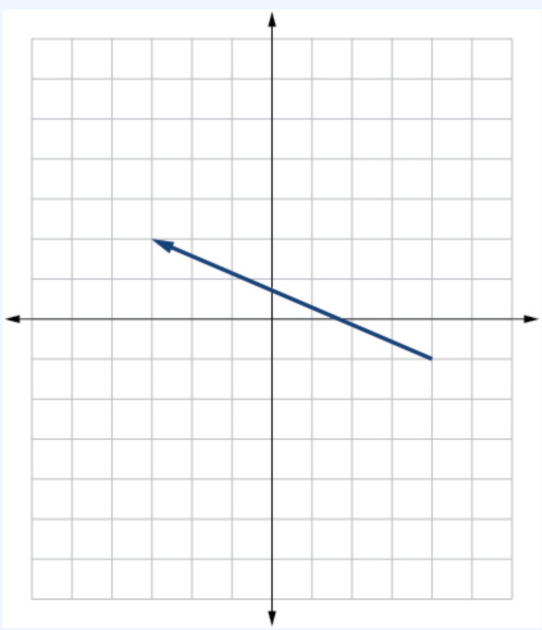
$4, 1$

48. Given initial point $P_1 = (2, 1)$ and terminal point $P_2 = (-1, 2)$, write the vector v in terms of i and j, then draw the vector on the graph.

49. Given initial point $P_1 = (4, -1)$ and terminal point $P_2 = (-3, 2)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Show Solution

$v = -7i + 3j$



50. Given initial point $P_1 = (3, 3)$ and terminal point $P_2 = (-3, 3)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Extensions

For the following exercises, use the given magnitude and direction in standard position, write the vector in component form.

51. $|v| = 6, \theta = 45°$

Show Solution
$3\sqrt{2}i + 3\sqrt{2}j$

52. $|v| = 8, \theta = 220°$

53. $|v| = 2, \theta = 300°$

Show Solution
$i - \sqrt{3}j$

54. $|v| = 5, \theta = 135°$

55. A 60-pound box is resting on a ramp that is inclined 12°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

Show Solution
a. 58.7; b. 12.5

56. A 25-pound box is resting on a ramp that is inclined 8°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

57. Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of 27° above the horizontal. Round to the nearest hundredth.

Show Solution
$x = 7.13$ pounds, $y = 3.63$ pounds

58. Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of 127° above the horizontal. Round to the nearest hundredth.

59. Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of 55° above the horizontal. Round to the nearest hundredth.

Show Solution
$x = 2.87$ pounds, $y = 4.10$ pounds

60. Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of 8° above the horizontal. Round to the nearest hundredth.

Real-World Applications

61. A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

Show Solution
4.635 miles, 17.764° N of E

62. A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

63. A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4

miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he have to walk?

Show Solution
17 miles. 10.318 miles

64. A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far would she have to walk?

65. A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to the walk, and in what direction?

Show Solution
Distance: 2.868. Direction: 86.474° North of West, or 3.526° West of North

66. A woman starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If she walked straight home, how far would she have to walk, and in what direction?

67. An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

Show Solution
4.924°. 659 km/hr

68. An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the

northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

69. An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

Show Solution
4.424°

70. An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

71. As part of a video game, the point $(5, 7)$ is rotated counterclockwise about the origin through an angle of 35°. Find the new coordinates of this point.

Show Solution
$(0.081, 8.602)$

72. As part of a video game, the point $(7, 3)$ is rotated counterclockwise about the origin through an angle of 40°. Find the new coordinates of this point.

73. Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

Show Solution

21.801°, relative to the car's forward direction

74. Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

75. A 50-pound object rests on a ramp that is inclined 19°. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

Show Solution
parallel: 16.28, perpendicular: 47.28 pounds

76. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it 45° from the horizontal. What single force is the resultant force acting on the body?

77. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it −135° from the horizontal, and 5 pounds acting on it directed 150° from the horizontal. What single force is the resultant force acting on the body?

Show Solution
19.35 pounds, 231.54° from the horizontal

78. The condition of equilibrium is when the sum of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 pounds acting on it 45° from the horizontal. What single force is needed to produce a state of equilibrium on the body?

79. Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward,

and 2 pounds acting on it 30° from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.

Show Solution
5.1583 pounds, 75.8° from the horizontal

Chapter Review Exercises

Non-right Triangles: Law of Sines

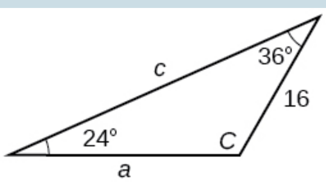
For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

1. $\beta = 50°, a = 105, b = 45$

Show Solution
Not possible

2. $\alpha = 43.1°, a = 184.2, b = 242.8$

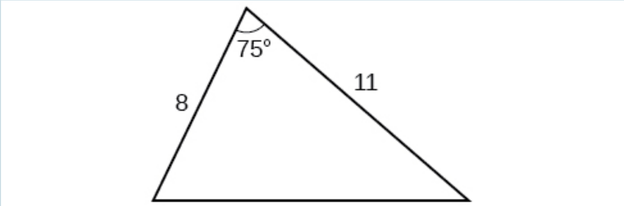
3. Solve the triangle.



Show Solution

$C = 120°, a = 23.1, c = 34.1$

4. Find the area of the triangle.



5. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 2.1 km apart, to be 25° and 49°, as shown in (Figure). Find the distance of the plane from point A and the elevation of the plane.

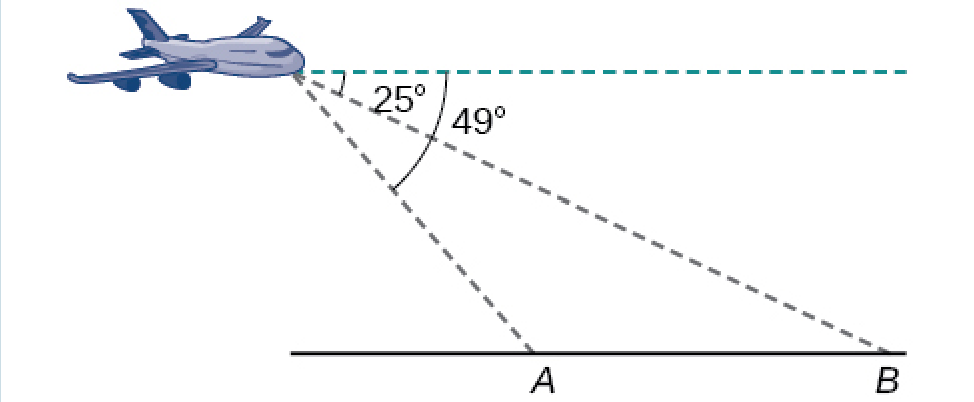


Figure 20.

Show Solution

distance of the plane from point $A:$ 2.2 km, elevation of the plane: 1.6 km

Non-right Triangles: Law of Cosines

6. Solve the triangle, rounding to the nearest tenth, assuming α is opposite side a, β is opposite side b, and γ s opposite side $c:\ a = 4, b = 6, c = 8.$

7. Solve the triangle in (Figure), rounding to the nearest tenth.

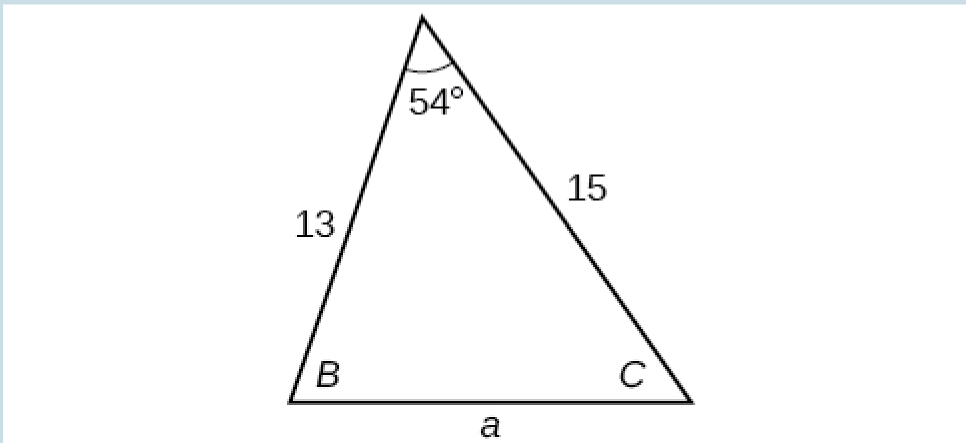


Figure 21.

Show Solution

$B = 71.0, C = 55.0, a = 12.8$

8. Find the area of a triangle with sides of length 8.3, 6.6, and 9.1.

9. To find the distance between two cities, a satellite calculates the distances and angle shown in (Figure) (not to scale). Find the distance between the cities. Round answers to the nearest tenth.

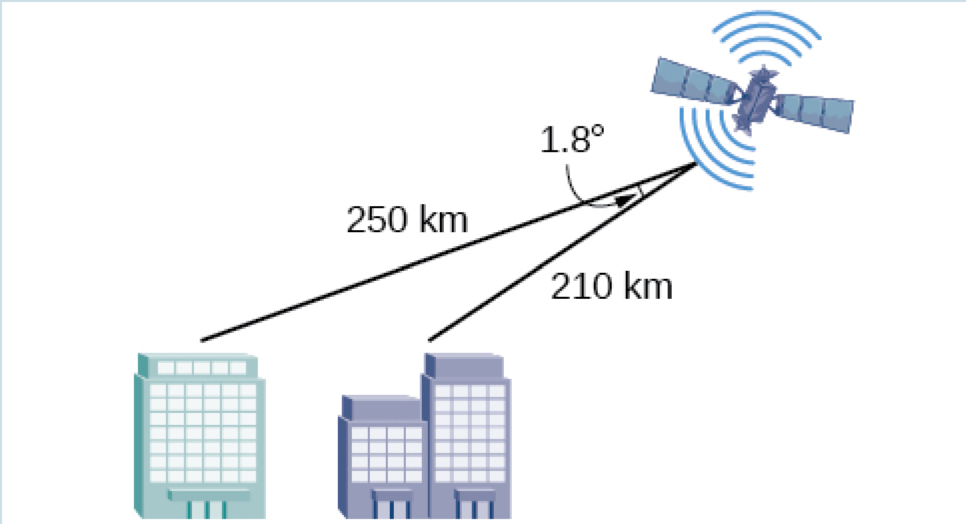


Figure 22.

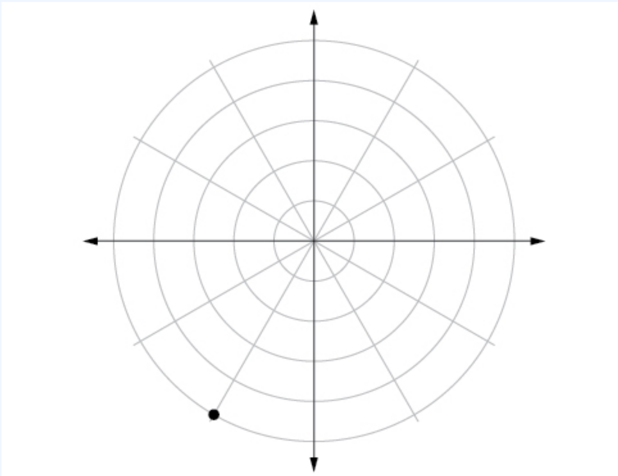
Show Solution
40.6 km

Polar Coordinates

10. Plot the point with polar coordinates $\left(3, \frac{\pi}{6}\right)$.

11. Plot the point with polar coordinates $\left(5, -\frac{2\pi}{3}\right)$

Show Solution



12. Convert $\left(6, -\frac{3\pi}{4}\right)$ to rectangular coordinates.

13. Convert $\left(-2, \frac{3\pi}{2}\right)$ to rectangular coordinates.

Show Solution
$(0, 2)$

14. Convert $(7, -2)$ to polar coordinates.

15. Convert $(-9, -4)$
to polar coordinates.

Show Solution
$(9.8489, 203.96)$

For the following exercises, convert the given Cartesian equation to a polar equation.

16. $x = -2$

17. $x^2 + y^2 = 64$

Show Solution

$r = 8$

18. $x^2 + y^2 = -2y$

For the following exercises, convert the given polar equation to a Cartesian equation.

19. $r = 7\cos\theta$

Show Solution

$x^2 + y^2 = 7x$

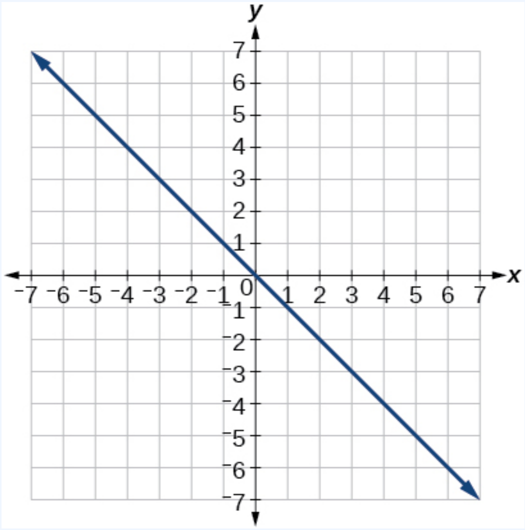
20. $r = \frac{-2}{4\cos\theta + \sin\theta}$

For the following exercises, convert to rectangular form and graph.

21. $\theta = \frac{3\pi}{4}$

Show Solution

$y = -x$



22. $r = 5\sec\theta$

Polar Coordinates: Graphs

For the following exercises, test each equation for symmetry.

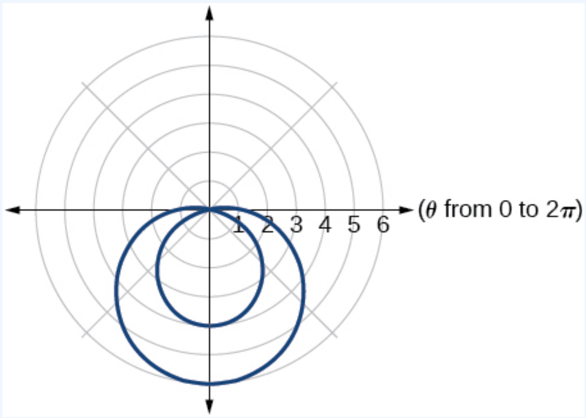
23. $r = 4 + 4\sin\theta$

Show Solution
symmetric with respect to the line $\theta = \frac{\pi}{2}$

24. $r = 7$

25. Sketch a graph of the polar equation $r = 1 - 5\sin\theta$. Label the axis intercepts.

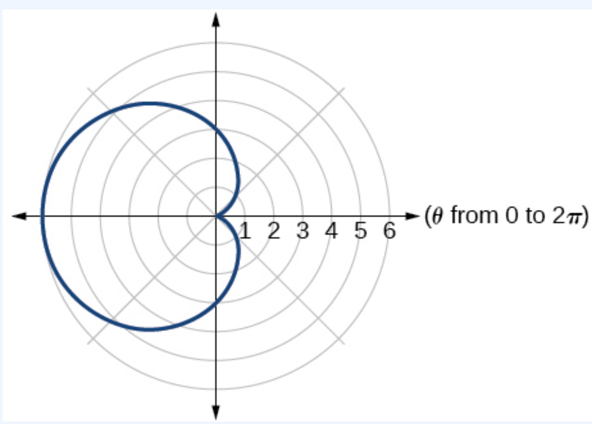
Show Solution



26. Sketch a graph of the polar equation $r = 5\sin\left(7\theta\right)$.

27. Sketch a graph of the polar equation $r = 3 - 3\cos\theta$

Show Solution



Polar Form of Complex Numbers

For the following exercises, find the absolute value of each complex number.

28. $-2 + 6i$

29. $4 - 3i$

Show Solution

5

Write the complex number in polar form.

30. $5+9i$

31. $\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Show Solution

$\text{cis}\left(-\frac{\pi}{3}\right)$

For the following exercises, convert the complex number from polar to rectangular form.

32. $z=5\text{cis}\left(\frac{5\pi}{6}\right)$

33. $z=3\text{cis}\left(40^\circ\right)$

Show Solution

$2.3+1.9i$

For the following exercises, find the product z_1z_2 in polar form.

34. $z_1=2\text{cis}\left(89^\circ\right)$

35. $z_2=5\text{cis}\left(23^\circ\right)$

36. $z_1=10\text{cis}\left(\frac{\pi}{6}\right)$

37. $z_2=6\text{cis}\left(\frac{\pi}{3}\right)$

Show Solution
$60\text{cis}\left(\frac{\pi}{2}\right)$

For the following exercises, find the quotient $\frac{z_1}{z_2}$ in polar form.

38. $z_1 = 12\text{cis}\left(55°\right)$

39. $z_2 = 3\text{cis}\left(18°\right)$

40. $z_1 = 27\text{cis}\left(\frac{5\pi}{3}\right)$

41. $z_2 = 9\text{cis}\left(\frac{\pi}{3}\right)$

Show Solution
$3\text{cis}\left(\frac{4\pi}{3}\right)$

For the following exercises, find the powers of each complex number in polar form.

42. Find z^4 when $z = 2\text{cis}\left(70°\right)$

43.Find z^2 when $z = 5\text{cis}\left(\frac{3\pi}{4}\right)$

Show Solution
$25\text{cis}\left(\frac{3\pi}{2}\right)$

For the following exercises, evaluate each root.

44. Evaluate the cube root of z when $z = 64\text{cis}\left(210°\right)$.

45. Evaluate the square root of z when $z = 25\text{cis}\left(\frac{3\pi}{2}\right)$.

Show Solution

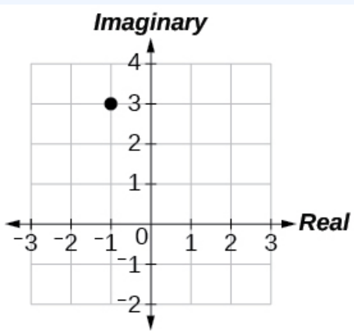
$5\text{cis}\left(\frac{3\pi}{4}\right), 5\text{cis}\left(\frac{7\pi}{4}\right)$

For the following exercises, plot the complex number in the complex plane.

46. $6 - 2i$

47. $-1 + 3i$

Show Solution



Parametric Equations

For the following exercises, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

48. $\begin{cases} x(t) = 3t - 1 \\ y(t) = \sqrt{t} \end{cases}$

49. $\begin{cases} x(t) = -\cos t \\ y(t) = 2\sin^2 t \end{cases}$

Show Solution

$x^2 + \frac{1}{2}y = 1$

50. Parameterize (write a parametric equation for) each Cartesian equation by using $x\left(t\right) = a\cos t$ and $y\left(t\right) = b\sin t$ for $\frac{x^2}{25} + \frac{y^2}{16} = 1.$

51. Parameterize the line from $\left(-2, 3\right)$ to $\left(4, 7\right)$ so that the line is at $\left(-2, 3\right)$ at $t = 0$ and $\left(4, 7\right)$ at $t = 1.$

Show Solution

$$\begin{cases} x(t) = -2 + 6t \\ y(t) = 3 + 4t \end{cases}$$

Parametric Equations: Graphs

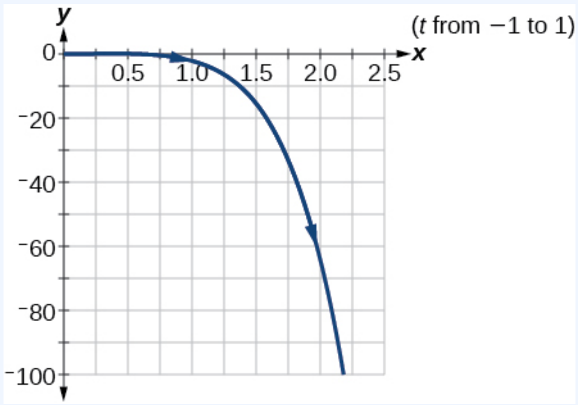
For the following exercises, make a table of values for each set of parametric equations, graph the equations, and include an orientation; then write the Cartesian equation.

52. $\begin{cases} x(t) = 3t^2 \\ y(t) = 2t - 1 \end{cases}$

53. $\begin{cases} x(t) = e^t \\ y(t) = -2e^{5\,t} \end{cases}$

Show Solution

$y = -2x^5$



54. $\begin{cases} x(t) = 3\cos t \\ y(t) = 2\sin t \end{cases}$

55. A ball is launched with an initial velocity of 80 feet per second at an angle of 40° to the horizontal. The ball is released at a height of 4 feet above the ground.

a. Find the parametric equations to model the path of the ball.
b. Where is the ball after 3 seconds?
c. How long is the ball in the air?

Show Solution

a. $\begin{cases} x(t) = (80\cos(40°))t \\ y(t) = -16t^2 + (80\sin(40°))t + 4 \end{cases}$
b. The ball is 14 feet high and 184 feet from where it was launched.
c. 3.3 seconds

Vectors

For the following exercises, determine whether the two vectors, u and v, are equal, where u has an initial point P_1 and a terminal point P_2, and v has an initial point P_3 and a terminal point P_4.

56. $P_1 = (-1, 4)\,, P_2 = (3, 1)\,, P_3 = (5, 5)$ and $P_4 = (9, 2)$

57. $P_1 = (6, 11)$, $P_2 = (-2, 8)$, $P_3 = (0, -1)$ and $P_4 = (-8, 2)$

Show Solution
not equal

For the following exercises, use the vectors $u = 2i - j$,$v = 4i - 3j$, and $w = -2i + 5j$ to evaluate the expression.

58. ***u*** – ***v***

59. 2***v*** – ***u*** + ***w***

Show Solution
4***i***

For the following exercises, find a unit vector in the same direction as the given vector.

60.***a*** = 8***i*** – 6***j***

61. ***b*** = −3***i*** – ***j***

Show Solution
$-\frac{3\sqrt{10}}{10}\boldsymbol{i} - \frac{\sqrt{10}}{10}\boldsymbol{j}$

For the following exercises, find the magnitude and direction of the vector.

62. $6, -2$

63. $-3, -3$

Show Solution

Magnitude: $3\sqrt{2}$, Direction: 225

For the following exercises, calculate $u \cdot v$.

64. ***u*** = −2***i*** + ***j*** and ***v*** = 3***i*** + 7***j***

65.***u*** = ***i*** + 4***j*** and ***v*** = 4***i*** + 3***j***

Show Solution

16

66. Given ***v*** $= -3, 4$ draw ***v***, 2***v***, and $\frac{1}{2}$ ***v***.

67. Given the vectors shown in (Figure), sketch ***u*** + ***v***, ***u*** − ***v*** and 3***v***.

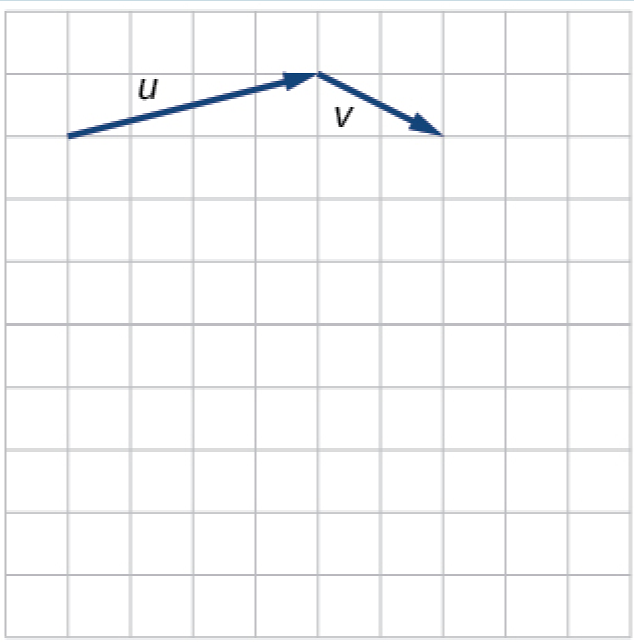
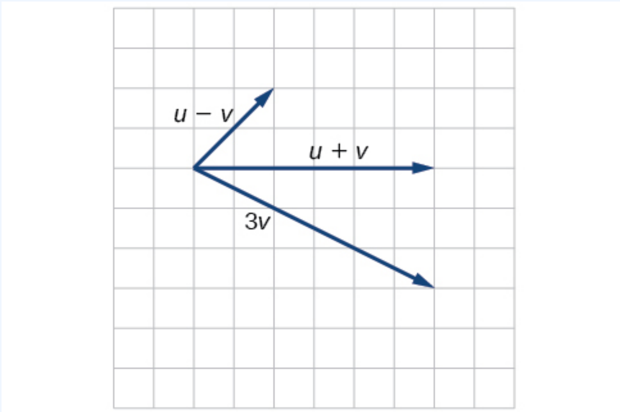


Figure 23.

Show Solution



68. Given initial point $P_1 = (3, 2)$ and terminal point $P_2 = (-5, -1)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

CHAPTER 4.10: VECTOR ADDITION AND SUBTRACTION: ANALYTICAL METHODS

Summary

- Define the rules of vector addition and subtraction using analytical methods.
- Apply analytical methods to determine vertical and horizontal component vectors.
- Apply analytical methods to determine the magnitude and direction of a resultant vector.

Analytical methods of vector addition and subtraction employ geometry and simple trigonometry rather than the ruler and protractor of graphical methods. Part of the graphical technique is retained, because vectors are still represented by arrows for easy visualization. However, analytical methods are more concise, accurate, and precise than graphical methods, which are limited by the accuracy with which a drawing can be made. Analytical methods are limited only by the accuracy and precision with which physical quantities are known.

You will be using trigonometry in this section.

Figure 11. Trig Tour

Here is a very nice PHET simulation to help review those concepts.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://ecampusontario.pressbooks.pub/sccmathtechmath1/?p=1106#iframe-phet-1 https://phet.colorado.edu/en/simulation/trig-tour

Resolving a Vector into Perpendicular Components

Analytical techniques and right triangles go hand-in-hand in physics because (among other things) motions along perpendicular directions are independent. We very often need to separate a vector into perpendicular components. For example, given a vector like $\vec{\text{A}}$ in Figure 1, we may wish to find which two perpendicular vectors, A_x and A_y , add to produce it. Notice these perpendicular component vectors will not have vector arrows in this text.

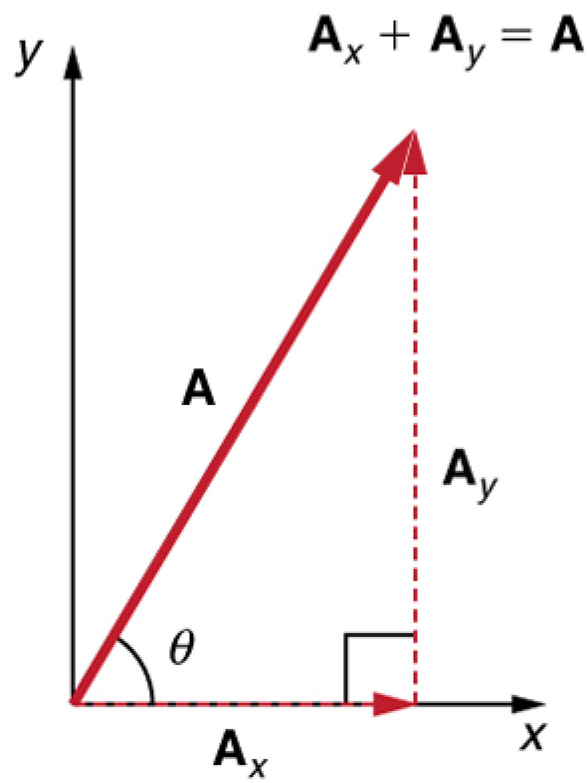


Figure 1. The vector **A**, with its tail at the origin of an x, y-coordinate system, is shown together with its x- and y-components, $\mathbf{A_x}$ and $\mathbf{A_y}$. These vectors form a right triangle. The analytical relationships among these vectors are summarized below.

A_x and A_y are defined to be the components of $\vec{\text{A}}$ along the x- and y-axes. The three vectors $\vec{\text{A}}$, A_x , and $$\text{A}_y$$ form a right triangle:

$$\mathbf{A_x + A_y = \vec{A}.}$$

Note that this relationship between vector components and the resultant vector holds only for vector quantities (which include both magnitude and direction). The relationship does not apply for the magnitudes alone. For example, if $\mathrm{A}_x = 3\ \mathrm{m}$ east, $\mathrm{A}_y = 4\ \mathrm{m}$ north, and $\vec{\mathrm{A}} = 5\ \mathrm{m}$ north-east, then it is true that the vectors $\mathrm{A}_x + \mathrm{A}_y = \vec{\mathrm{A}}$. However, it is *not* true that the sum of the magnitudes of the vectors is also equal. That is,

$$\mathbf{3\ m + 4\ m \neq 5\ m}$$

Thus,

$$\mathbf{A}_{\boldsymbol{x}} + \mathbf{A}_{\boldsymbol{y}} \neq \vec{\mathbf{A}}$$

If the vector $\vec{\mathrm{A}}$ is known, then its magnitude ***A*** and its angle ***θ*** (its direction) are known. To find A_x and A_y , its x- and y-components, we use the following relationships for a right triangle.

$$\mathbf{A}_{\boldsymbol{x}} = \boldsymbol{A}\cos\ \boldsymbol{\theta}$$

and

$$\mathbf{A}_{\boldsymbol{y}} = \boldsymbol{A}\sin\ \boldsymbol{\theta}.$$

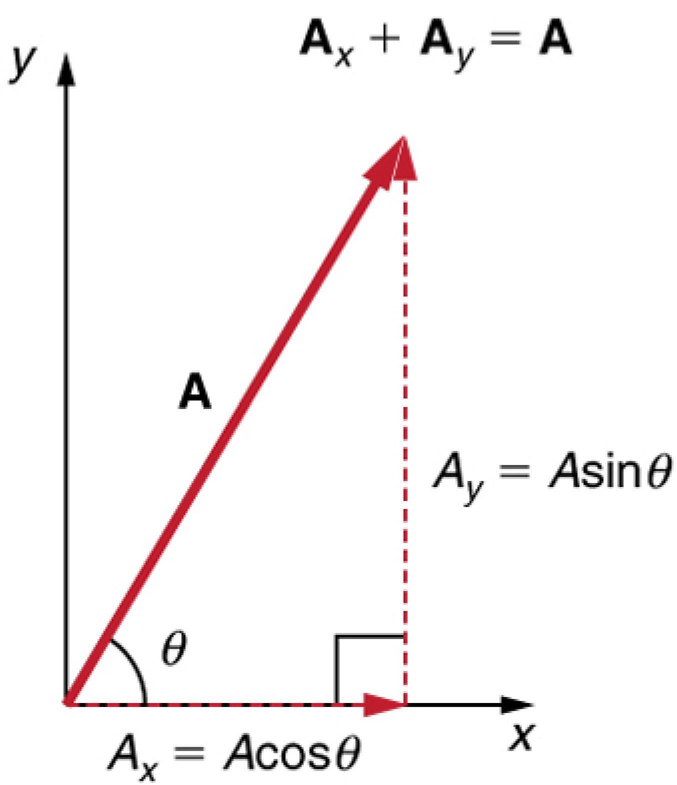


Figure 2. The magnitudes of the vector components $\mathbf{A_x}$ and $\mathbf{A_y}$ can be related to the resultant vector **A** and the angle **θ** with trigonometric identities. Here we see that ***A*x=*A cos θ*** and ***A*y=*A sinθ***.

Suppose, for example, that $\vec{\mathrm{A}}$ is the vector representing the total displacement of the person walking in a city considered in Chapter 3.1 Kinematics in Two Dimensions: An Introduction and Chapter 3.2 Vector Addition and Subtraction: Graphical Methods.

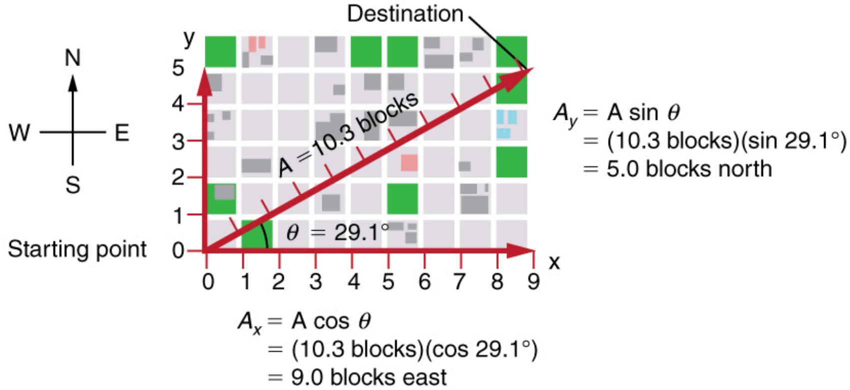


Figure 3. We can use the relationships $A_x = A \cos\theta$ and $A_y = A \sin\theta$ to determine the magnitude of the horizontal and vertical component vectors in this example.

Then $A = 10.3$ blocks and $\theta = 29.1°$, so that

$$\mathbf{A_x = A\ \cos\theta = (10.3\ blocks)(\cos 29.1^o) = 9.0\ blocks}$$
$$\mathbf{A_y = A\ \sin\theta = (10.3\ blocks)(\sin 29.1^o) = 5.0\ blocks.}$$

Calculating a Resultant Vector

If the perpendicular components A_x and A_y of a vector $\vec{\mathrm{A}}$ are known, then $\vec{\mathrm{A}}$ can also be found analytically. To find the magnitude A and direction θ of a vector from its perpendicular components A_x and A_y, we use the following relationships:

$$\boldsymbol{A} = \sqrt{\mathbf{A_x}^2 + \mathbf{A_y}^2}$$
$$\boldsymbol{\theta} = \mathbf{\tan^{-1}(A_y / A_x).}$$

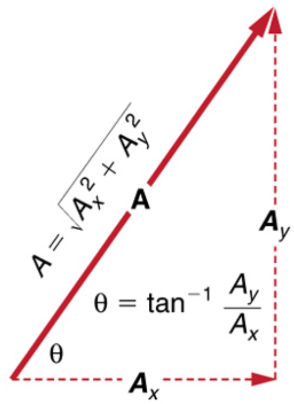


Figure 4. The magnitude and direction of the resultant vector can be determined once the horizontal and vertical components $\boldsymbol{A_x}$ and $\boldsymbol{A_y}$ have been determined.

Note that the equation $A = \sqrt{\mathrm{A}_x{}^2 + \mathrm{A}_y{}^2}$ is just the Pythagorean theorem relating the legs of a right triangle to the length of the hypotenuse. For example, if A_x and A_y are 9 and 5 blocks, respectively, then $A = \sqrt{9^2 + 5^2} = 10.3$ blocks, again consistent with the example of the person walking in a city. Finally, the direction is $\theta = \tan^{-1}(5/9) = 29.1^0$, as before.

DETERMINING VECTORS AND VECTOR COMPONENTS WITH ANALYTICAL METHODS

Equations $\mathbf{A_x} = \boldsymbol{A}\cos\boldsymbol{\theta}$ and $\mathbf{A_y} = \boldsymbol{A}\sin\boldsymbol{\theta}$ are used to find the perpendicular components of a vector—that is, to go from ***A*** and ***θ*** to $\mathbf{A_x}$ and $\mathbf{A_y}$. Equations

$\boldsymbol{A = \sqrt{A_x{}^2 \; + \; A_y{}^2}}$ and $\boldsymbol{\theta = \tan^{-1} (A_y \; / \; A_x)}$ are used to find a vector from its perpendicular components—that is, to go from $\mathbf{A_x}$ and $\mathbf{A_y}$ to ***A*** and ***θ***. Both processes are crucial to analytical methods of vector addition and subtraction.

Adding Vectors Using Analytical Methods

To see how to add vectors using perpendicular components, consider Figure 5, in which the vectors $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ are added to produce the resultant $\vec{\mathrm{R}}$.

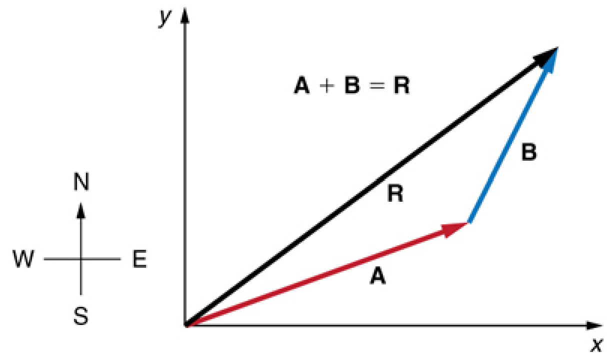


Figure 5. Vectors **A** and **B** are two legs of a walk, and **R** is the resultant or total displacement. You can use analytical methods to determine the magnitude and direction of **R**.

If $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ represent two legs of a walk (two displacements), then $\vec{\mathrm{R}}$ is the total displacement. The person taking the walk ends up at the tip of $\vec{\mathrm{R}}$. There are many ways to arrive at the same point. In particular, the person could have walked first in the *x*-direction and then in the *y*-direction. Those paths are the *x*– and *y*-components of the resultant, R_x and R_y . If we know R_x and R_y , we can find ***R*** and ***θ*** using the equations $\boldsymbol{A = \sqrt{A_x{}^2 \; + \; A_y{}^2}}$ and $\boldsymbol{\theta = \tan^{-1}} (\mathrm{A}_{\boldsymbol{y}} \; / \; \mathrm{A}_{\boldsymbol{x}})$. When you use the analytical method of vector addition, you can determine the components or the magnitude and direction of a vector.

Step 1. *Identify the x- and y-axes that will be used in the problem. Then, find the components of each vector to be added along the chosen perpendicular axes.* Use the equations $\mathrm{A}_x = A\cos\theta$ and $\mathrm{A}_y = A\sin\theta$ to find the components. In Figure 6, these components are A_x , A_y , B_x , and B_y . The angles that vectors $\vec{\mathrm{A}}[l/atex]and[latex]\vec{\mathrm{B}}$ make with the *x*-axis are $\boldsymbol{\theta_{\mathrm{A}}}$ and $\boldsymbol{\theta_{\mathrm{B}}}$, respectively.

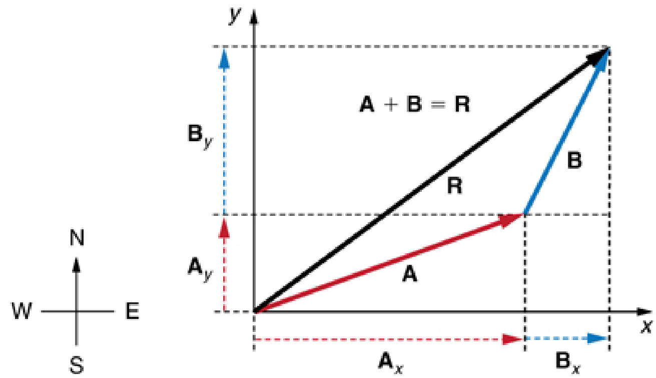


Figure 6. To add vectors **A** and **B**, first determine the horizontal and vertical components of each vector. These are the dotted vectors $\mathbf{A_x}$, $\mathbf{A_y}$, $\mathbf{B_x}$and $\mathbf{B_y}$ shown in the image.

Step 2. *Find the components of the resultant along each axis by adding the components of the individual vectors along that axis.* That is, as shown in Figure 7,

$$\mathbf{R_x} = \mathbf{A_x} + \mathbf{B_x}$$

and

$$\mathbf{R_y} = \mathbf{A_y} + \mathbf{B_y}.$$

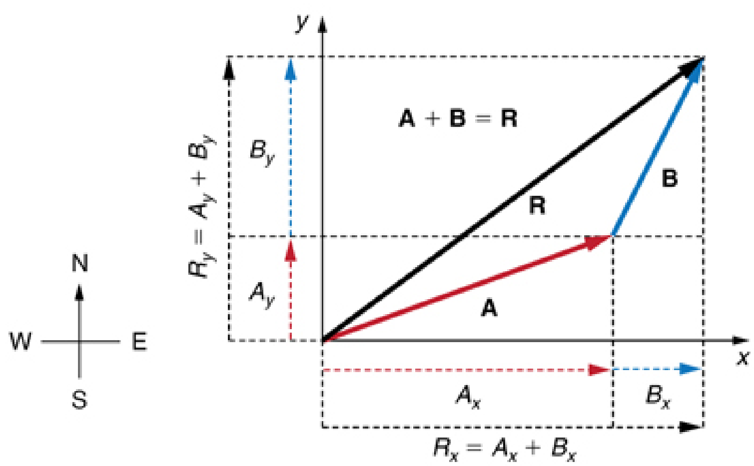


Figure 7. The magnitude of the vectors $\mathbf{A_x}$ and $\mathbf{B_x}$ add to give the magnitude $\boldsymbol{R_x}$ of the resultant vector in the horizontal direction. Similarly, the magnitudes of the vectors $\mathbf{A_y}$ and $\mathbf{B_y}$add to give the magnitude $\boldsymbol{R_y}$ of the resultant vector in the vertical direction.

Components along the same axis, say the *x*-axis, are vectors along the same line and, thus, can be added to one another like ordinary numbers. The same is true for components along the *y*-axis. (For example, a 9-block eastward walk could be taken in two legs, the first 3 blocks east and the second 6 blocks east, for a total of 9, because they are along the same direction.) So resolving vectors into components along common axes makes it easier to add them. Now that the components of $\vec{R}$ are known, its magnitude and direction can be found.

Step 3. *To get the magnitude* ***R*** *of the resultant, use the Pythagorean theorem:*

$$\mathbf{R} = \sqrt{\mathbf{R_x}^2 + \mathbf{R_y}^2}.$$

Step 4. *To get the direction of the resultant:*

$$\theta = \tan^{-1}(\mathbf{R_y} / \mathbf{R_x}).$$

The following example illustrates this technique for adding vectors using perpendicular components.

Example 1: Adding Vectors Using Analytical Methods

Add the vector $\vec{\mathbf{A}}$ to the vector $\vec{\mathbf{B}}$ shown in Figure 8, using perpendicular components along the *x*– and *y*-axes. The *x*– and *y*-axes are along the east–west and north–south directions, respectively. Vector $\vec{\mathbf{A}}$ represents the first leg of a walk in which a person walks **53.0 m** in a direction **20.0°** north of east. Vector $\vec{\mathbf{B}}$ represents the second leg, a displacement of **34.0 m** in a direction **63.0°** north of east.

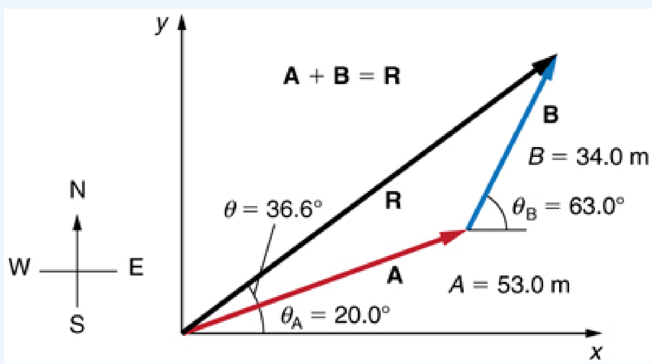


Figure 8. Vector **A** has magnitude **53.0 m** and direction **20.0^0** north of the x-axis. Vector **B** has magnitude **34.0 m** and direction **63.0^0** north of the x-axis. You can use analytical methods to determine the magnitude and direction of **R**.

Strategy

The components of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ along the *x*– and *y*-axes represent walking due east and due north to get to the same ending point. Once found, they are combined to produce the resultant.

Solution

Following the method outlined above, we first find the components of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ along the *x*– and *y*-axes. Note that ***A*=53.0 m**, **θ_A=20.0°**, ***B*=34.0 m**, and **θ_B=63.0°**. We find the *x*-components by using $\mathbf{A_x = A\cos\theta}$, which gives

$$\mathbf{A_x = A\cos\ \theta_A = (53.0\ m)(cos20.0^o)}$$
$$\mathbf{= (53.0\ m)(0.940) = 49.8\ m}$$

and

$$\mathbf{B_x = B\cos\ \theta_B = (34.0\ m)(cos63.0^o)}$$
$$\mathbf{= (34.0\ m)(0.454) = 15.4\ m.}$$

Similarly, the *y*-components are found using $\mathbf{A}_y = A\sin\theta$:

$$\mathbf{A_y = A\sin\ \theta_A = (53.0\ m)(sin20.0^o)}$$
$$\mathbf{= (53.0\ m)(0.342) = 18.1\ m}$$

and

$$\mathbf{B_y = B\sin\ \theta_B = (34.0\ m)(sin63.0^o)}$$
$$\mathbf{= (34.0\ m)(0.891) = 30.3\ m.}$$

The *x*– and *y*-components of the resultant are thus

$$\mathbf{R_x = A_x\ +\ B_x = 49.8\ m\ +\ 15.4\ m = 65.2\ m}$$

and

$$\mathbf{R_y = A_y\ +\ B_y = 18.1\ m\ +\ 30.3\ m = 48.4\ m.}$$

Now we can find the magnitude of the resultant by using the Pythagorean theorem:

$$\mathbf{R = \sqrt{R_x^2 + R_y^2} = \sqrt{(65.2)^2 + (48.4)^2\ m}}$$

so that

$$\mathbf{R = 81.2\ m.}$$

Finally, we find the direction of the resultant:

$$\mathbf{\theta = tan^{-1}(R_y\ /\ R_x) = +tan^{-1}(48.4/65.2).}$$

Thus,

$$\mathbf{\theta = tan^{-1}(0.742) = 36.6^o.}$$

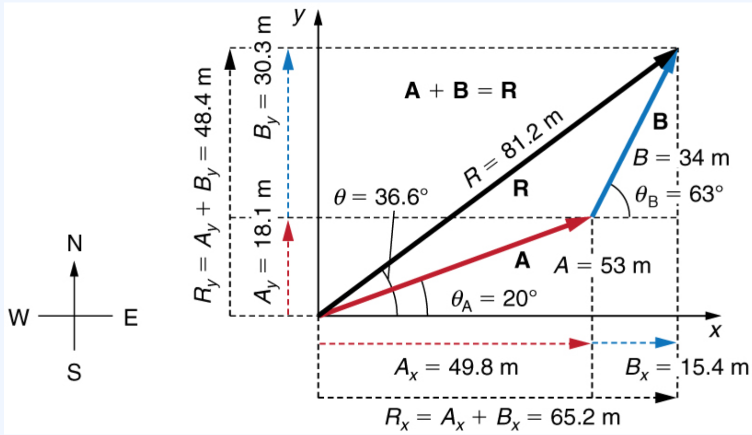


Figure 9. Using analytical methods, we see that the magnitude of **R** is **81.2 m** and its direction is **36.6^0** north of east.

Discussion

This example illustrates the addition of vectors using perpendicular components. Vector subtraction using perpendicular components is very similar—it is just the addition of a negative vector.

Subtraction of vectors is accomplished by the addition of a negative vector. That is, $\vec{\mathrm{A}} - \vec{\mathrm{B}} \equiv \vec{\mathrm{A}} + (-\vec{\mathrm{B}})$. Thus, *the method for the subtraction of vectors using perpendicular components is identical to that for addition*. The components of $\vec{\mathrm{B}}$ are the negatives of the components of $\vec{\mathrm{B}}$. The *x*– and *y*-components of the resultant $\vec{\mathrm{A}} - \vec{\mathrm{B}} = \vec{\mathrm{R}}$ are thus

$$\mathbf{R_x} = \mathbf{A_x} \ + \ (-\mathbf{B_x})$$

and

$$\mathbf{R_y} = \mathbf{A_y} \ + \ (-\mathbf{B_y})$$

and the rest of the method outlined above is identical to that for addition. (See Figure 10.)

Analyzing vectors using perpendicular components is very useful in many areas of physics, because perpendicular quantities are often independent of one another. The next module, Chapter 3.4 Projectile Motion, is one of many in which using perpendicular components helps make the picture clear and simplifies the physics.

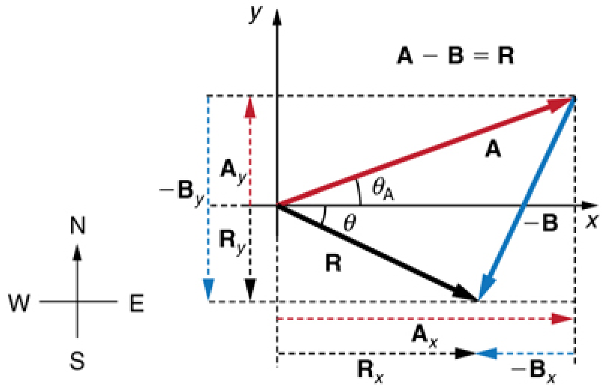


Figure 10. The subtraction of the two vectors shown in Figure 5. The components of **-B** are the negatives of the components of **B**. The method of subtraction is the same as that for addition.

PHET EXPLORATIONS: VECTOR ADDITION

Learn how to add vectors. Drag vectors onto a graph, change their length and angle, and sum them together. The magnitude, angle, and components of each vector can be displayed in several formats. Please note that this simulation uses Flash so it might not work on all machines.

Figure 11. Vector Addition

Summary

- The analytical method of vector addition and subtraction involves using the Pythagorean theorem and trigonometric identities to determine the magnitude and direction of a resultant vector.
- The steps to add vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ using the analytical method are as follows:

Step 1: Determine the coordinate system for the vectors. Then, determine the horizontal and vertical components of each vector using the equations

$$\mathbf{A_x} = \boldsymbol{A}\,\mathbf{cos}\ \boldsymbol{\theta}$$
$$\mathbf{B_x} = \boldsymbol{B}\,\mathbf{cos}\ \boldsymbol{\theta}$$

and

$$\mathbf{A_y} = \boldsymbol{A}\,\mathbf{sin}\ \boldsymbol{\theta}$$
$$\mathbf{B_y} = \boldsymbol{B}\,\mathbf{sin}\ \boldsymbol{\theta}.$$

Step 2: Add the horizontal and vertical components of each vector to determine the components $\mathbf{R}_x$ and $\mathbf{R}_y$ of the resultant vector, $\vec{\mathbf{R}}$:

$$\mathbf{R_x} = \mathbf{A_x} + \mathbf{B_x}$$

and

$$\mathbf{R_y} = \mathbf{A_y} + \mathbf{B_y}.$$

Step 3: Use the Pythagorean theorem to determine the magnitude, $\boldsymbol{R}$, of the resultant vector $\vec{\mathbf{R}}$:

$$\boldsymbol{R} = \sqrt{\mathbf{R_x}^2 + \mathbf{R_y}^2}.$$

Step 4: Use a trigonometric identity to determine the direction, θ, of $\boldsymbol{R}$:

$$\boldsymbol{\theta} = \mathbf{tan^{-1}(R_y\ /\ R_x)}.$$

Conceptual Questions

1: Suppose you add two vectors $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$. What relative direction between them produces the resultant with the greatest magnitude? What is the maximum magnitude? What relative direction between them produces the resultant with the smallest magnitude? What is the minimum magnitude?

2: Give an example of a nonzero vector that has a component of zero.

3: Explain why a vector cannot have a component greater than its own magnitude.

4: If the vectors $\vec{A}$ and $\vec{B}$ are perpendicular, what is the component of $\vec{A}$ along the direction of $\vec{B}$? What is the component of $\vec{B}$ along the direction of $\vec{A}$?

Problems & Exercises

1: Find the following for path C in Figure 12: (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

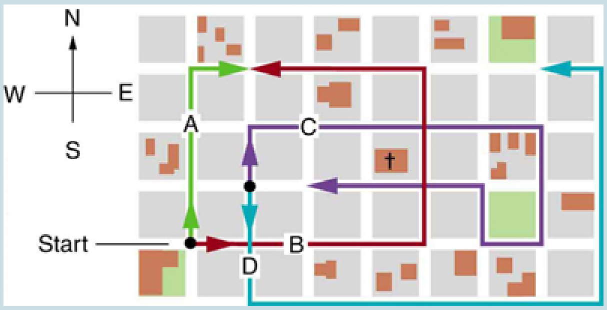


Figure 12. The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.

2: Find the following for path D in Figure 12: (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

3: Find the north and east components of the displacement from San Francisco to Sacramento shown in Figure 13.

Figure 13.

4: Solve the following problem using analytical techniques: Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$, as in Figure 14, then this problem asks you to find their sum $\vec{\mathrm{R}} = \vec{\mathrm{A}} + \vec{\mathrm{B}}$.)

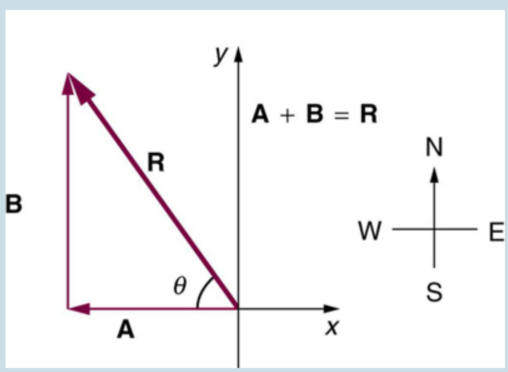


Figure 14. The two displacements **A** and **B** add to give a total displacement **R** having magnitude ***R*** and direction ***θ***.

Note that you can also solve this graphically. Discuss why the analytical technique for solving this problem is potentially more accurate than the graphical technique.

5: Repeat Exercise 4 using analytical techniques, but reverse the order of the two legs of the walk and show that you get the same final result. (This problem shows that adding them in reverse order gives the same result—that is, $\vec{\mathrm{B}}+\vec{\mathrm{A}}=\vec{\mathrm{A}}+\vec{\mathrm{B}}$.) Discuss how taking another path to reach the same point might help to overcome an obstacle blocking your other path.

6: You drive 7.50 km in a straight line in a direction 15° east of north. (a) Find the distances you would have to drive straight east and then straight north to arrive at the same point. (This determination is equivalent to find the components of the displacement along the east and north directions.) (b) Show that you still arrive at the same point if the east and north legs are reversed in order.

7: Do Exercise 4 again using analytical techniques and change the second leg of the walk to 25.0 m straight south. (This is equivalent to subtracting $\vec{\mathrm{B}}$ from $\vec{\mathrm{A}}$ —that is, finding $\vec{\mathrm{R}}'=\vec{\mathrm{A}}-\vec{\mathrm{B}}$) (b) Repeat again, but now you first walk 25.0 m north and then 18.0 m east. (This is equivalent to subtract $\vec{\mathrm{A}}$ from $\vec{\mathrm{B}}$ —that is, to find $\vec{\mathrm{A}}=\vec{\mathrm{B}}+\vec{\mathrm{C}}$. Is that consistent with your result?)

Glossary

analytical method

the method of determining the magnitude and direction of a resultant vector using the Pythagorean theorem and trigonometric identities

Solutions

Problems & Exercises

1: (a) 13 x 120 m = 1560 m = 1.56 km (b) 120 m east

2: (a) 13 x 120 m = 1560 m = 1.56 km (b) magnitude = 646 m at 21.8 ^{o} North of East

3: North-component 87.0 km , east-component 87 km

4: 30.8 m , 35.8 degrees west of north

5: 30.8 m , 35.8 degrees west of north

7: (a) $\mathbf{30.8\ m, 54.2^{o}}$ south of west (b) $\mathbf{30.8\ m, 54.2^{o}}$ north of east

CHAPTER 5: FACTORING AND QUADRATICS

CHAPTER 5.1: GREATEST COMMON FACTOR

The opposite of multiplying polynomials together is factoring polynomials. Factored polynomials help to solve equations, learn behaviours of graphs, work with fractions and more. Because so many concepts in algebra depend on us being able to factor polynomials, it is important to have very strong factoring skills.

In this section, the focus is on factoring using the greatest common factor or GCF of a polynomial. When you previously multiplied polynomials, you multiplied monomials by polynomials by distributing, solving problems such as $4x^2(2x^2 - 3x + 8)$ to yield $8x^4 - 12x^3 + 32x$. For factoring, you will work the same problem backwards. For instance, you could start with the polynomial $8x^2 - 12x^3 + 32x$ and work backwards to $4x(2x - 3x^2 + 8)$.

To do this, first identify the GCF of a polynomial. Look at finding the GCF of several numbers. To find the GCF of several numbers, look for the largest number that each of the numbers can be divided by.

Example 1

Find the GCF of 15, 24, 27.

First, break all these numbers into their primes.

$$\begin{array}{rcl} 15 & = & 3 \times 5 \\ 24 & = & 2 \times 2 \times 2 \times 3 \text{ or } 2^3 \times 3 \\ 27 & = & 3 \times 3 \times 3 \text{ or } 3^3 \end{array}$$

By observation, the only number that each can be divided by is 3. Therefore, the GCF = 3.

Example 2

Find the GCF of $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$.

First, break all these numbers into their primes. (Use • to designate multiplication instead of ×.)

$$\begin{array}{lcl} 24x^4y^2z & = & 2^3 \cdot 3 \cdot x^4 \cdot y^2 \cdot z \\ 18x^2y^4 & = & 2 \cdot 3^2 \cdot x^2 \cdot y^4 \\ 12x^3yz^5 & = & 2^2 \cdot 3 \cdot x^3 \cdot y \cdot z^5 \end{array}$$

By observation, what is shared between all three monomials is $2 \cdot 3 \cdot x^2 \cdot y$ or $6x^2y$.

Questions

Factor out the common factor in each of the following polynomials.

1. $9 + 8b^2$
2. $x - 5$
3. $45x^2 - 25$
4. $1 + 2n^2$
5. $56 - 35p$
6. $50x - 80y$
7. $7ab - 35a^2b$
8. $27x^2y^5 - 72x^3y^2$
9. $-3a^2b + 6a^3b^2$
10. $8x^3y^2 + 4x^3$
11. $-5x^2 - 5x^3 - 15x^4$
12. $-32n^9 + 32n^6 + 40n^5$
13. $28m^4 + 40m^3 + 8$
14. $-10x^4 + 20x^2 + 12x$
15. $30b^9 + 5ab - 15a^2$
16. $27y^7 + 12y^2x + 9y^2$
17. $-48a^2b^2 - 56a^3b - 56a^5b$
18. $30m^6 + 15mn^2 - 25$
19. $20x^8y^2z^2 + 15x^5y^2z + 35x^3y^3z$
20. $3p + 12q - 15q^2r^2$
21. $-18n^5 + 3n^3 - 21n + 3$
22. $30a^8 + 6a^5 + 27a^3 + 21a^2$

23. $-40x^{11} - 20x^{12} + 50x^{13} - 50x^{14}$
24. $-24x^6 - 4x^4 + 12x^3 + 4x^2$
25. $-32mn^8 + 4m^6n + 12mn^4 + 16mn$
26. $-10y^7 + 6y^{10} - 4y^{10}x - 8y^8x$

Answers to odd questions

1. $9 + 8b^2$
3. $5(9x^2 - 5)$
5. $7(8 - 5p)$
7. $7ab(1 - 5a)$
9. $3a^2b(-1 + 2ab)$
11. $5x^2(-1 - x - 3x^2) \text{ or } -5x^2(1 + x + 3x^2)$
13. $4(7m^4 + 10m^3 + 2)$
15. $5(6b^9 + ab - 3a^2)$
17. $-8a^2b(6b + 7a + 7a^3)$
19. $5x^3y^2z(4x^5z + 3x^2 + 7y)$
21. $3(-6n^5 + n^3 - 7n + 1)$
23. $-10x^{11}(4 + 2x - 5x^2 + 5x^3)$
25. $4mn(-8n^7 + m^5 + 3n^3 + 4)$

CHAPTER 5.2: FACTORING BY GROUPING

First thing to do when factoring is to factor out the GCF. This GCF is often a monomial, like in the problem $5xy + 10xz$ where the GCF is the monomial $5x$, so you would have $5x(y + 2z)$. However, a GCF does not have to be a monomial; it could be a binomial. Consider the following two examples.

Example 1

Find and factor out the GCF for $3ax - 7bx$.

By observation, one can see that both have x in common.

This means that $3ax - 7bx = x(3a - 7b)$.

Example 2

Find and factor out the GCF for $3a(2a + 5b) - 7b(2a + 5b)$.

Both have $(2a + 5b)$ as a common factor.

This means that if you factor out $(2a + 5b)$, you are left with $3a - 7b$.

The factored polynomial is written as $(2a + 5b)(3a - 7b)$.

In the same way as factoring out a GCF from a binomial, there is a process known as grouping to factor out common binomials from a polynomial containing four terms.

Find and factor out the GCF for $10ab + 15b^2 + 4a + 6b$.

To do this, first split the polynomial into two binomials.

$10ab + 15b^2 + 4a + 6b$ becomes $10ab + 15b^2$ and $4a + 6b$.

Now find the common factor from each binomial.

$10ab + 15b^2$ has a common factor of $5b$ and becomes $5b(2a + 3b)$.

$4a + 6b$ has a common factor of 2 and becomes $2(2a + 3b)$.

This means that $10ab + 15b^2 + 4a + 6b = 5b(2a + 3b) + 2(2a + 3b)$.

$5b(2a + 3b) + 2(2a + 3b)$ can be factored as $(2a + 3b)(5b + 2)$.

Questions

Factor the following polynomials.

1. $40r^3 - 8r^2 - 25r + 5$
2. $35x^3 - 10x^2 - 56x + 16$
3. $3n^3 - 2n^2 - 9n + 6$
4. $14v^3 + 10v^2 - 7v - 5$
5. $15b^3 + 21b^2 - 35b - 49$
6. $6x^3 - 48x^2 + 5x - 40$
7. $35x^3 - 28x^2 - 20x + 16$
8. $7n^3 + 21n^2 - 5n - 15$
9. $7xy - 49x + 5y - 35$
10. $42r^3 - 49r^2 + 18r - 21$
11. $16xy - 56x + 2y - 7$
12. $3mn - 8m + 15n - 40$
13. $2xy - 8x^2 + 7y^3 - 28y^2x$
14. $5mn + 2m - 25n - 10$
15. $40xy + 35x - 8y^2 - 7y$
16. $8xy + 56x - y - 7$
17. $10xy + 30 + 25x + 12y$
18. $24xy + 25y^2 - 20x - 30y^3$
19. $3uv + 14u - 6u^2 - 7v$

20. $56ab + 14 - 49a - 16b$

Answers to odd questions

1. $8r^2(5r-1) - 5(5r-1)$
$(5r-1)(8r^2-5)$
3. $n^2(3n-2) - 3(3n-2)$
$(3n-2)(n^2-3)$
5. $3b^2(5b+7) - 7(5b+7)$
$(5b+7)(3b^2-7)$
7. $7x^2(5x-4) - 4(5x-4)$
$(5x-4)(7x^2-4)$
9. $7x(y-7) + 5(y-7)$
$(y-7)(7x+5)$
11. $8x(2y-7) + 1(2y-7)$
$(2y-7)(8x+1)$
13. $2x(y-4x) + 7y^2(y-4x)$
$(y-4x)(2x+7y^2)$
15. $5x(8y+7) - y(8y+7)$
$(8y+7)(5x-y)$
17. $12y + 10xy + 30 + 25x$
$2y(6+5x) + 5(6+5x)$
$(6+5x)(2y+5)$
19. $-6u^2 + 3uv + 14u - 7v$
$-3u(2u-v) + 7(2u-v)$
$(2u-v)(-3u+7)$

CHAPTER 5.3: FACTORING TRINOMIALS WHERE A = 1

Factoring expressions with three terms, or trinomials, is a very important type of factoring to master, since this kind of expression is often a quadratic and occurs often in real life applications. The strategy to master these is to turn the trinomial into the four-term polynomial problem type solved in the previous section. The tool used to do this is central to the Master Product Method. To better understand this, consider the following example.

Example 1

Factor the trinomial $x^2 + 2x - 24$.

Start by multiplying the coefficients from the first and the last terms. This is $1 \cdot -24$, which yields -24.

The next task is to find all possible integers that multiply to -24 and their sums.

multiply to -24	sum of these integers
$-1 \cdot 24$	23
$-2 \cdot 12$	10
$-3 \cdot 8$	5
$-4 \cdot 6$	2
$-6 \cdot 4$	-2
$-8 \cdot 3$	-5
$-12 \cdot 2$	-10
$-24 \cdot 1$	-23

Look for the pair of integers that multiplies to -24 and adds to 2, so that it matches the equation that you started with.

For this example, the pair is $-4 \cdot 6$, which adds to 2.

Now take the original trinomial $x^2 + 2x - 24$ and break the $2x$ into $-4x$ and $6x$.

Rewrite the original trinomial as $x^2 - 4x + 6x - 24$.

Now, split this into two binomials as done in the previous section and factor.

$$x^2 - 4x \text{ yields } x(x-4) \text{ and } 6x - 24 \text{ yields } 6(x-4).$$

$$x^2 - 4x + 6x - 24 \text{ becomes } x(x-4) + 6(x-4).$$

$$x(x-4) + 6(x-4) \text{ factors to } (x-4)(x+6).$$

$$x^2 + 2x - 24 = (x-4)(x+6)$$

Example 2

Factor the trinomial $x^2 + 9x + 18$.

Start by multiplying the coefficients from the first and the last terms. This is $1 \cdot 18$, which yields 18.

The next task is to find all possible integers that multiply to 18 and their sums.

multiply to 18	sum of these integers
$1 \cdot 18$	19
$2 \cdot 9$	11
$3 \cdot 6$	9
$6 \cdot 3$	9
$9 \cdot 2$	11
$18 \cdot 1$	19

Look for the pair of integers that multiplies to 18 and adds to 9, so that it matches the equation that you started with.

For this example, the pair is $3 \cdot 6$, which adds to 9.

Now take the original trinomial $x^2 + 9x + 18$ and break the $9x$ into $3x$ and $6x$.

Rewrite the original trinomial as $x^2 + 3x + 6x + 18$.

Now, split this into two binomials as done in the previous section and factor.

$$x^2 + 3x \text{ yields } x(x+3) \text{ and } 6x + 18 \text{ yields } 6(x+3).$$

$$x^2 + 3x + 6x + 18 \text{ becomes } x(x+3) + 6(x+3).$$

$$x(x+3) + 6(x+3) \text{ factors to } (x+3)(x+6).$$

$$x^2 + 9x + 18 = (x + 3)(x + 6)$$

Please note the following is also true:

multiply to 18	sum of these integers
$-1 \cdot -18$	-19
$-2 \cdot -9$	-11
$-3 \cdot -6$	-9
$-6 \cdot -3$	-9
$-9 \cdot -2$	-11
$-18 \cdot -1$	-19

This means that solutions can be found where the middle term is $19x$, $11x$, $9x$, $-19x$, $-11x$ or $-9x$.

Questions

Factor each of the following trinomials.

1. $p^2 + 17p + 72$
2. $x^2 + x - 72$
3. $n^2 - 9n + 8$
4. $x^2 + x - 30$
5. $x^2 - 9x - 10$
6. $x^2 + 13x + 40$
7. $b^2 + 12b + 32$
8. $b^2 - 17b + 70$
9. $u^2 - 8uv + 15v^2$
10. $m^2 - 3mn - 40n^2$
11. $m^2 + 2mn - 8n^2$
12. $x^2 + 10xy + 16y^2$
13. $x^2 - 11xy + 18y^2$
14. $u^2 - 9uv + 14v^2$
15. $x^2 + xy - 12y^2$
16. $x^2 + 14xy + 45y^2$

Answer to odd questions

1. $9 \times 8 = 72$

$9 + 8 = 17$
$p^2 + 9p + 8p + 72$
$p(p+9) + 8(p+9)$
$(p+9)(p+8)$

3. $-8 \times -1 = 8$

$-8 + -1 = -9$
$n^2 - n - 8n + 8$
$n(n-1) - 8(n-1)$
$(n-1)(n-8)$

5. $-10 \times 1 = -10$

$-10 + 1 = -9$
$x^2 - 10x + x - 10$
$x(x-10) + 1(x-10)$
$(x-10)(x+1)$

7. $4 \times 8 = 32$

$4 + 8 = 12$
$b^2 + 4b + 8b + 32$
$b(b+4) + 8(b+4)$
$(b+4)(b+8)$

9. $-3 \times -5 = 15$

$-3 + -5 = -8$
$u^2 - 3uv - 5uv + 15v^2$
$u(u-3v) - 5v(u-3v)$
$(u-3v)(u-5v)$

11. $4 \times -2 = -8$

$4 + -2 = 2$
$m^2 + 4mn - 2mn - 8n^2$
$m(m+4n) - 2n(m+4n)$
$(m+4n)(m-2n)$

13. $-9 \times -2 = 18$

$-9 + -2 = -11$
$x^2 - 9xy - 2xy + 18y^2$
$x(x-9y) - 2y(x-9y)$
$(x-9y)(x-2y)$

15. $4 \times -3 = -12$

$4 + -3 = 1$

$$x^2 + 4xy - 3xy - 12y^2$$
$$x(x + 4y) - 3y(x + 4y)$$
$$(x + 4y)(x - 3y)$$

CHAPTER 5.4: FACTORING TRINOMIALS WHERE A ≠ 1

Factoring trinomials where the leading term is not 1 is only slightly more difficult than when the leading coefficient is 1. The method used to factor the trinomial is unchanged.

Example 1

Factor the trinomial $3x^2 + 11x + 6$.

Start by multiplying the coefficients from the first and the last terms. This is $3 \cdot 6$, which yields 18.

The next task is to find all possible integers that multiply to 18 and their sums.

multiply to 18	sum of these integers
$1 \cdot 18$	19
$2 \cdot 9$	11
$3 \cdot 6$	9
$6 \cdot 3$	9
$9 \cdot 2$	11
$18 \cdot 1$	19

Look for the pair of integers that multiplies to 18 and adds to 11, so that it matches the equation that you started with.

For this example, the pair is $2 \cdot 9$, which adds to 11.

Now take the original trinomial $3x^2 + 11x + 6$ and break the $11x$ into $2x$ and $9x$.

Rewrite the original trinomial as $3x^2 + 2x + 9x + 6$.

Now, split this into two binomials as done in the previous section and factor.

$3x^2 + 2x$ yields $x(3x + 2)$ and $9x + 6$ yields $3(3x + 2)$.

$3x^2 + 2x + 9x + 6$ becomes $x(3x + 2) + 3(3x + 2)$.

$x(3x + 2) + 3(3x + 2)$ factors to $(3x + 2)(x + 3)$.

$$3x^2 + 11x + 6 = (3x + 2)(x + 3)$$

The master product method works for any integer breakup of the polynomial. Slightly more complicated are questions that involve two different variables in the original polynomial.

Example 2

Factor the trinomial $4x^2 - xy - 5y^2$.

Start by multiplying the coefficients from the first and the last terms. This is $4 \cdot -5$, which yields −20.

The next task is to find all possible integers that multiply to −20 and their sums.

multiply to -20	sum of these integers
$-1 \cdot 20$	19
$-2 \cdot 10$	8
$-4 \cdot 5$	1
$-5 \cdot 4$	-1
$-10 \cdot 2$	-8
$-20 \cdot 1$	-19

Look for the pair of integers that multiplies to −20 and adds to −11, so that it matches the equation that you started with.

For this example, the pair is $-5 \cdot 4$, which adds to −1.

Now take the original trinomial $4x^2 - xy - 5y^2$ and break the $-xy$ into $-5xy$ and $4xy$.

Rewrite the original trinomial as $4x^2 - 5xy + 4xy - 5y^2$.

Now, split this into two binomials as done in the previous section and factor.

$4x^2 - 5xy$ yields $x(4x - 5y)$ and $4xy - 5y^2$ yields $y(4x - 5y)$.

$4x^2 - xy - 5y^2$ becomes $x(4x - 5y) + y(4x - 5y)$.

$x(4x - 5y) + y(4x - 5y)$ factors to $(x + y)(4x - 5y)$.

$$4x^2 - xy - 5y^2 = (x + y)(4x - 5y)$$

There are a number of variations potentially encountered when factoring trinomials. For instance, the original terms might be mixed up. There could be something like $-10x+3x^2+8$ that is not in descending powers of x. This requires reordering in descending powers before beginning to factor.

$$-10x+3x^2+8 \longrightarrow 3x^2-10x+8 \text{ (factorable form)}$$

It might also be necessary to factor out a common factor before starting. The polynomial above can be written as $30x^2-100x+80$, in which a common factor of 10 should be factored out prior to factoring.

$$\text{This turns } 30x^2-100x+80 \text{ into } 10(3x^2-10x+8).$$

There are also slight variations on the common factored binomial that can be illustrated by factoring the trinomial $3x^2-10x+8$.

Example 3

Factor the trinomial $3x^2-10x+8$.

Start by multiplying the coefficients from the first and the last terms. This is $3 \cdot 8$, which yields 24.

The next task is to find all possible integers that multiply to 24 and their sums (knowing that the middle coefficient must be negative).

multiply to 24	sum of these integers
$-1 \cdot -24$	-25
$-2 \cdot -12$	-14
$-3 \cdot -8$	-11
$-4 \cdot -6$	-10
$-6 \cdot -4$	-10
$-8 \cdot -3$	-11
$-12 \cdot -2$	-14
$-24 \cdot -1$	-25

Look for the pair of integers that multiplies to 24 and adds to −10, so that it matches the equation that you started with.

For this example, the pair is $-4 \cdot -6$, which adds to −10.

Now take the original trinomial $3x^2-10x+8$ and break the $-10x$ into $-4x$ and $-6x$.

Rewrite the original trinomial as $3x^2-4x-6x+8$.

Now, split this into two binomials as done in the previous section and factor.

$$3x^2 - 4x \text{ yields } x(3x-4)\text{, but } -6x+8 \text{ yields } 2(-3x+4).$$

$$x(3x-4) \text{ and } 2(-3x+4) \text{ are a close match, but their signs are different.}$$

The way to deal with this is to factor out a negative, specifically, −2 instead of 2.

$$-6x+8 \text{ can be factored two ways: } 2(-3x+4) \text{ and } -2(3x-4).$$

Choose the second factoring, so the common factor matches.

$$3x^2-10x+8 \text{ becomes } x(3x-4)+-2(3x-4).$$

$$x(3x-4)+-2(3x-4) \text{ factors to } (3x-4)(x-2).$$

$$3x^2-10x+8=(3x-4)(x-2)$$

Example 4

Factor the following trinomials, which are both variations of the trinomial seen before in 7.4.3:

1. $3x^2-14x+8$
   The pair of numbers that can be used to break it up is −2 and −12.
   $$\begin{array}{lll} 3x^2-14x+8 \text{ breaks into} & 3x^2-2x-12x+8 & \\ & x(3x-2)-4(3x-2) & \text{Common factor is } (3x-2) \\ & (3x-2)(x-4) & \text{Left over when factored} \end{array}$$
2. $3x^2-11x+8$
   The pair of numbers that can be used to break it up is −3 and −8.
   $$\begin{array}{lll} 3x^2-11x+8 \text{ breaks into} & 3x^2-3x-8x+8 & \\ & 3x(x-1)-8(x-1) & \text{Common factor is } (x-1) \\ & (x-1)(3x-8) & \text{Left over when factored} \end{array}$$

Questions

Factor each of the following trinomials.

1. $7x^2-19x-6$

2. $3n^2 - 2n - 8$
3. $7b^2 + 15b + 2$
4. $21v^2 - 11v - 2$
5. $5a^2 + 13a - 6$
6. $5n^2 - 18n - 8$
7. $2x^2 - 5x + 2$
8. $3r^2 - 4r - 4$
9. $2x^2 + 19x + 35$
10. $3x^2 + 4x - 15$
11. $2b^2 - b - 3$
12. $2k^2 + 5k - 12$
13. $3x^2 + 17xy + 10y^2$
14. $7x^2 - 2xy - 5y^2$
15. $3x^2 + 11xy - 20y^2$
16. $12u^2 + 16uv - 3v^2$
17. $4k^2 - 17k + 4$
18. $4r^2 + 3r - 7$
19. $4m^2 - 9mn - 9n^2$
20. $4x^2 - 6xy + 30y^2$
21. $4x^2 + 13xy + 3y^2$
22. $6u^2 + 5uv - 4v^2$
23. $10x^2 + 19xy - 2y^2$
24. $6x^2 - 13xy - 5y^2$

Answers to odd questions

1. $-21 \times 2 = -42$

$-21 + 2 = -19$

$7x^2 - 21x + 2x - 6$

$7x(x - 3) + 2(x - 3)$

$(x - 3)(7x + 2)$

3. $14 \times 1 = 14$

$14 + 1 = 15$

$7b^2 + 14b + b + 2$

$7b(b + 2) + 1(b + 2)$

$(b + 2)(7b + 1)$

5. $15 \times -2 = -30$

$15 + -2 = 13$

$5a^2+15a-2a-6$

$5a(a+3)-2(a+3)$

$(a+3)(5a-2)$

7. $-1\times -4=4$

$-1+-4=-5$

$2x^2-x-4x+2$

$x(2x-1)-2(2x-1)$

$(2x-1)(x-2)$

9. $14\times 5=70$

$14+5=19$

$2x^2+14x+5x+35$

$2x(x+7)+5(x+7)$

$(x+7)(2x+5)$

11. $-3\times 2=-6$

$-3+2=-1$

$2b^2-3b+2b-3$

$b(2b-3)+1(2b-3)$

$(2b-3)(b+1)$

13. $15\times 2=30$

$15+2=17$

$3x^2+15xy+2xy+10y^2$

$3x(x+5y)+2y(x+5y)$

$(x+5y)(3x+2y)$

15. $15\times -4=-60$

$15+-4=11$

$3x^2+15xy-4xy-20y^2$

$3x(x+5y)-4y(x+5y)$

$(x+5y)(3x-4y)$

17. $-16\times -1=16$

$-16+-1=-17$

$4k^2-16k-k+4$

$4k(k-4)-1(k-4)$

$(k-4)(4k-1)$

19. $-12\times 3=-36$

$-12+3=-9$

$4m^2-12mn+3mn-9n^2$

$4m(m-3n)+3n(m-3n)$
$(m-3n)(4m+3n)$

21. $12\times 1=12$

$12+1=13$
$4x^2+12xy+xy+3y^2$
$4x(x+3y)+y(x+3y)$
$(x+3y)(4x+y)$

23. $20\times -1=-20$

$20+-1=19$
$10x^2+20xy-xy-2y^2$
$10x(x+2y)-1(x+2y)$
$(x-2y)(10x-y)$

CHAPTER 5.5: FACTORING SPECIAL PRODUCTS

Now transition from multiplying special products to factoring special products. If you can recognize them, you can save a lot of time. The following is a list of these special products (note that $a^2 + b^2$ cannot be factored):

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) \\ (a+b)^2 &= a^2 + 2ab + b^2 \\ (a-b)^2 &= a^2 - 2ab + b^2 \\ a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\ a^3 + b^3 &= (a+b)(a^2 - ab + b^2) \end{aligned}$$

The challenge is therefore in recognizing the special product.

Example 1

Factor $x^2 - 36$.

This is a difference of squares. $(x-6)(x+6)$ is the solution.

Example 2

Factor $x^2 - 6x + 9$.

This is a perfect square. $(x-3)(x-3)$ or $(x-3)^2$ is the solution.

Example 3

Factor $x^2 + 6x + 9$.

This is a perfect square. $(x+3)(x+3)$ or $(x+3)^2$ is the solution.

Example 4

Factor $4x^2 + 20xy + 25y^2$.

This is a perfect square. $(2x+5y)(2x+5y)$ or $(2x+5y)^2$ is the solution.

Example 5

Factor $m^3 - 27$.

This is a difference of cubes. $(m-3)(m^2+3m+9)$ is the solution.

Example 6

Factor $125p^3 + 8r^3$.

This is a difference of cubes. $(5p + 2r)(25p^2 - 10pr + 4r^2)$ is the solution.

Questions

Factor each of the following polynomials.

1. $r^2 - 16$
2. $x^2 - 9$
3. $v^2 - 25$
4. $x^2 - 1$
5. $p^2 - 4$
6. $4v^2 - 1$
7. $3x^2 - 27$
8. $5n^2 - 20$
9. $16x^2 - 36$
10. $125x^2 + 45y^2$
11. $a^2 - 2a + 1$
12. $k^2 + 4k + 4$
13. $x^2 + 6x + 9$
14. $n^2 - 8n + 16$
15. $25p^2 - 10p + 1$
16. $x^2 + 2x + 1$
17. $25a^2 + 30ab + 9b^2$
18. $x^2 + 8xy + 16y^2$
19. $8x^2 - 24xy + 18y^2$
20. $20x^2 + 20xy + 5y^2$
21. $8 - m^3$
22. $x^3 + 64$
23. $x^3 - 64$
24. $x^3 + 8$
25. $216 - u^3$
26. $125x^3 - 216$

27. $125a^3 - 64$
28. $64x^3 - 27$
29. $64x^3 + 27y^3$
30. $32m^3 - 108n^3$

Answers to odd questions

1. $(r-4)(r+4)$
3. $(v-5)(v+5)$
5. $(p-2)(p+2)$
7. $3(x^2-9)$

$3(x-3)(x+3)$

9. $4(4x^2-9)$

$4(2x-3)(2x+3)$

11. $(a-1)^2$
13. $(x+3)^2$
15. $(5p-1)^2$
17. $(5a+3b)^2$
19. $2(4x^2-12xy+9y^2)$

$2(2x-3y)^2$

21. $(2-m)(4+2m+m^2)$
23. $(x-4)(x^2+4x+16)$
25. $(6-u)(36+6u+u^2)$
27. $(5a-4)(25a^2+20a+16)$
29. $(4x+3y)(16x^2-12xy+9y^2)$

CHAPTER 5.6: FACTORING QUADRATICS OF INCREASING DIFFICULTY

Factoring equations that are more difficult involves factoring equations and then checking the answers to see if they can be factored again.

Example 1

Factor $y^4 - 81x^4$.

This is a standard difference of squares that can be rewritten as $(y^2)^2 - (9x^2)^2$, which factors to $(y^2 - 9x^2)(y^2 + 9x^2)$. This is not completely factored yet, since $(y^2 - 9x^2)$ can be factored once more to give $(y - 3x)(y + 3x)$.

$$\text{Therefore, } y^4 - 81x^4 = (y^2 + 9x^2)(y - 3x)(y + 3x).$$

This multiple factoring of an equation is also common in mixing differences of squares with differences of cubes.

Example 2

Factor $x^6 - 64y^6$.This is a standard difference of squares that can be rewritten as $(x^3)^2 + (8x^3)^2$, which factors to $(x^3 - 8y^3)(x^3 + 8x^3)$. This is not completely factored yet, since both $(x^3 - 8y^3)$ and $(x^3 + 8x^3)$ can be factored again.

$$(x^3 - 8y^3) = (x - 2y)(x^2 + 2xy + y^2) \text{ and}$$
$$(x^3 + 8y^3) = (x + 2y)(x^2 - 2xy + y^2)$$

This means that the complete factorization for this is:

$$x^6 - 64y^6 = (x-2y)(x^2+2xy+y^2)(x+2y)(x^2-2xy+y^2)$$

Example 3

A more challenging equation to factor looks like $x^6 + 64y^6$. This is not an equation that can be put in the factorable form of a difference of squares. However, it can be put in the form of a sum of cubes.

$$x^6 + 64y^6 = (x^2)^3 + (4y^2)^3$$

In this form, $(x^2)^3 + (4y^2)^3$ factors to $(x^2+4y^2)(x^4+4x^2y^2+64y^4)$.

Therefore, $x^6 + 64y^6 = (x^2+4y^2)(x^4+4x^2y^2+64y^4)$.

Example 4

Consider encountering a sum and difference of squares question. These can be factored as follows: $(a+b)^2 - (2a-3b)^2$ factors as a standard difference of squares as shown below:

$$(a+b)^2 - (2a-3b)^2 = [(a+b)-(2a-3b)][(a+b)+(2a-3b)]$$

Simplifying inside the brackets yields:

$$[a+b-2a+3b][a+b+2a-3b]$$

Which reduces to:

$$[-a+4b][3a-2b]$$

Therefore:

$$(a+b)^2-(2a-3b)^2=[-a-4b][3a-2b]$$

Examples 5

Consider encountering the following difference of cubes question. This can be factored as follows:

$(a+b)^3-(2a-3b)^3$ factors as a standard difference of squares as shown below:

$$(a+b)^3-(2a-3b)^3$$
$$=[(a+b)-(2a+3b)][(a+b)^2+(a+b)(2a+3b)+(2a+3b)^2]$$

Simplifying inside the brackets yields:

$$[a+b-2a-3b][a^2+2ab+b^2+2a^2+5ab+3b^2+4a^2+12ab+9b^2]$$

Sorting and combining all similar terms yields:

$$\begin{array}{rrr} & [\quad a+\quad b] & [\ a^2+\ \ 2ab+\quad b^2] \\ & [-2a-3b] & [2a^2+\ \ 5ab+\ \ 3b^2] \\ + & & [4a^2+12ab+\ \ 9b^2] \\ \hline & [-a-2b] & [7a^2+19ab+13b^2] \end{array}$$

Therefore, the result is:

$$(a+b)^3-(2a-3b)^3=[-a-2b][7a^2+19ab+13b^2]$$

Questions

Completely factor the following equations.

1. x^4-16y^4
2. $16x^4-81y^4$
3. x^4-256y^4
4. $625x^4-81y^4$
5. $81x^4-16y^4$

6. $x^4 - 81y^4$
7. $625x^4 - 256y^4$
8. $x^4 - 81y^4$
9. $x^6 - y^6$
10. $x^6 + y^6$
11. $x^6 - 64y^6$
12. $64x^6 + y^6$
13. $729x^6 - y^6$
14. $729x^6 + y^6$
15. $729x^6 + 64y^6$
16. $64x^6 - 15625y^6$
17. $(a+b)^2 - (c-d)^2$
18. $(a+2b)^2 - (3a-4b)^2$
19. $(a+3b)^2 - (2c-d)^2$
20. $(3a+b)^2 - (a-b)^2$
21. $(a+b)^3 - (c-d)^3$
22. $(a+3b)^3 + (4a-b)^3$

Answers to odd questions

1. $(x^2 - 4y^2)(x^2 + 4y^2)$

$(x-2y)(x+2y)(x^2+4y^2)$

3. $(x^2 - 16y^2)(x^2 + 16y^2)$

$(x-4y)(x+4y)(x^2+16y^2)$

5. $(9x^2 - 4y^2)(9x^2 + 4y^2)$

$(3x+2y)(3x-2y)(9x^2+4y^2)$

7. $(25x^2 - 16y^2)(25x^2 + 16y^2)$

$(5x-4y)(5x+4y)(25x^2+16y^2)$

9. $(x^3 - y^3)(x^3 + y^3)$

$(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)$

11. $(x^3 - 8y^3)(x^3 + 8y^3)$

$(x-2y)(x^2+2xy+4y^2)(x+2y)(x^2-2xy+4y^2)$

13. $(27x^3 - y^3)(27x^3 + y^3)$

$(3x-y)(9x^2+3xy+y^2)(3x+y)(9x^2-3xy+y^2)$

15. $(9x^2)^3 + (4y^2)^3$

$(9x^2+4y^2)(81x^4-36x^2y^2+16y^4)$

17. $[(a+b)-(c-d)][(a+b)+(c-d)]$

$[a+b-c+d][a+b+c-d]$

19. $[(a+3b)-(2c-d)][(a+3b)+(2c-d)]$
$[a+3b-2c+d][a+3b+2c-d]$

21. $[(a+b)-(c-d)][(a+b)^2+(a+b)(c-d)+(c-d)^2]$
$[a+b-c+d][a^2+2ab+b^2+ac-ad+bc-bd+c^2-2cd+d^2]$

CHAPTER 5.7: CHOOSING THE CORRECT FACTORING STRATEGY

With so many different tools used to factor, it is prudent to have a section to determine the best strategy to factor.

Factoring Hints

1. Look for any factor to simplify the polynomial before you start!
2. If you have two terms, look for a sum or difference of squares or cubes.

$$a^2 - b^2 = (a+b)(a-b)a^3 - b^3 = (a-b)(a^2+ab+b^2)a^3 + b^3 = (a+b)(a^2 - ab + b2)$$

1. If you have three terms, see if the master product method works.
2. If you have four terms, see if factoring by grouping works.

Questions

Factor each completely.

1. $24ac - 18ab + 60dc - 45db$
2. $2x^2 - 11x + 15$
3. $5u^2 - 9uv + 4v^2$
4. $16x^2 + 48xy + 36y^2$
5. $-2x^3 + 128y^3$
6. $20uv - 60u^3 - 5xv + 15xu^2$
7. $54u^3 - 16$
8. $54 - 128x^3$
9. $n^2 - n$
10. $5x^2 - 22x - 15$
11. $x^2 - 4xy + 3y^2$

12. $45u^2 - 150uv + 125v^2$
13. $m^2 - 4n^2$
14. $12ab - 18a + 6nb - 9n$
15. $36b^2c - 16ad - 24b^2d + 24ac$
16. $3m^3 - 6m^2n - 24n^2m$
17. $128 + 54x^3$
18. $64m^3 + 27n^3$
19. $n^3 + 7n^2 + 10n$
20. $64m^3 - n^3$
21. $27x^3 - 64$
22. $16a^2 - 9b^2$
23. $5x^2 + 2x$
24. $2x^2 - 10x + 12$

Answers to odd questions

1. $6a(4c-3b) + 15d(4c-3b)$

$(4c-3b)(6a+15d)$

$3(4c-3b)(2a+5d)$

3. $-5 \times -4 = 20$

$-5 + -4 = -9$

$5u^2 - 5uv - 4uv + 4v^2$

$5u(u-v) - 4v(u-v)$

$(u-v)(5u-4v)$

5. $-2(x^3 - 64y^3)$

$-2(x-4y)(x^2+4xy+16y^2)$

7. $2(27u^3 - 8)$

$2(3u-2)(9u^2+6u+4)$

9. $n(n-1)$

11. $x^2 - 3xy - xy + 3y^2$

$x(x-3y) - y(x-3y)$

$(x-3y)(x-y)$

13. $(m-2n)(m+2n)$

15. $36b^2c - 24b^2d + 24ac - 16ad$

$12b^2(3c-2d) + 8a(3c-2d)$

$(3c-2d)(12b^2+8a)$

$4(3c-2d)(3b^2+2a)$

17. $2(64+27x^3)$

$2(4+3x)(16-12x+9x^2)$

19. $5\times 2=10$

$5+2=7$

$n(n^2+7n+10)$

$n(n^2+5n+2n+10)$

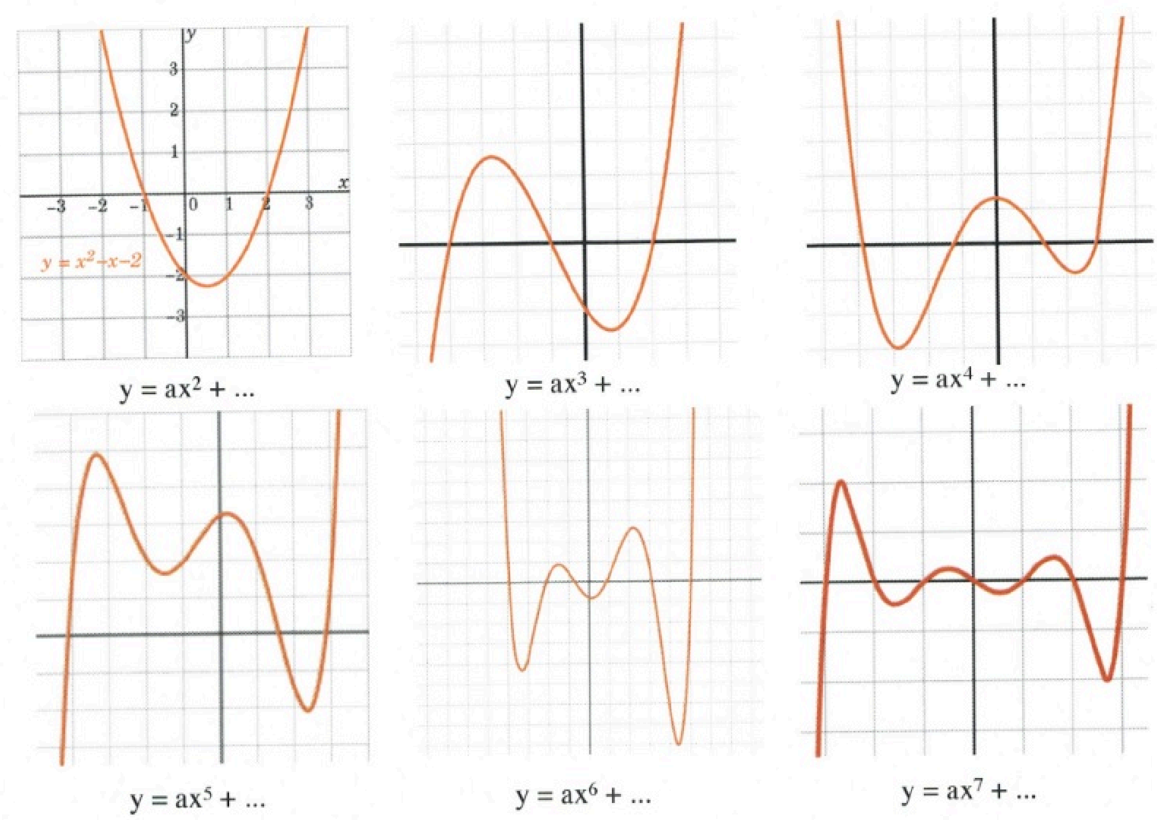
$n(n(n+5)+2(n+5))$

$n(n+5)(n+2)$

21. $(3x-4)(9x^2+12x+16)$
23. $x(5x+2)$

CHAPTER 5.8: SOLVING QUADRIATIC EQUATIONS BY FACTORING

Solving quadratics is an important algebraic tool that finds value in many disciplines. Typically, the quadratic is in the form of $y = ax^2 + bx + c$, which when graphed is a parabola. Of special importance are the x-values that are found when $y = 0$, which show up when graphed as the parabola crossing the x-axis. For a trinomial, there can be as many as three x-axis crossings. The following show the possible number of x-axis intercepts for 2nd degree (quadratic) to 7th degree (septic) functions. Please note that if the fifth degree polynomial were shifted down a few values, it would also show 5 x-axis intercepts. It is these x-axis intercepts that are of interest.



The approach to finding these x-intercepts is elementary: set $y = 0$ in the original equation and factor it. Once the equation is factored, then find the x-values that solve for 0. This is shown in the next few examples.

Example 1

Solve the following quadratic equation: $(2x-3)(5x+1)=0$.

In this problem there are two separate binomials: $(2x-3)$ and $(5x+1)$. Since their product is equal to 0, there will be two solutions: the value for x that makes $2x-3=0$ and the value for x that makes $5x+1=0$.

These are:

$$\begin{array}{rrrrrcrrrrr} 2x & - & 3 & = & 0 & \text{and} & 5x & + & 1 & = & 0 \\ & + & 3 & & +3 & & & - & 1 & & -1 \\ & & 2x & = & 3 & & & & 5x & = & -1 \\ \\ & & x & = & \dfrac{3}{2} & & & & x & = & -\dfrac{1}{5} \end{array}$$

Example 2

Solve the following polynomial equation: $(x-3)(x+3)(x-1)(x+1)=0$.

For this polynomial, there are four different solutions:

$$(x-3)=0, (x+3)=0, (x-1)=0, (x+1)=0$$

Solving for these four x-values gives us:

$$\begin{array}{rrrrrrrrrrrrrrrrrrrr} x & - & 3 & = & 0 & x & + & 3 & = & 0 & x & - & 1 & = & 0 & x & + & 1 & = & 0 \\ & + & 3 & & +3 & & - & 3 & & -3 & & + & 1 & & +1 & & - & 1 & & -1 \\ & & x & = & 3 & & & x & = & -3 & & & x & = & 1 & & & x & = & -1 \end{array}$$

The solutions are: $x=\pm 3, \pm 1$.

It would be nice if there were only given factored equations to solve, but that is not how it goes. You are generally required to factor the equation first before it can be solved.

Example 3

Solve the following quadratic equation: $4x^2 + x - 3 = 0$.

First, factor $4x^2 + x - 3$ and get $(4x - 3)(x + 1) = 0$.

Now, solve for $4x - 3 = 0$ and $x + 1 = 0$.

Solving these two binomials yields:

$$\begin{array}{rrrrr} 4x & - & 3 & = & 0 \\ & + & 3 & & +3 \\ & & 4x & = & 3 \\ \\ & & x & = & \dfrac{3}{4} \end{array} \qquad \begin{array}{rrrrr} x & + & 1 & = & 0 \\ & - & 1 & & -1 \\ & & x & = & -1 \\ \\ & & x & = & -1 \end{array}$$

Questions

Solve each of the following polynomials by using factoring.

1. $(k - 7)(k + 2) = 0$
2. $(a + 4)(a - 3) = 0$
3. $(x - 1)(x + 4) = 0$
4. $(2x + 5)(x - 7) = 0$
5. $6x^2 - 150 = 0$
6. $p^2 + 4p - 32 = 0$
7. $2n^2 + 10n - 28 = 0$
8. $m^2 - m - 30 = 0$
9. $7x^2 + 26x + 15 = 0$
10. $2b^2 - 3b - 2 = 0$
11. $x^2 - 4x - 8 = -8$
12. $v^2 - 8v - 3 = -3$
13. $x^2 - 5x - 1 = -5$
14. $a^2 - 6a + 6 = -2$

15. $7x^2 + 17x - 20 = -8$
16. $4n^2 - 13n + 8 = 5$
17. $x^2 - 6x = 16$
18. $7n^2 - 28n = 0$
19. $4k^2 + 22k + 23 = 6k + 7$
20. $a^2 + 7a - 9 = -3 + 6a$
21. $9x^2 - 46 + 7x = 7x + 8x^2 + 3$
22. $x^2 + 10x + 30 = 6$
23. $40p^2 + 183p - 168 = p + 5p^2$
24. $24x^2 + 11x - 80 = 3x$

Answers to odd questions

1. $$\begin{array}{rrrrr} k & - & 7 & = & 0 \\ & + & 7 & & +7 \\ & & k & = & 7 \end{array} \qquad \begin{array}{rrrrr} k & + & 2 & = & 0 \\ & - & 2 & & -2 \\ & & k & = & -2 \end{array}$$

3. $$\begin{array}{rrrrr} x & - & 1 & = & 0 \\ & + & 1 & & +1 \\ & & x & = & 1 \end{array} \qquad \begin{array}{rrrrr} x & + & 4 & = & 0 \\ & - & 4 & & -4 \\ & & x & = & -4 \end{array}$$

5. $$\begin{array}{rcr} 6(x^2 - 25) & = & 0 \\ 6(x-5)(x+5) & = & 0 \\ & & \\ x & = & 5 \\ x & = & -5 \end{array}$$

7. $$\begin{array}{rcr} 2(n^2 + 5n - 14) & = & 0 \\ 2(n+7)(n-2) & = & 0 \\ & & \\ n & = & -7 \\ n & = & 2 \end{array}$$

9. $$\begin{array}{rcr} (x+3)(7x+5) & = & 0 \\ & & \\ x & = & -3 \\ x & = & -\dfrac{5}{7} \end{array}$$

11. $$\begin{array}{rrrrrrr} x^2 & - & 4x & - & 8 & = & -8 \\ & & & + & 8 & & +8 \\ & & x^2 & - & 4x & = & 0 \\ & & x(x & - & 4) & = & 0 \\ & & & & & & \\ & & & & x & = & 0 \\ & & & & x & = & 4 \end{array}$$

13.
$$\begin{array}{rrrrrrrr} & x^2 & - & 5x & - & 1 & = & -5 \\ & & & & + & 5 & & +5 \\ & x^2 & - & 5x & + & 4 & = & 0 \\ (x & - & 4) & (x & - & 1) & = & 0 \\ \\ & & & & & x & = & 4 \\ & & & & & x & = & 1 \end{array}$$

15.
$$\begin{array}{rrrrrrrr} & 7x^2 & + & 17x & - & 20 & = & -8 \\ & & & & + & 8 & & +8 \\ & 7x^2 & + & 17x & - & 12 & = & 0 \\ (7x & - & 4) & (x & + & 3) & = & 0 \\ \\ & & & & & x & = & \frac{4}{7} \\ \\ & & & & & x & = & -3 \end{array}$$

17.
$$\begin{array}{rrrrrrrr} & x^2 & - & 6x & & & = & 16 \\ & & & & - & 16 & & -16 \\ & x^2 & - & 6x & - & 16 & = & 0 \\ (x & - & 8) & (x & + & 2) & = & 0 \\ \\ & & & & & x & = & 8 \\ & & & & & x & = & -2 \end{array}$$

19.
$$\begin{array}{rrrrrrrrrr} & 4k^2 & + & 22k & + & 23 & = & 6k & + & 7 \\ & & - & 6k & - & 7 & & -6k & - & 7 \\ & 4k^2 & + & 16k & + & 16 & = & 0 & & \\ & 4(k^2 & + & 4k & + & 4) & = & 0 & & \\ 4(k & + & 2) & (k & + & 2) & = & 0 & & \\ \\ & & & & & k & = & -2 & & \end{array}$$

21.
$$\begin{array}{rrrrrrrrrrrr} & 9x^2 & - & 46 & + & 7x & = & 7x & + & 8x^2 & + & 3 \\ & -8x^2 & - & 3 & - & 7x & & -7x & - & 8x^2 & - & 3 \\ & & & x^2 & - & 49 & = & 0 & & & & \\ (x & - & 7) & (x & + & 7) & = & 0 & & & & \\ \\ & & & & & x & = & 7 & & & & \\ & & & & & x & = & -7 & & & & \end{array}$$

23.

$$\begin{array}{rrrrrrrrr} & 40p^2 & + & 183p & - & 168 & = & p & + & 5p^2 \\ & -5p^2 & - & p & & & & -p & - & 5p^2 \\ & 35p^2 & + & 182p & - & 168 & = & 0 & & \\ & 7(5p^2 & + & 26p & - & 24) & = & 0 & & \\ 7(p & + & 6) & (5p & - & 4) & = & 0 & & \\ & & & & & p & = & -6 & & \\ & & & & & p & = & \dfrac{4}{5} & & \end{array}$$

CHAPTER 5.9: AGE WORD PROBLEMS

One application of linear equations is what are termed age problems. When solving age problems, generally the age of two different people (or objects) both now and in the future (or past) are compared. The objective of these problems is usually to find each subject's current age. Since there can be a lot of information in these problems, a chart can be used to help organize and solve. An example of such a table is below.

Person or Object	Current Age	Age Change

Example 1

Joey is 20 years younger than Becky. In two years, Becky will be twice as old as Joey. Fill in the age problem chart, but do not solve.

- The first sentence tells us that Joey is 20 years younger than Becky (this is the current age)
- The second sentence tells us two things:

1. The age change for both Joey and Becky is plus two years
2. In two years, Becky will be twice the age of Joey in two years

Person or Object	Current Age	Age Change (+2)
Joey (J)	B – 20	B – 20 + 2 B – 18
Becky (B)	B	B = 2

Using this last statement gives us the equation to solve:

$$B + 2 = 2(B - 18)$$

Example 2

Carmen is 12 years older than David. Five years ago, the sum of their ages was 28. How old are they now?

- The first sentence tells us that Carmen is 12 years older than David (this is the current age)
- The second sentence tells us the age change for both Carmen and David is five years ago (−5)

Filling in the chart gives us:

Person or Object	Current Age	Age Change (−5)
Carmen (C)	D + 12	D + 12 – 5 D + 7
David (D)	D	D – 5

The last statement gives us the equation to solve:

$$\text{Five years ago, the sum of their ages was 28}$$

$$\begin{array}{rrrrrrrrl} (D & + & 7) & + & (D & - & 5) & = & 28 \\ & & & & 2D & + & 2 & = & 28 \\ & & & & & - & 2 & & -2 \\ & & & & & & 2D & = & 26 \\ & & & & & & D & = & \dfrac{26}{2} = 13 \end{array}$$

Therefore, Carmen is David's age (13) + 12 years = 25 years old.

Example 3

The sum of the ages of Nicole and Kristin is 32. In two years, Nicole will be three times as old as Kristin. How old are they now?

- The first sentence tells us that the sum of the ages of Nicole (N) and Kristin (K) is 32. So N + K = 32, which means that N = 32 − K or
  K = 32 − N (we will use these equations to eliminate one variable in our final equation)
- The second sentence tells us that the age change for both Nicole and Kristen is in two years (+2)

Filling in the chart gives us:

Person or Object	Current Age	Age Change (+2)
Nicole (N)	N	N + 2
Kristin (K)	32 – N	(32 – N) + 2 34 – N

The last statement gives us the equation to solve:

In two years, Nicole will be three times as old as Kristin

$$\begin{array}{rrrcrrr} N & + & 2 & = & 3(34 & - & N) \\ N & + & 2 & = & 102 & - & 3N \\ +3N & - & 2 & & -2 & + & 3N \\ & & 4N & = & 100 & & \\ & & & & & & \\ & & N & = & \dfrac{100}{4} & = & 25 \end{array}$$

If Nicole is 25 years old, then Kristin is 32 − 25 = 7 years old.

Example 4

Louise is 26 years old. Her daughter Carmen is 4 years old. In how many years will Louise be double her daughter's age?

- The first sentence tells us that Louise is 26 years old and her daughter is 4 years old
- The second line tells us that the age change for both Carmen and Louise is to be calculated (x)

Filling in the chart gives us:

Person or Object	Current Age	Age Change
Louise (L)	26	$26 = x$
Daughter (D)	4	$D = x$

The last statement gives us the equation to solve:

In how many years will Louise be double her daughter's age?

$$\begin{array}{rcrcrcr} 26 & + & x & = & 2(4 & + & x) \\ 26 & + & x & = & 8 & + & 2x \\ -26 & - & 2x & & -26 & - & 2x \\ & & -x & = & -18 & & \\ & & x & = & 18 & & \end{array}$$

In 18 years, Louise will be twice the age of her daughter.

Questions

For Questions 1 to 8, write the equation(s) that define the relationship.

1. Rick is 10 years older than his brother Jeff. In 4 years, Rick will be twice as old as Jeff.
2. A father is 4 times as old as his son. In 20 years, the father will be twice as old as his son.
3. Pat is 20 years older than his son James. In two years, Pat will be twice as old as James.
4. Diane is 23 years older than her daughter Amy. In 6 years, Diane will be twice as old as Amy.
5. Fred is 4 years older than Barney. Five years ago, the sum of their ages was 48.
6. John is four times as old as Martha. Five years ago, the sum of their ages was 50.
7. Tim is 5 years older than JoAnn. Six years from now, the sum of their ages will be 79.
8. Jack is twice as old as Lacy. In three years, the sum of their ages will be 54.

Solve Questions 9 to 20.

9. The sum of the ages of John and Mary is 32. Four years ago, John was twice as old as Mary.
10. The sum of the ages of a father and son is 56. Four years ago, the father was 3 times as old as the son.
11. The sum of the ages of a wood plaque and a bronze plaque is 20 years. Four years ago, the bronze plaque was one-half the age of the wood plaque.
12. A man is 36 years old and his daughter is 3. In how many years will the man be 4 times as old as his daughter?
13. Bob's age is twice that of Barry's. Five years ago, Bob was three times older than Barry. Find the age of both.
14. A pitcher is 30 years old, and a vase is 22 years old. How many years ago was the pitcher twice as old as the vase?
15. Marge is twice as old as Consuelo. The sum of their ages seven years ago was 13. How old are they now?
16. The sum of Jason and Mandy's ages is 35. Ten years ago, Jason was double Mandy's age. How old are they now?
17. A silver coin is 28 years older than a bronze coin. In 6 years, the silver coin will be twice as old as the bronze coin. Find the present age of each coin.
18. The sum of Clyde and Wendy's ages is 64. In four years, Wendy will be three times as old as Clyde. How old are they now?
19. A sofa is 12 years old and a table is 36 years old. In how many years will the table be twice as old as the sofa?
20. A father is three times as old as his son, and his daughter is 3 years younger than his son. If the sum of all three ages 3 years ago was 63 years, find the present age of the father.

Answers to odd questions.

1. $$\begin{aligned} R &= J+10 \\ R+4 &= 2(J+4) \end{aligned}$$
3. $$\begin{aligned} P &= J+20 \\ P+2 &= 2(J+2) \end{aligned}$$
5. $$\begin{aligned} F &= B+4 \\ (F-5)+(B-5) &= 48 \end{aligned}$$
7. $$\begin{aligned} T &= 5+J \\ (T+6)+(J+6) &= 79 \end{aligned}$$

9.

$$\begin{array}{rrrrrrr}
 & J & + & m & = & 32 & \\
 & & - & m & & -m & \\
 & & & J & = & 32 & - & m \\
\\
 & & J & - & 4 & = & 2(m & - & 4) \\
(32 & - & m) & - & 4 & = & 2m & - & 8 \\
32 & - & m & - & 4 & = & 2m & - & 8 \\
 & & 28 & - & m & = & 2m & - & 8 \\
 & & +8 & + & m & & +m & + & 8 \\
 & & & & \frac{36}{3} & = & \frac{3m}{3} \\
\\
 & & & & m & = & 12 \\
\\
 & & & & \therefore J & = & 32 & - & m \\
 & & & & J & = & 32 & - & 12 \\
 & & & & J & = & 20
\end{array}$$

11.

$$\begin{array}{rrrrrrrrr}
 & & w & + & B & = & 20 \\
 & & -w & & & & & - & w \\
 & & & & B & = & 20 & - & w \\
\\
 & & B & - & 4 & = & \frac{1}{2}(w & - & 4) \\
\\
20 & - & w & - & 4 & = & \frac{1}{2}(w & - & 4) \\
\\
 & & [16 & - & w & = & \frac{1}{2}(w & - & 4)] & (2)
\end{array}$$

$$\begin{array}{rrrrrrr}
32 & - & 2w & = & w & - & 4 \\
+4 & + & 2w & & +2w & + & 4 \\
 & & \frac{36}{3} & = & \frac{3w}{3} \\
\\
 & & w & = & 12 \\
\\
 & & B & = & 20 & - & w \\
 & & B & = & 20 & - & 12 \\
 & & B & = & 8
\end{array}$$

13.

$$\begin{array}{rrrrrrr}
 & & B_{\text{o}} & = & 2B_{\text{y}} \\
\\
B_{\text{o}} & - & 5 & = & 3(B_{\text{y}} & - & 5) \\
2B_{\text{y}} & - & 5 & = & 3B_{\text{y}} & - & 15 \\
-3B_{\text{y}} & + & 5 & & -3B_{\text{y}} & + & 5 \\
 & & -B_{\text{y}} & = & -10 \\
 & & B_{\text{y}} & = & 10 \\
\\
 & & \therefore B_{\text{o}} & = & 2B_{\text{y}} \\
 & & B_{\text{o}} & = & 2(10) \\
 & & B_{\text{o}} & = & 20
\end{array}$$

15.

$$\begin{array}{rrrrrrrr} & & & & & m & = & 2c \\ \\ (m & - & 7) & + & (c & - \quad 7) & = & 13 \\ & & m & + & c & - \quad 14 & = & 13 \\ & & 2c & + & c & - \quad 14 & = & 13 \\ & & & & & + \quad 14 & & +14 \\ & & & & & \dfrac{3c}{3} & = & \dfrac{27}{3} \\ \\ & & & & & c & = & 9 \\ \\ & & & & & \therefore m & = & 2c \\ & & & & & m & = & 2(9) \\ & & & & & m & = & 18 \end{array}$$

17.

$$\begin{array}{rrrrrrrrr} & & & & S & = & 28 & + & B \\ \\ & & S & + & 6 & = & 2(B & + & 6) \\ 28 & + & B & + & 6 & = & 2B & + & 12 \\ & & B & + & 34 & = & 2B & + & 12 \\ & & -B & - & 12 & = & -B & - & 12 \\ & & & & 22 & = & B & & \\ \\ & & & & S & = & 28 & + & B \\ & & & & S & = & 28 & + & 22 \\ & & & & S & = & 50 & & \end{array}$$

19.

$$\begin{array}{rrrrrrr} & & S & = & 12 & & \\ & & T & = & 36 & & \\ \\ T & + & x & = & 2(S & + & x) \\ 36 & + & x & = & 2(12 & + & x) \\ 36 & + & x & = & 24 & + & 2x \\ -24 & - & x & & -24 & - & x \\ & & x & = & 12 & & \end{array}$$

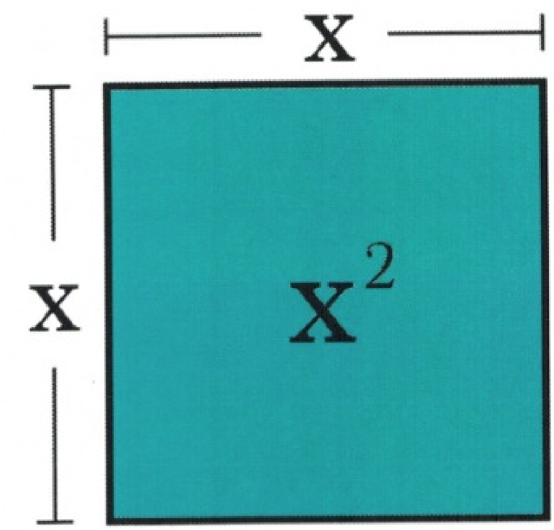
CHAPTER 5.10: COMPLETING THE SQUARE

How To "Complete the Square" Visually[1]

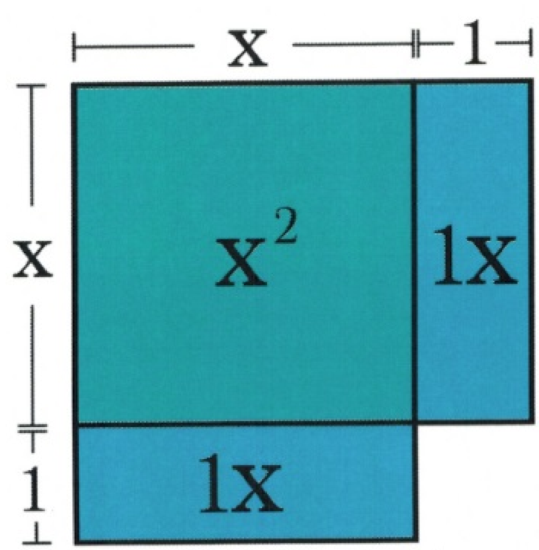
Let's use an area model to visualize how to complete the square of the following equation:

$$y = x^2 + 2x + 12$$

The area model used by Brett Berry is fairly straightforward, having multiple variations and forms that can be found online. The standard explanation begins by representing x^2 as a square whose sides are both x units in length and make an area of x^2.

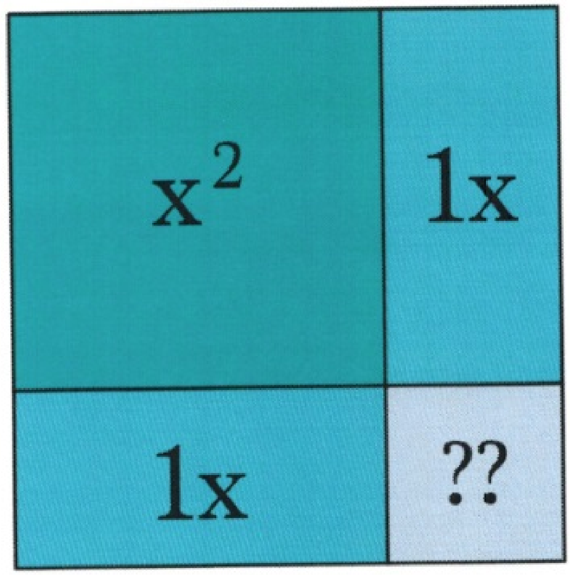


Next, add $2x$ to the block defined as x^2. This is done by taking the $2x$ block and cutting it in half, then add to both sides of your original square x. This acts to continue the sides of x in two directions by $1x$.



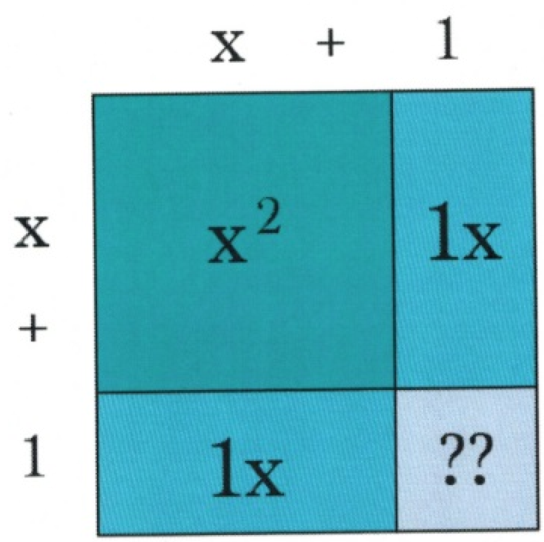
In this example, the square length on each side has increased by 1, but as you can see from the diagram, this larger square is missing the corner piece. To complete the square, add a small piece to complete the visual square. The question is, what is the area of this missing piece?

1. Adapted from Brett Berry

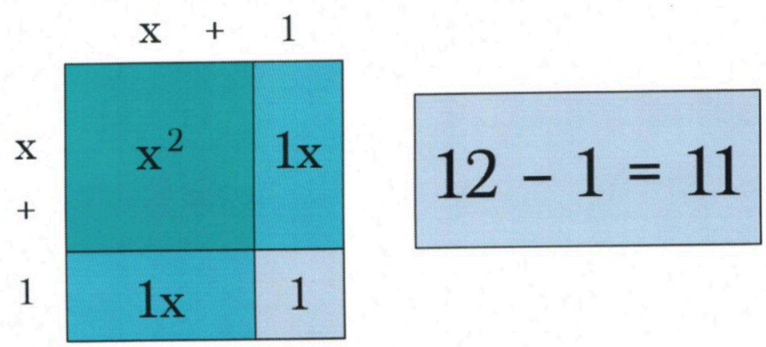


Since the blue blocks adjacent to our missing piece are both 1 unit wide, deduce that the missing block has an area of 1 × 1 = 1.

Also note that, by adding together the outermost units of the square, the area of the square becomes the desired binomial squared $(x+1)^2$.



Now, all that's left to do is literally complete the square and adjust for the extra units. To do this, first, fill in the area of the purple square, which is known to be 1. Since the original equation had a constant of 12, subtract 1 from 12 to account for the 1 added to the square.



The square is now complete! The square is $(x+1)^2$ with 11 leftover. The extra 11 can simply be added to the end of our binomial squared: $y = (x+1)^2 + 11$.

In the problems most likely be required to solve, $y = 0$, so the original equation will not be written as $y = x^2 + 2x + 12$; rather, it will be $0 = x^2 + 2x + 12$.

Example 1

Solve for x in the equation $0 = x^2 + 8x + 12$.

The first step is to complete the square. Rather than drawing out a sketch to show the process of completing the square, simply take half the middle term and rewrite $x^2 + 8x$ as $(x + 4)^2$.

When squared out $(x + 4)^2$, it is $x^2 + 8x + 16$.

Note that this is 4 larger than the original $0 = x^2 + 8x + 12$. This means that $(x + 4)^2 - 4$ is the same as $0 = x^2 + 8x + 12$.

The equation needed to be solved has now become $0 = (x + 4)^2 - 4$. First, add 4 to each side:

$$\begin{array}{rcrcrcr} 0 & = & (x & + & 4)^2 & - & 4 \\ +4 & = & & & & + & 4 \\ 4 & = & (x & + & 4)^2 & & \end{array}$$

Now take the square root from both sides:

$$\begin{array}{rcrcl} (4)^{\frac{1}{2}} & = & [(x & + & 4)^2]^{\frac{1}{2}} \\ \pm 2 & = & x & + & 4 \end{array}$$

Subtracting 4 from both sides leaves $x = -4 \pm 2$, which gives the solutions $x = -6$ and $x = -2$.

It is always wise to check answers in the original equation, which for these two yield:

$$\begin{array}{rcrcrcl} x & = & -6: & & & & \\ \\ 0 & = & x^2 & + & 8x & + & 12 \\ 0 & = & (-6)^2 & + & 8(-6) & + & 12 \\ 0 & = & 36 & - & 48 & + & 12\checkmark \end{array}$$

$$\begin{array}{rcrcrcl} x & = & -2: & & & & \\ \\ 0 & = & x^2 & + & 8x & + & 12 \\ 0 & = & (-2)^2 & + & 8(-2) & + & 12 \\ 0 & = & 4 & - & 16 & + & 12\checkmark \end{array}$$

Sometimes, it is required to complete the square where there is some value ≠ 1 in front of the x^2. For example:

Example 2

Solve for x in the equation $0 = 2x^2 + 12x - 7$.

The first step is to factor 2 from both terms in $2x^2 + 12x$, which then leaves $0 = 2(x^2 + 6x) - 7$.

Isolating $x^2 + 6x$ yields $x^2 + 6x = \dfrac{7}{2}$.

As before, complete the square for $x^2 + 6x$, which yields $(x + 3)^2$. When squared out $(x + 3)^2$, you get $x^2 + 6x + 9$.

Now add 9 to the other side of the equation:

$$x^2 + 6x + 9 = \frac{7}{2} + 9$$

Simplifying this yields:

$$(x+3)^2 = \frac{25}{2}$$

Now take the square root from both sides:

$$[(x+3)^2]^{\frac{1}{2}} = \left(\frac{25}{2}\right)^{\frac{1}{2}}$$

Which leaves:

$$x + 3 = \pm\left(\frac{25}{2}\right)^{\frac{1}{2}}$$

Subtract 3 from both sides:

$$x = -3 \pm \left(\frac{25}{2}\right)^{\frac{1}{2}}$$

Rationalizing the denominator yields:

$$x = -3 + \frac{5\sqrt{2}}{2} \text{ or } x = -3 - \frac{5\sqrt{2}}{2}$$

When checking these answers in the original equation, both solutions are valid.

Questions

Find the value that completes the square and then rewrite as a perfect square.

1. $x^2 - 30x + \_\_$
2. $a^2 - 24a + \_\_$
3. $m^2 - 36m + \_\_$
4. $x^2 - 34x + \_\_$
5. $x^2 - 15x + \_\_$

6. $r^2 - 19r + \_\_$
7. $y^2 - y + \_\_$
8. $p^2 - 17p + \_\_$

Solve each equation by completing the square.

9. $x^2 - 16x + 55 = 0$
10. $n^2 - 4n - 12 = 0$
11. $v^2 - 4v - 21 = 0$
12. $b^2 + 8b + 7 = 0$
13. $x^2 - 8x = -6$
14. $x^2 - 13 = 4x$
15. $3k^2 + 24k = -1$
16. $4a^2 + 36a = -2$

Answers to odd questions

1. $$\begin{aligned} \frac{30}{2} &= 15 \\ 15^2 &= 225 \\ \therefore x^2 &- 30x + 225 \text{ or } (x-15)^2 \end{aligned}$$
3. $$\begin{aligned} \frac{36}{2} &= 18 \\ 18^2 &= 324 \\ \therefore m^2 &- 36m + 324 \text{ or } (m-18)^2 \end{aligned}$$
5. $$\begin{aligned} \frac{15}{2} &= 7.5 \\ 7.5^2 &= 56.25 \\ \therefore x^2 &- 15x + 56.25 \text{ or } \left(x - \frac{15}{2}\right)^2 \end{aligned}$$
7. $$\begin{aligned} &\frac{1}{2} \\ \left(\frac{1}{2}\right)^2 &= \frac{1}{4} \\ \therefore y^2 &- y + \frac{1}{4} \text{ or } \left(y - \frac{1}{2}\right)^2 \end{aligned}$$

9.
$$\begin{array}{rrrrrrl} x^2 & - & 16x & + & 55 & = & 0 \\ & & & - & 55 & & -55 \\ & & x^2 & - & 16x & = & -55 \end{array}$$

$$\begin{array}{rrrrrrrrr} x^2 & - & 16x & + & 64 & = & 64 & - & 55 \\ & & (x & - & 8)^2 & = & 9 & & \end{array}$$

$$\sqrt{(x-8)^2} = \sqrt{9}$$

$$\begin{array}{rrrrrrr} x & - & 8 & = & \pm & 3 & \\ & + & 8 & & + & 8 & \\ & & x & = & 8 & \pm & 3 \\ & & x & = & 5, & 11 & \end{array}$$

11.
$$\begin{array}{rrrrrrl} v^2 & - & 4v & - & 21 & = & 0 \\ & & & + & 21 & & +21 \\ & & v^2 & - & 4v & = & 21 \end{array}$$

$$\begin{array}{rrrrrrrrr} v^2 & - & 4v & + & 4 & = & 21 & + & 4 \\ & & (v & - & 2)^2 & = & 25 & & \end{array}$$

$$\sqrt{(v-2)^2} = \sqrt{25}$$

$$\begin{array}{rrrrrrr} v & - & 2 & = & \pm & 5 & \\ & + & 2 & & + & 2 & \\ & & v & = & 2 & \pm & 5 \\ & & v & = & 7, & -3 & \end{array}$$

13.
$$\begin{array}{rrrrrrrrr} x^2 & - & 8x & + & 16 & = & -6 & + & 16 \\ & & (x & - & 4)^2 & = & 10 & & \end{array}$$

$$\sqrt{(x-4)^2} = \sqrt{10}$$

$$\begin{array}{rrrrrrr} x & - & 4 & = & \pm & \sqrt{10} & \\ & + & 4 & & + & 4 & \\ & & x & = & 4 & \pm & \sqrt{10} \end{array}$$

15.
$$\begin{array}{rrrrrrrrr} & & \frac{3}{3}(k^2 & + & 8k) & = & \frac{-1}{3} & & \\ & & k^2 & + & 8k & = & -\frac{1}{3} & & \\ k^2 & + & 8k & + & 16 & = & -\frac{1}{3} & + & 16 \\ & & (k & + & 4)^2 & = & 15\frac{2}{3} & & \end{array}$$

$$\sqrt{(k+4)^2} = \sqrt{15\frac{2}{3}}$$

$$\begin{array}{rrrrrr} k & + & 4 & = & \pm & \sqrt{\dfrac{47}{3}} \\ & - & 4 & & - & 4 \\ & & k & = & -4 & \pm \sqrt{\dfrac{47}{3}} \end{array}$$

CHAPTER 5.11: THE QUADRATIC EQUATION

A rule of thumb about factoring: after spending several minutes trying to factor an equation, if its taking to long, use the quadratic equation to generate solutions instead.

Look at the equation $ax^2 + bx + c = 0$, the values of x that make this equation equal to zero can be found by:

$$x = \frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a}$$

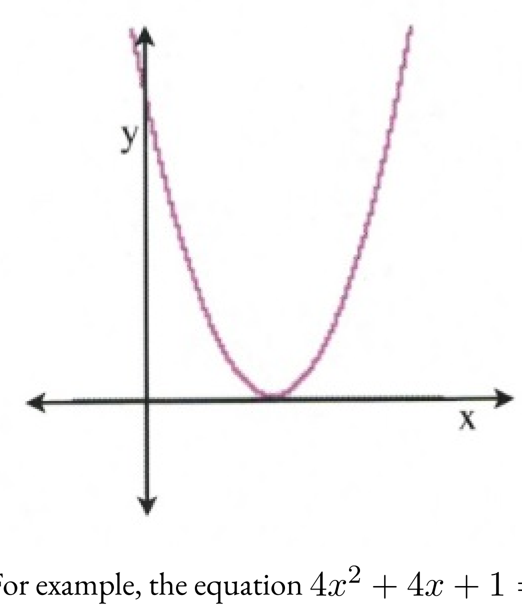
One of the key factors here is the value found from $(b^2 - 4ac)^{\frac{1}{2}}$. The interior of this radical $b^2 - 4ac$ can have three possible values: negative, positive, or zero.

$b^2 - 4ac$ is called the discriminant, and it defines how many solutions of x there will be and what type of solutions they are.

If $b^2 - 4ac = 0$, then there is exactly one solution:

$$x = \frac{-b \pm 0}{2a} = \frac{-b}{2a}$$

The meaning of this is that the parabolic curve that can be drawn from the equation will only touch the x -axis at one spot, and so there is only one solution for that quadratic. This can be seen from the image to the right: the quadratic curve touches the x-axis at only one position, which means that there is only one solution for x.



For example, the equation $4x^2 + 4x + 1 = 0$ has one solution. Check:

$$\begin{array}{llllllll} a=4 & b^2 & - & 4ac & = & (4)^2 & - & 4(4)(1) \\ b=4 & & & & = & 16 & - & 16 \\ c=1 & & & & = & 0 & & \end{array}$$

The solution ends up being $x = \dfrac{-(4)}{2(4)}$ or $x = -\dfrac{1}{2}$.

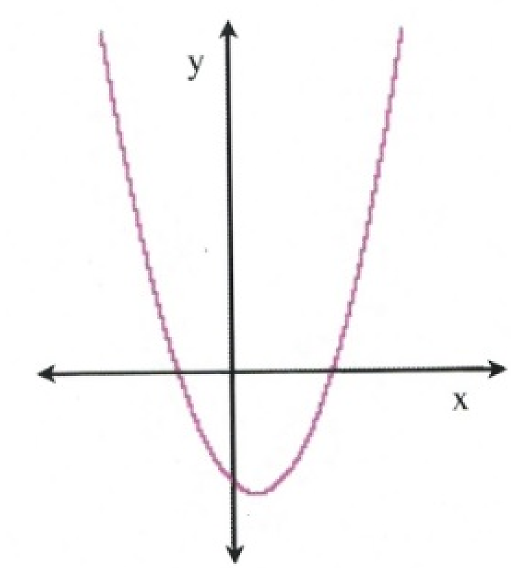
If $b^2 - 4ac =$ any positive value, then there are exactly two solutions:

$$x = \frac{-b \pm \text{some positive number}}{2a}$$

or simply

$$\frac{-b + \text{some positive number}}{2a} \quad \text{and} \quad \frac{-b - \text{ some positive number}}{2a}$$

The meaning of this is that the parabolic curve that can be drawn from the equation will now touch (and cross) the x-axis at two positions, and so there are now two solutions for the quadratic. This can be seen from the image to the right: the quadratic curve crosses the x-axis at two positions, which means that there are now two solutions for x.



For example, the equation $3x^2 + 4x + 1 = 0$ has two solutions. Check:

$$\begin{array}{llllllll} a=3 & b^2 & - & 4ac & = & (4)^2 & - & 4(3)(1) \\ b=4 & & & & = & 16 & - & 12 \\ c=1 & & & & = & 4 & & \end{array}$$

When 4 is put back into the quadratic equation and root 4 is taken, the solution now becomes ±2.

For this quadratic:

$$x = \frac{-4 \pm 2}{2(3)} = \frac{-4 \pm 2}{6}$$

The solutions are $x = \dfrac{-6}{6} = -1$ and $x = \dfrac{-2}{6} = -\dfrac{1}{3}$.

There exists one last possible solution for a quadratic, which happens when $b^2 - 4ac =$ any negative value. When this occurs, there are exactly two solutions, which are defined as imaginary roots or solutions or, more properly, complex roots, since the solution involves taking the root of a negative value.

The example provided shows that the quadratic never touches or crosses the x-axis, yet it is possible to generate a solution if using imaginary numbers when solving a negative radical discriminant $b^2 - 4ac$.

For example, the equation $5x^2 + 2x + 1 = 0$ has two complex or imaginary solutions. Check:

$$\begin{array}{llcl} a=5 & b^2 - 4ac & = & (2)^2 - 4(5)(1) \\ b=2 & & = & 4 - 20 \\ c=1 & & = & -16 \end{array}$$

When −16 is put back into the quadratic equation and the root of −16 is taken, the solution becomes $\pm 4i$.

For this quadratic:

$$x = \frac{-2 \pm 4i}{2(5)} = \frac{-2 \pm 4i}{10}$$

The solutions are $x = \dfrac{-1+2i}{5}$ and $x = \dfrac{-1-2i}{5}$.

Note: these solutions are complex conjugates of each other.

It is often useful to check the discriminants of a quadratic equation to define the nature of the roots for the quadratic before proceeding to a full solution.

Example 1

Find the values of x that solve the equation $x^2 + 6x - 7 = 0$.

$$
\begin{array}{lrcl}
a = 1 & x & = & \dfrac{-6 \pm [6^2 - 4(1)(-7)]^{\frac{1}{2}}}{2(1)} \\[2ex]
b = 6 & x & = & \dfrac{-6 \pm [36 + 28]^{\frac{1}{2}}}{2} \\[2ex]
c = -7 & x & = & \dfrac{-6 \pm [64]^{\frac{1}{2}}}{2} \\[2ex]
\text{Which reduces to} & x & = & \dfrac{-6 \pm 8}{2} \\[2ex]
\text{And yields} & x & = & -7, 1
\end{array}
$$

Example 2

Find the values of x that solve the equation $9x^2 + 6x + 1 = 0$.

$$
\begin{array}{lll}
a = 9 & x = & \dfrac{-(-6) \pm [(-6)^2 - 4(9)(1)]^{\frac{1}{2}}}{2(9)} \\
b = 6 & x = & \dfrac{6 \pm [36 - 36]^{\frac{1}{2}}}{18} \\
c = 1 & x = & \dfrac{6 \pm [0]^{\frac{1}{2}}}{18} \\
\text{Which reduces to} & x = & \dfrac{1}{3}
\end{array}
$$

In case you are curious:

How to Derive the Quadratic Formula

$$ax^2 + bx + c = 0 \quad \text{Separate constant from variables}$$

$$-c - c \quad \text{Subtract } c \text{ from both sides}$$

$$\frac{ax^2}{a} + \frac{bx}{a} = \frac{-c}{a} \quad \text{Divide each term by } a$$

$$x^2 + \frac{b}{a}x = \frac{-c}{a} \quad \text{Find the number that completes the square}$$

$$\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \quad \text{Add to both sides}$$

$$\frac{b^2}{4a^2} - \frac{c}{a}\left(\frac{4a}{4a}\right) = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2} \quad \text{Get the common denominator on the right}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2} \quad \text{Factor}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \text{Solve using the even root property}$$

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{Simplify roots}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \quad \text{Subtract } \frac{b}{2a} \text{ from both sides}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Our solution}$$

Questions

Use the quadratic discriminant to determine the nature of the roots.

a. $4x^2 + 2x - 5 = 0$
b. $9x^2 - 6x + 1 = 0$
c. $2x^2 + 3x - 5 = 0$
d. $3x^2 + 5x = 3$

e. $3x^2 + 5x = 2$
f. $x^2 - 8x + 16 = 0$
g. $a^2 - 56 = -10a$
h. $x^2 + 4 = 4x$
i. $5x^2 = -26 + 10x$
j. $n^2 = -21 + 10n$

Solve each of the following using the quadratic equation.

1. $4a^2 + 3a - 6 = 0$
2. $3k^2 + 2k - 3 = 0$
3. $2x^2 - 8x - 2 = 0$
4. $6n^2 + 8n - 1 = 0$
5. $2m^2 - 3m + 6 = 0$
6. $5p^2 + 2p + 6 = 0$
7. $3r^2 - 2r - 1 = 0$
8. $2x^2 - 2x - 15 = 0$
9. $4n^2 - 3n + 10 = 0$
10. $b^2 + 6b + 9 = 0$
11. $v^2 - 4v - 5 = -8$
12. $x^2 + 2x + 6 = 4$

Answers to odd questions

a. $2^2 - 4(4)(-5) \Rightarrow 4 + 80 = 84 \qquad \therefore 2 \text{ real solutions}$
c. $(3)^2 - 4(2)(-5) \Rightarrow 9 + 40 = 49 \qquad \therefore 2 \text{ real solutions}$
e. $3x^2 + 5x - 2 \Rightarrow (5)^2 - 4(3)(-2) \Rightarrow 25 + 24 = 49 \qquad \therefore 2 \text{ real solutions}$
g. $a^2+10a-56 \Rightarrow (10)^2-4(1)(-56) \Rightarrow 100+224 = 324 \qquad \therefore 2 \text{ real solutions}$
i. $5x^2 - 10x + 26 \Rightarrow (-10)^2 - 4(5)(26) \Rightarrow 100 - 520 = -420$
$\therefore 2 \text{ non-real solutions}$

1. $$\begin{aligned} a &= 4 \\ b &= 3 \\ c &= -6 \end{aligned}$$

$$\begin{aligned} a &= \frac{-3 \pm \sqrt{3^2 - 4(4)(-6)}}{2(4)} \\ a &= \frac{-3 \pm \sqrt{9 + 96}}{8} \\ a &= \frac{-3 \pm \sqrt{105}}{8} \end{aligned}$$

3. $$\begin{aligned} a &= 2 \\ b &= -8 \\ c &= -2 \end{aligned}$$

$$\begin{aligned} x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(-2)}}{2(2)} \\ x &= \frac{8 \pm \sqrt{64 + 16}}{4} \\ x &= \frac{8 \pm \sqrt{80}}{4} \\ x &= \frac{8 \pm 4\sqrt{5}}{4} \Rightarrow 2 \pm \sqrt{5} \end{aligned}$$

5. $$\begin{aligned} a &= 2 \\ b &= -3 \\ c &= 6 \end{aligned}$$

$$\begin{aligned} m &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(6)}}{2(2)} \\ m &= \frac{3 \pm \sqrt{9 - 48}}{4} \\ m &= \frac{3 \pm \sqrt{-39}}{4} \end{aligned}$$

A negative square root means there are 2 non-real solutions or no real solution.

7.
$$\begin{aligned} a &= 3 \\ b &= -2 \\ c &= -1 \end{aligned}$$

$$\begin{aligned} r &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-1)}}{2(3)} \\ r &= \frac{2 \pm \sqrt{4 + 12}}{6} \\ r &= \frac{2 \pm \sqrt{16}}{6} \\ r &= \frac{2 \pm 4}{6} \Rightarrow 1, -\frac{1}{3} \end{aligned}$$

9.
$$\begin{aligned} a &= 4 \\ b &= -3 \\ c &= 10 \end{aligned}$$

$$\begin{aligned} n &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(10)}}{2(4)} \\ n &= \frac{3 \pm \sqrt{9 - 160}}{8} \\ n &= \frac{3 \pm \sqrt{-151}}{8} \end{aligned}$$

$\therefore$ 2 non-real solutions

11.
$$\begin{array}{ccccccccccc} v^2 & - & 4v & - & 5 & = & -8 & & & & \\ & & & + & 8 & & +8 & & & & \\ & & & & 0 & = & v^2 & - & 4v & + & 3 \end{array}$$

$$\begin{aligned} a &= 1 \\ b &= -4 \\ c &= 3 \end{aligned}$$

$$v = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$$

$$v = \frac{4 \pm \sqrt{16 - 12}}{2}$$

$$v = \frac{4 \pm \sqrt{4}}{2}$$

$$v = \frac{4 \pm 2}{2} \Rightarrow 2 \pm 1$$

$$v = 3, 1$$

CHAPTER 5.12: SOLVING QUADRATIC EQUATIONS USING SUBSTITUTION

Factoring trinomials in which the leading term is not 1 is only slightly more difficult than when the leading coefficient is 1. The method used to factor the trinomial is unchanged.

Example 1

Solve for x in $x^4 - 13x^2 + 36 = 0$.

First start by converting this trinomial into a form that is more common. Here, it would be a lot easier when factoring $x^2 - 13x + 36 = 0$. There is a standard strategy to achieve this through substitution.

First, let $u = x^2$. Now substitute u for every x^2, the equation is transformed into $u^2 - 13u + 36 = 0$.

$u^2 - 13u + 36 = 0$ factors into $(u - 9)(u - 4) = 0$.

Once the equation is factored, replace the substitutions with the original variables, which means that, since $u = x^2$, then $(u - 9)(u - 4) = 0$ becomes $(x^2 - 9)(x^2 - 4) = 0$.

To complete the factorization and find the solutions for x, then $(x^2 - 9)(x^2 - 4) = 0$ must be factored once more. This is done using the difference of squares equation: $a^2 - b^2 = (a + b)(a - b)$.

Factoring $(x^2 - 9)(x^2 - 4) = 0$ thus leaves $(x - 3)(x + 3)(x - 2)(x + 2) = 0$.

Solving each of these terms yields the solutions $x = \pm 3, \pm 2$.

This same strategy can be followed to solve similar large-powered trinomials and binomials.

Example 2

Factor the binomial $x^6 - 7x^3 - 8 = 0$.

Here, it would be a lot easier if the expression for factoring was $x^2 - 7x - 8 = 0$.

First, let $u = x^3$, which leaves the factor of $u^2 - 7u - 8 = 0$.

$u^2 - 7u - 8 = 0$ easily factors out to $(u-8)(u+1) = 0$.

Now that the substituted values are factored out, replace the u with the original x^3. This turns $(u-8)(u+1) = 0$ into $(x^3 - 8)(x^3 + 1) = 0$.

The factored $(x^3 - 8)$ and $(x^3 + 1)$ terms can be recognized as the difference of cubes.

These are factored using $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ and $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

And so, $(x^3 - 8)$ factors out to $(x-2)(x^2 + 2x + 4)$ and $(x^3 + 1)$ factors out to $(x+1)(x^2 - x + 1)$.

Combining all of these terms yields:

$$(x-2)(x^2 + 2x + 4)(x+1)(x^2 - x + 1) = 0$$

The two real solutions are $x = 2$ and $x = -1$. Checking for any others by using the discriminant reveals that all other solutions are complex or imaginary solutions.

Questions

Factor each of the following polynomials and solve what you can.

1. $x^4 - 5x^2 + 4 = 0$
2. $y^4 - 9y^2 + 20 = 0$
3. $m^4 - 7m^2 - 8 = 0$
4. $y^4 - 29y^2 + 100 = 0$
5. $a^4 - 50a^2 + 49 = 0$
6. $b^4 - 10b^2 + 9 = 0$
7. $x^4 + 64 = 20x^2$

8. $6z^6 - z^3 = 12$
9. $z^6 - 216 = 19z^3$
10. $x^6 - 35x^3 + 216 = 0$

Answers to odd questions

1. let $u = x^2$

$\therefore u^2 - 5u + 4 = 0$
factors to $(u-4)(u-1) = 0$
replace $u : (x^2-4)(x^2-1) = 0$

$(x-2)(x+2)(x-1)(x+1) = 0$
$x = \pm 2, \pm 1$

3. $u = m^2$

$\therefore u^2 - 7u - 8 = 0$
$(u-8)(u+1) = 0$
$(m^2-8)(m^2+1) = 0$

$(m+\sqrt{8})(m-\sqrt{8})(m^2+1) = 0$
$m = \pm\sqrt{8}$ or $\pm 2\sqrt{2}$
$m^2 + 1$ has 2 non-real solutions

5. let $u = a^2$

$\therefore u^2 - 50u + 49 = 0$
$(u-49)(u-1) = 0$
$(a^2-49)(a^2-1) = 0$

$(a-7)(a+7)(a-1)(a+1) = 0$
$a = \pm 7, \pm 1$

7. $x^4 - 20x^2 + 64 = 0$

let $u = x^2$
$\therefore u^2 - 20u + 64 = 0$
$(u-16)(u-4) = 0$
$(x^2-16)(x^2-4) = 0$

$(x-4)(x+4)(x-2)(x+2) = 0$
$x = \pm 4, \pm 2$

9. $z^6 - 19z^3 - 216 = 0$

let $u = z^3$
$\therefore u^2 - 19u - 216 = 0$

$(u - 27)(u + 8) = 0$
$(z^3 - 27)(z^3 + 8) = 0$

$(z - 3)(z^2 + 3z + 9)(z + 2)(z^2 - 2z + 4) = 0$
$z = 3, -2$
2 non-real solutions each for the 2nd and 4th factors

CHAPTER 5.13: GRAPHING QUADRATIC EQUATIONS—VERTEX AND INTERCEPT METHOD

One useful strategy that is used to get a quick sketch of a quadratic equation is to identify 3 key points of the quadratic: its vertex and the two intercept points. From these 3 points, it's possible to sketch out a rough graph of what the quadratic graph looks like.

The **intercepts** are where the quadratic equation crosses the x-axis and are found when the quadratic is set to equal 0. So instead of the quadratic looking like $y = ax^2 + bx + c$, it is instead factored from the form $0 = ax^2 + bx + c$ to get its x-intercepts (roots). For expedience, you can get these values using the quadratic equation.

$$x = \frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a}$$

The **vertex** is found by using the quadratic equation where the discriminant equals zero, which gives us the x-coordinate of $x = \dfrac{-b}{2a}$. The y-coordinate of the vertex is then found by placing the x-coordinate of the vertex $\left(x = \dfrac{-b}{2a}\right)$ back into the original quadratic $(y = ax^2 + bx + c)$ and solving for y.

The vertex then takes the form of $\left[\dfrac{-b}{2a}, a\left(\dfrac{-b}{2a}\right)^2 + \left(\dfrac{-b}{2a}\right)x + c\right]$, or simply as $\left[\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right]$.

What is new here is finding the vertex, so consider the following examples.

Example 1

Find the vertex of $y = x^2 + 6x - 7$.

For this equation, $a = 1$, $b = 6$ and $c = -7$.

This means that the x-coordinate of the vertex $x = \dfrac{-b}{2a}$ will give us the value $x = \dfrac{-(6)}{2(1)} = -3$.

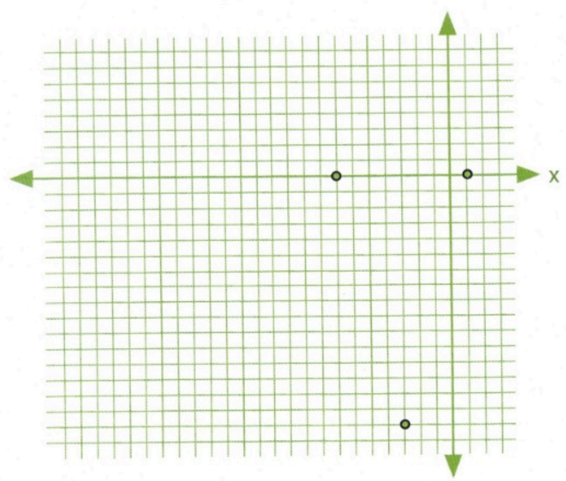
We now use this x-coordinate to find the y-coordinate.

$$\begin{array}{rcrcrcr} y & = & ax^2 & + & bx & + & c \\ y & = & 1(-3)^2 & + & 6(-3) & - & 7 \\ y & = & 9 & - & 18 & - & 7 \\ y & = & -16 & & & & \end{array}$$

The vertex is at $x = -3$ and $y = -16$ and can be given by the coordinate $(-3, -16)$.

The x-intercepts or roots of the quadratic in Example 10.6.1 are found by factoring $x^2 + 6x - 7 = 0$.

For this problem, the quadratic factors to $(x + 7)(x - 1) = 0$, which means the roots are $x = -7$ and $x = 1$. Putting all this data together gives us the vertex coordinate $(-3, -16)$ and the two x-intercept coordinates $(-7, 0)$ and $(1, 0)$. These are the values used to create the rough sketch.



Trying to sketch this curve will be somewhat challenging if there is to be any semblance of accuracy.

When this happens, it is quite easy to fill in some of the places where there may have been coordinates by using a data table.

For this graph, choose values from $x = 2$ to $x = -8$

.

First, find the value of y when $x = 2$:

$$\begin{array}{rcrcrcr} y & = & x^2 & + & 6x & - & 7 \\ y & = & 1(2)^2 & + & 6(2) & - & 7 \\ y & = & 4 & + & 12 & - & 7 \\ y & = & 9 & & & & \end{array}$$

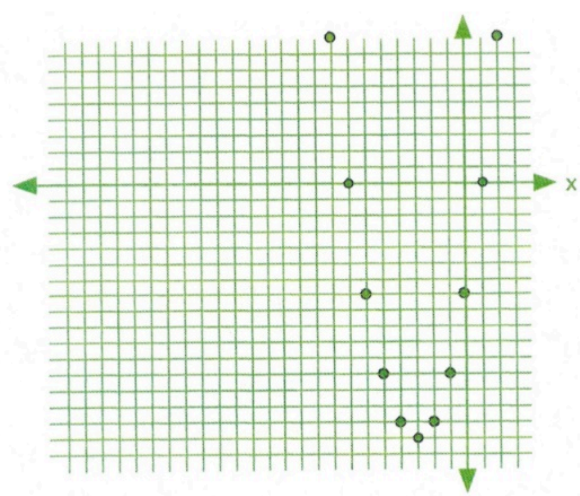
Put this value in the table and then carry on to complete all of it.

x	y
2	9
1	0
0	−7
−1	−12
−2	−15
−3	−16
−4	−15
−5	−12
−6	−7
−7	0
−8	9

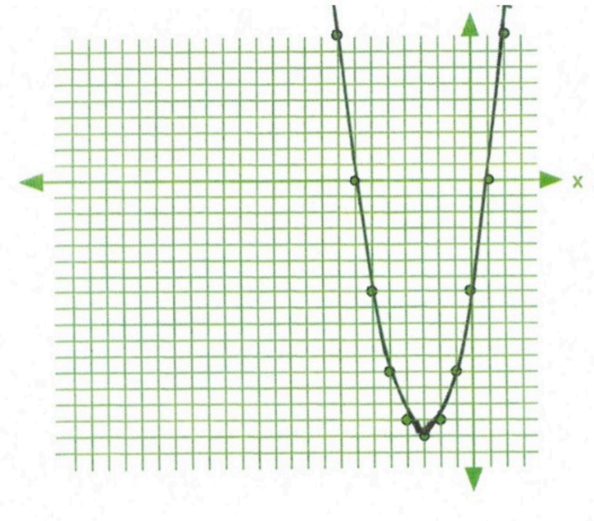
Placing all of these coordinates on the graph will generate a graph showing increased detail, as shown below. All that remains is to draw a curve that connects the points on the graph. The level of detail required to draw the curve only depends on the unique characteristics of the curve itself.

Remember:

For the quadratic equation $y = ax^2 + bx + c$, the x-coordinate of the vertex is $x = \dfrac{-b}{2a}$ and the y-coordinate of the vertex is $y = a\left(\dfrac{-b}{2a}\right)^2 + \left(\dfrac{-b}{2a}\right)x + c$.



The following questions will ask you to sketch the quadratic function using the vertex and the x-intercepts and then later to draw a data table to find the coordinates of data points from which to draw a curve.



Both approaches are quite valuable, the difference is only in the details, which if required can use both techniques to general a curve in increased detail.

Example 2

Find the vertex of $y = x^2 - 6x - 7$.

In the equation, $a = 1$, $b = -6$, and $c = -7$.

This means that the x-coordinate of the vertex $x = \dfrac{-b}{2a}$ will give us the value $x = \dfrac{-(-6)}{2}(1)$ or 3.

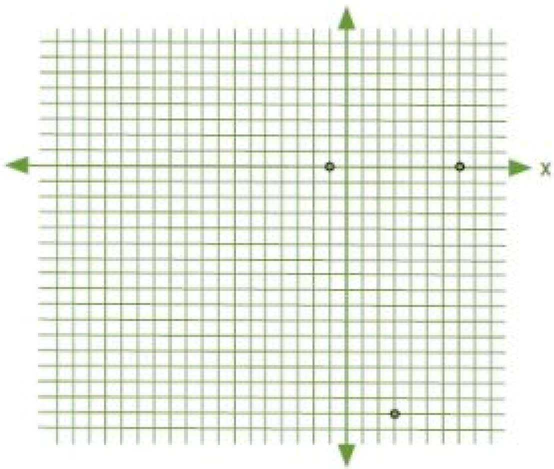
We now use this x-coordinate to find the y-coordinate.

$$\begin{array}{rrrrrrr} y & = & ax^2 & + & bx & + & c \\ y & = & 1(3)^2 & - & 6(3) & - & 7 \\ y & = & 9 & - & 18 & - & 7 \\ y & = & -16 & & & & \end{array}$$

The vertex is at $x = +3$ and $y = -16$ and can be given by the coordinate $(+3, -16)$.

The x-intercepts or roots of this quadratic are found by factoring $x^2 + 6x - 7 = 0$.

For this problem, the quadratic factors to $(x - 7)(x + 1) = 0$, which means the roots are $x = +7$ and $x = -1$. Putting all this data together gives us the vertex coordinate $(-3, -16)$ and the two x-intercept coordinates $(7, 0)$ and $(-1, 0)$.



Trying to sketch this curve will be somewhat challenging if there is to be any semblance of accuracy.

When this happens, it is quite easy to fill in some of the places where there may have been coordinates by using a data table.

For this graph, choose values for $x = 0$ to $x = 6$. First, find the value of y when $x = 0$:

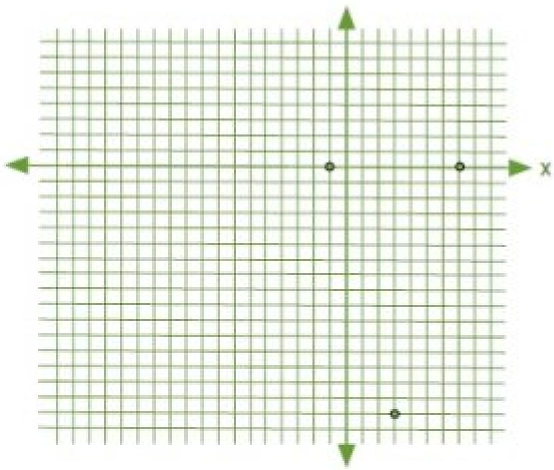
$$\begin{array}{rcrcrcr} y & = & x^2 & - & 6x & - & 7 \\ y & = & (0)^2 & - & 6(0) & - & 7 \\ y & = & 0 & - & 0 & - & 7 \\ y & = & -7 & & & & \end{array}$$

Put this value in the table and then carry on to complete all of it.

x	y
5	7
4	0
3	−5
2	−8
1	−9
0	−8
−1	−5
−2	0
−3	7

Placing all of these coordinates on the graph will generate a graph showing increased detail as shown below. All that remains is to draw a curve that connects the points on the graph. The level of detail you require to draw the curve only depends on the unique characteristics of the curve itself.

Remember:



For the quadratic equation $y = ax^2 + bx + c$, the x-coordinate of the vertex is $x = \dfrac{-b}{2a}$ and the y-coordinate of the vertex is $y = a\left(\dfrac{-b}{2a}\right)^2 + \left(\dfrac{-b}{2a}\right)x + c$.

The following questions will ask you to sketch the quadratic function using the vertex and the x-intercepts, and then later to draw a data table to find the coordinates of data points with which to draw a curve.

Both approaches are quite valuable. The difference is only in the detail. If required, you can use both techniques to generate a curve in increased detail.

Questions

Find the vertex and intercepts of the following quadratics. Use this information to graph the quadratic.

1. $y = x^2 - 2x - 8$
2. $y = x^2 - 2x - 3$
3. $y = 2x^2 - 12x + 10$
4. $y = 2x^2 - 12x + 16$
5. $y = -2x^2 + 12x - 18$
6. $y = -2x^2 + 12x - 10$
7. $y = -3x^2 + 24x - 45$
8. $y = -2(x^2 + 2x) + 6$

First, find the line of symmetry for each of the following equations. Then, construct a data table for each equation. Use this table to graph the equation.

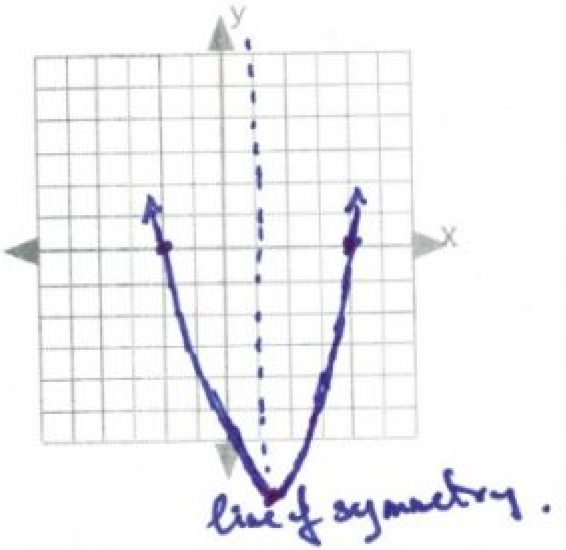
9. $y = 3x^2 - 6x - 5$
10. $y = 2x^2 - 4x - 3$
11. $y = -x^2 + 4x + 2$
12. $y = -3x^2 - 6x + 2$

Answers to odd questions

1. intercepts: $\begin{aligned} y &= 0 \\ 0 &= x^2 - 2x - 8 \\ 0 &= (x-4)(x+2) \\ x &= 4, -2 \end{aligned}$ vertex: $\left[\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right]$ $(1, -9)$ line of symmetry:

$$\begin{aligned} x &= \frac{-b}{2a} \\ x &= \frac{-(-2)}{2(1)} \Rightarrow \frac{2}{2} \text{ or } 1 \end{aligned}$$

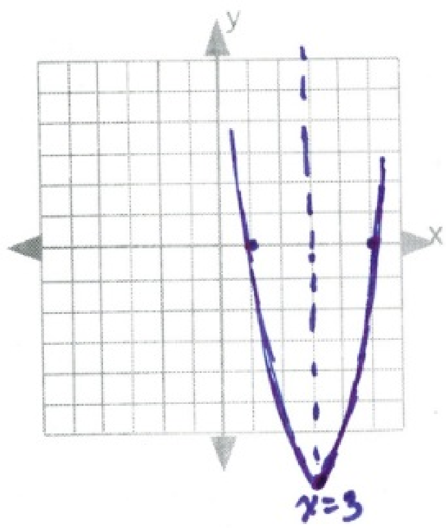
$$\begin{aligned} \therefore f(1) &= 1^2 - 2(1) - 8 \\ f(1) &= -9 \end{aligned}$$



3. intercepts: $\begin{aligned} 0 &= 2x^2 - 12x + 10 \\ 0 &= 2(x^2 - 6x + 5) \\ 0 &= 2(x-5)(x-1) \\ x &= 5, 1 \end{aligned}$ line of symmetry: $\begin{aligned} x &= \frac{-b}{2a} \\ x &= \frac{-6}{2(1)} \Rightarrow \frac{6}{2} \text{ or } 3 \end{aligned}$ vertex:

$$\begin{aligned}f(3) &= 2(3)^2 - 12(3) + 10\\ f(3) &= 18 - 36 + 10\\ f(3) &= -8\end{aligned}$$

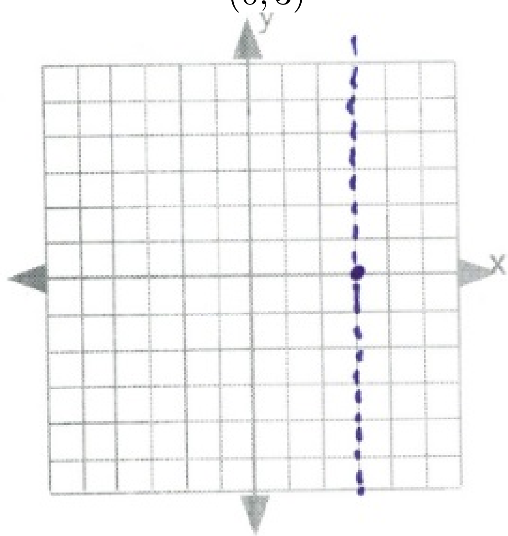
$(3, -8)$



5. intercepts: $\begin{aligned}0 &= -2x^2 + 12x - 18\\ 0 &= -2(x^2 - 6x + 9)\\ 0 &= -2(x-3)(x-3)\\ x &= 3\end{aligned}$ line of symmetry: $\begin{aligned}x &= \frac{-b}{2a}\\ x &= \frac{-12}{2(-2)} \Rightarrow \frac{-12}{-4} \text{ or } 3\end{aligned}$

$$\begin{aligned}f(3) &= -2(3)^2 - 12(3) - 18\\ f(3) &= -18 + 36 - 18\\ \text{vertex:} f(3) &= 0\end{aligned}$$

$(0, 3)$

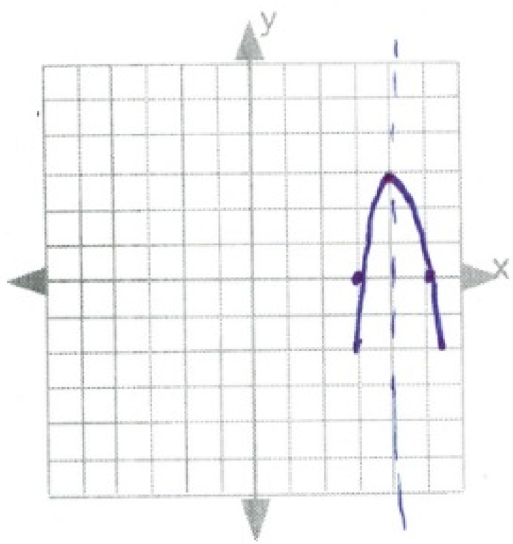


7. intercepts:
$$\begin{aligned} 0 &= -3x^2 + 24x - 45 \\ 0 &= -3(x^2 - 8x + 15) \\ 0 &= -3(x-3)(x-5) \\ x &= 3, 5 \end{aligned}$$

line of symmetry:
$$\begin{aligned} x &= \frac{-b}{2a} \\ x &= \frac{-24}{2(-3)} \Rightarrow \frac{-24}{-6} \text{ or } 4 \end{aligned}$$

$$\begin{aligned} f(4) &= -3(4)^2 + 24(4) - 45 \\ f(4) &= -48 + 96 - 45 \\ \text{vertex:} f(4) &= 3 \end{aligned}$$

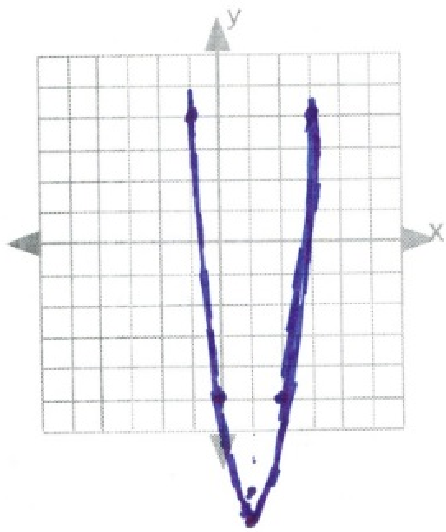
$(4, 3)$



9. line of symmetry:

$$x = \frac{-b}{2a} \Rightarrow \frac{-(-6)}{2(3)} \Rightarrow \frac{6}{6} \text{ or } 1$$

x	y
3	4
2	−5
1	−9
0	−5
−1	4

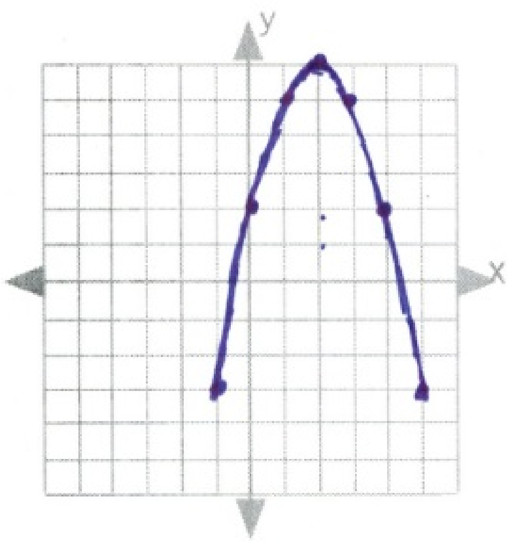


line of symmetry:

11.

$$x = \frac{-b}{2a} \Rightarrow \frac{-4}{2(-1)} \Rightarrow \frac{-4}{-2} \text{ or } 2$$

x	y
5	−3
4	2
3	5
2	6
1	5
0	2
−1	−3



y
x

CHAPTER 6: RATIONAL EXPRESSIONS AND EQUATIONS

CHAPTER 6.1: REDUCING RATIONAL EXPRESSIONS

Definition: A rational expression can be defined as a simple fraction where either the numerator, denominator or both are polynomials. Care must be taken when solving rational expressions in that one must be careful of solutions yielding 0 in the denominator.

Rational expressions are expressions written as a quotient of polynomials. Examples of rational expressions include:

$$\frac{x^2-x-12}{x^2-9x+20} \quad \text{or} \quad \frac{3}{x-2} \quad \text{or} \quad \frac{a-b}{b^2-a^2} \quad \text{or} \quad \frac{3}{13}$$

As rational expressions are a special type of fraction, it is important to remember that you cannot have zero in the denominator of a fraction. For this reason, rational expressions have what are called excluded values that make the denominator equal zero if used.

Example 1

Find the excluded values for the following rational expression:

$$\frac{x^2-1}{3x^2+5x}$$

For this expression, the excluded values are found by solving $3x^2+5x \neq 0$.

Factor $3x^2+5x$ first to yield $x(3x+5) \neq 0$.

There are now have two parts of this that can be made to equal 0:

$$x \neq 0 \qquad \text{and} \qquad 3x+5 \neq 0$$

Solving the second yields:

$$\begin{array}{rrrrr} 3x & + & 5 & \neq & 0 \\ & - & 5 & & -5 \\ & & 3x & \neq & -5 \\ & & x & \neq & -\dfrac{5}{3} \end{array}$$

As with our previous polynomials, evaluating rational expressions is easily accomplished by substituting the value for the variable and using order of operations.

Example 2

Evaluate the following rational expression for $x = -6$:

$$\begin{aligned} & \frac{x^2-4}{x^2+6x+8} \\ = & \frac{(-6)^2-4}{(-6)^2+6(-6)+8} \\ = & \frac{36-4}{36+6(-6)+8} \\ = & \frac{32}{36-36+8} \\ = & \frac{32}{8} \\ = & 4 \end{aligned}$$

Just as the expression was reduced in the previous example, often a rational expression can be reduced, even without knowing the value of the variable. When it is reduced, divide out common factors. This has already been seen with monomials when the properties of exponents was discussed. If the problem only has monomials, you can reduce the coefficients and subtract exponents on the variables.

Example 3

Reduce the following rational expression:

$$\begin{aligned} & \frac{15x^4y^2}{25x^2y^6} \\ = & \frac{3\cdot 5x^{4-2}y^{2-6}}{5\cdot 5} \\ = & \frac{3x^2y^{-4}}{5} \\ = & \frac{3x^2}{5y^4} \end{aligned}$$

If there is more than just one term in either the numerator or the denominator, you might need to first factor the numerator and denominator.

Example 4

Reduce the following rational expressions:

1. $$\begin{aligned} & \frac{28}{8x^2-16} \\ = & \frac{28}{8(x^2-2)} \\ = & \frac{7}{2(x^2-2)} \end{aligned}$$

2. $$\begin{aligned} & \frac{9x-3}{18x-6} \\ = & \frac{3(3x-1)}{6(3x-1)} \\ = & \frac{1}{2} \end{aligned}$$

3. $$\begin{aligned} & \frac{x^2-25}{x^2+8x+15} \\ = & \frac{(x+5)(x-5)}{(x+3)(x+5)} \\ = & \frac{x-5}{x+3} \end{aligned}$$

Questions

Evaluate Questions 1 to 6.

1. $\dfrac{4v+2}{6}$ when $v=4$
2. $\dfrac{b-3}{3b-9}$ when $b=-2$
3. $\dfrac{x-3}{x^2-4x+3}$ when $x=-4$
4. $\dfrac{a+2}{a^2+3a+2}$ when $a=-1$
5. $\dfrac{b+2}{b^2+4b+4}$ when $b=0$
6. $\dfrac{n^2-n-6}{n-3}$ when $n=4$

For each of Questions 7 to 14, state the excluded values.

7. $\dfrac{3k^2+30k}{k+10}$

8. $\dfrac{27p}{18p^2 - 36p}$
9. $\dfrac{10m^2 + 8m}{10m}$
10. $\dfrac{10x + 16}{6x + 20}$
11. $\dfrac{r^2 + 3r + 2}{5r + 10}$
12. $\dfrac{6n^2 - 21n}{6n^2 + 3n}$
13. $\dfrac{b^2 + 12b + 32}{b^2 + 4b - 32}$
14. $\dfrac{10v^2 + 30v}{35v^2 - 5v}$

For Questions 15 to 32, simplify each expression.

15. $\dfrac{21x^2}{18x}$
16. $\dfrac{12n}{4n^2}$
17. $\dfrac{24a}{40a^2}$
18. $\dfrac{21k}{24k^2}$
19. $\dfrac{18m - 24}{60}$
20. $\dfrac{n - 9}{9n - 81}$
21. $\dfrac{x + 1}{x^2 + 8x + 7}$
22. $\dfrac{28m + 12}{36}$
23. $\dfrac{n^2 + 4n - 12}{n^2 - 7n + 10}$
24. $\dfrac{b^2 + 14b + 48}{b^2 + 15b + 56}$
25. $\dfrac{9v + 54}{v^2 - 4v - 60}$
26. $\dfrac{k^2 - 12k + 32}{k^2 - 64}$
27. $\dfrac{2n^2 + 19n - 10}{9n + 90}$
28. $\dfrac{3x^2 - 29x + 40}{5x^2 - 30x - 80}$

29. $\dfrac{2x^2-10x+8}{3x^2-7x+4}$
30. $\dfrac{7n^2-32n+16}{4n-16}$
31. $\dfrac{7a^2-26a-45}{6a^2-34a+20}$
32. $\dfrac{4k^3-2k^2-2k}{k^3-18k^2+9k}$

Answers to odd questions

1. $\dfrac{4(4)+2}{6} \Rightarrow \dfrac{16+2}{6} \Rightarrow \dfrac{18}{6} \Rightarrow 3$
3. $\dfrac{-4-3}{(-4)^2-4(-4)+3} \Rightarrow \dfrac{-7}{16+16+3} \Rightarrow -\dfrac{7}{35} \Rightarrow -\dfrac{1}{5}$
5. $\dfrac{b+2}{b^2+4b+4} \Rightarrow \dfrac{2}{4} \Rightarrow \dfrac{1}{2}$
7. $\begin{array}{rrrr} k & + & 10 & \neq & 0 \\ & - & 10 & & -10 \\ & & k & \neq & -10 \end{array}$
9. $\begin{array}{rrr} 10m & \neq & 0 \\ m & \neq & 0 \end{array}$
11. $\begin{array}{rrr} 5(r & + & 2) \\ r & \neq & -2 \end{array}$
13. $(b-4)(b+8)$
$$\begin{array}{rrr} b & \neq & 4 \\ b & \neq & -8 \end{array}$$
15. $\dfrac{21x^2}{18x} \Rightarrow \dfrac{3\cdot 7\cdot x\cdot x}{3\cdot 6\cdot x} \Rightarrow \dfrac{7x}{6}$
17. $\dfrac{24a}{40a^2} \Rightarrow \dfrac{3\cdot 8\cdot a}{5\cdot 8\cdot a\cdot a} \Rightarrow \dfrac{3}{5a}$
19. $\dfrac{18m-24}{60} \Rightarrow \dfrac{6(3m-4)}{6(10)} \Rightarrow \dfrac{3m-4}{10}$
21. $\dfrac{x+1}{x^2+8x+7} \Rightarrow \dfrac{x+1}{(x+1)(x+7)} \Rightarrow \dfrac{1}{x+7}$
23. $\dfrac{n^2+4n-12}{n^2-7n+10} \Rightarrow \dfrac{(n+6)(n-2)}{(n-5)(n-2)} \Rightarrow \dfrac{n+6}{n-5}$
25. $\dfrac{9v+54}{v^2-4v-60} \Rightarrow \dfrac{9(v-6)}{(v-10)(v+6)} \Rightarrow \dfrac{9}{v-10}$
27. $\dfrac{2n^2+19n-10}{9n+90} \Rightarrow \dfrac{(2n-1)(n+10)}{9(n+10)} \Rightarrow \dfrac{2n-1}{9}$
29. $\dfrac{2x^2-10x+8}{3x^2-7x+4} \Rightarrow \dfrac{2(x-4)(x-1)}{(3x-4)(x-1)} \Rightarrow \dfrac{2(x-4)}{3x-4}$

31. $\dfrac{7a^2-26a-45}{6a^2-34a+20} \Rightarrow \dfrac{(a-5)(7a+9)}{2(3a-2)(a-5)} \Rightarrow \dfrac{7a+9}{2(3a-2)}$

CHAPTER 6.2: MULTIPLICATION AND DIVISION OF RATIONAL EXPRESSIONS

Multiplying and dividing rational expressions is very similar to the process used to multiply and divide fractions.

Example 1

Reduce and multiply $\frac{15}{49}$ and $\frac{14}{45}$.

$$\frac{15}{49} \cdot \frac{14}{45} \text{ reduces to } \frac{1}{7} \cdot \frac{2}{3}, \text{ which equals } \frac{2}{21}$$

(15 and 45 reduce to 1 and 3, and 14 and 49 reduce to 2 and 7)

This process of multiplication is identical to division, except the first step is to reciprocate any fraction that is being divided.

Example 2

Reduce and divide $\frac{25}{18}$ by $\frac{15}{6}$.

$$\frac{25}{18} \div \frac{15}{6} \text{ reciprocates to } \frac{25}{18} \cdot \frac{6}{15}, \text{ which reduces to } \frac{5}{3} \cdot \frac{1}{3}, \text{ which equals } \frac{5}{9}$$

(25 and 15 reduce to 5 and 3, and 6 and 18 reduce to 1 and 3)

When multiplying with rational expressions, follow the same process: first, divide out common factors, then multiply straight across.

Example 3

Reduce and multiply $\dfrac{25x^2}{9y^8}$ and $\dfrac{24y^4}{55x^7}$.

$$\frac{25x^2}{9y^8}\cdot\frac{24y^4}{55x^7}\text{ reduces to }\frac{5}{3y^4}\cdot\frac{8}{11x^5},\text{ which equals }\frac{40}{33x^5y^4}$$

(25 and 55 reduce to 5 and 11, 24 and 9 reduce to 8 and 3, x^2 and x^7 reduce to x^5, y^4 and y^8 reduce to y^4)

Remember: when dividing fractions, reciprocate the dividing fraction.

Example 4

Reduce and divide $\dfrac{a^4b^2}{a}$ by $\dfrac{b^4}{4}$.

$$\frac{a^4b^2}{a}\div\frac{b^4}{4}\text{ reciprocates to }\frac{a^4b^2}{a}\cdot\frac{4}{b^4},\text{ which reduces to }\frac{a^3}{1}\cdot\frac{4}{b^2},\text{ which equals }\frac{4a^3}{b^2}$$

(After reciprocating, $4a^4b^2$ and b^4 reduce to $4a^3$ and b^2)

In dividing or multiplying some fractions, the polynomials in the fractions must be factored first.

Example 5

Reduce, factor and multiply $\frac{x^2-9}{x^2+x-20}$ and $\frac{x^2-8x+16}{3x+9}$.

$$\frac{x^2-9}{x^2+x-20}\cdot\frac{x^2-8x+16}{3x+9} \text{ factors to } \frac{(x+3)(x-3)}{(x-4)(x+5)}\cdot\frac{(x-4)(x-4)}{3(x+3)}$$

Dividing or cancelling out the common factors $(x+3)$ and $(x-4)$ leaves us with $\frac{x-3}{x+5}\cdot\frac{x-4}{3}$, which results in $\frac{(x-3)(x-4)}{3(x+5)}$.

Example 6

Reduce, factor and multiply or divide the following fractions:

$$\frac{a^2+7a+10}{a^2+6a+5}\cdot\frac{a+1}{a^2+4a+4}\div\frac{a-1}{a+2}$$

Factoring each fraction and reciprocating the last one yields:

$$\frac{(a+5)(a+2)}{(a+5)(a+1)}\cdot\frac{(a+1)}{(a+2)(a+2)}\cdot\frac{(a+2)}{(a-1)}$$

Dividing or cancelling out the common polynomials leaves us with:

$$\frac{1}{a-1}$$

Questions

Simplify each expression.

1. $\frac{8x^2}{9}\cdot\frac{9}{2}$

2. $\dfrac{8x}{3} \div \dfrac{4x}{7}$
3. $\dfrac{5x^2}{4} \cdot \dfrac{6}{5}$
4. $\dfrac{10p}{5} \div \dfrac{8}{10}$
5. $\dfrac{(m-6)}{7(7m-5)} \cdot \dfrac{5m(7m-5)}{m-6}$
6. $\dfrac{7(n-2)}{10(n+3)} \div \dfrac{n-2}{(n+3)}$
7. $\dfrac{7r}{7r(r+10)} \div \dfrac{r-6}{(r-6)^2}$
8. $\dfrac{6x(x+4)}{(x-3)} \cdot \dfrac{(x-3)(x-6)}{6x(x-6)}$
9. $\dfrac{x-10}{35x+21} \div \dfrac{7}{35x+21}$
10. $\dfrac{v-1}{4} \cdot \dfrac{4}{v^2-11v+10}$
11. $\dfrac{x^2-6x-7}{x+5} \cdot \dfrac{x+5}{x-7}$
12. $\dfrac{1}{a-6} \cdot \dfrac{8a+80}{8}$
13. $\dfrac{4m+36}{m+9} \cdot \dfrac{m-5}{5m^2}$
14. $\dfrac{2r}{r+6} \div \dfrac{2r}{7r+42}$
15. $\dfrac{n-7}{6n-12} \cdot \dfrac{12-6n}{n^2-13n+42}$
16. $\dfrac{x^2+11x+24}{6x^3+18x^2} \cdot \dfrac{6x^3+6x^2}{x^2+5x-24}$
17. $\dfrac{27a+36}{9a+63} \div \dfrac{6a+8}{2}$
18. $\dfrac{k-7}{k^2-k-12} \cdot \dfrac{7k^2-28k}{8k^2-56k}$
19. $\dfrac{x^2-12x+32}{x^2-6x-16} \cdot \dfrac{7x^2+14x}{7x^2+21x}$
20. $\dfrac{9x^3+54x^2}{x^2+5x-14} \cdot \dfrac{x^2+5x-14}{10x^2}$
21. $(10m^2+100m) \cdot \dfrac{18m^3-36m^2}{20m^2-40m}$
22. $\dfrac{n-7}{n^2-2n-35} \div \dfrac{9n+54}{10n+50}$
23. $\dfrac{x^2-1}{2x-4} \cdot \dfrac{x^2-4}{x^2-x-2} \div \dfrac{x^2+x-2}{3x-6}$
24. $\dfrac{a^3+b^3}{a^2+3ab+2b^2} \cdot \dfrac{3a-6b}{3a^2-3ab+3b^2} \div \dfrac{a^2-4b^2}{a+2b}$

Answers to odd questions

1. $\dfrac{8x^2}{9} \cdot \dfrac{9}{2} \Rightarrow \dfrac{2 \cdot 4 \cdot x^2}{9} \cdot \dfrac{9}{2} \Rightarrow 4x^2$
3. $\dfrac{5x^2}{4} \cdot \dfrac{6}{5} \Rightarrow \dfrac{5 \cdot x^2}{2 \cdot 2} \cdot \dfrac{2 \cdot 3}{5} \Rightarrow \dfrac{3x^2}{2}$
5. $\dfrac{(m-6)}{7(7m-5)} \cdot \dfrac{5m(7m-5)}{m-6} \Rightarrow \dfrac{5m}{7}$
7. $\dfrac{7r}{7r(r+10)} \div \dfrac{r-6}{(r-6)^2} \Rightarrow \dfrac{7r}{7r(r+10)} \cdot \dfrac{(r-6)^2}{r-6} \Rightarrow \dfrac{7r}{7r(r+10)} \cdot \dfrac{(r-6)(r-6)}{r-6} \Rightarrow$
$\dfrac{r-6}{r+10}$
9. $\dfrac{x-10}{35x+21} \div \dfrac{7}{35x+21} \Rightarrow \dfrac{x-10}{7(5x+3)} \cdot \dfrac{7(5x+3)}{7} \Rightarrow \dfrac{x-10}{7}$
11. $\dfrac{x^2-6x-7}{x+5} \cdot \dfrac{x+5}{x-7} \Rightarrow \dfrac{(x-7)(x+1)}{(x+5)} \cdot \dfrac{(x+5)}{(x-7)} \Rightarrow x+1$
13. $\dfrac{4m+36}{m+9} \cdot \dfrac{m-5}{5m^2} \Rightarrow \dfrac{4(m+9)}{m+9} \cdot \dfrac{m-5}{5m^2} \Rightarrow \dfrac{4(m-5)}{5m^2}$
15. $\dfrac{n-7}{6n-12} \cdot \dfrac{12-6n}{n^2-13n+42} \Rightarrow \dfrac{(n-7)}{6(n-2)} \cdot \dfrac{6(2-n)}{(n-6)(n-7)} \Rightarrow \dfrac{-1(n-2)}{(n-2)(n-6)} \Rightarrow$
$\dfrac{-1}{n-6}$
17. $\dfrac{27a+36}{9a+63} \div \dfrac{6a+8}{2} \Rightarrow \dfrac{9(3a+4)}{9(a+7)} \cdot \dfrac{2}{2(3a+4)} \Rightarrow \dfrac{1}{a+7}$
19. $\dfrac{x^2-12x+32}{x^2-6x-16} \cdot \dfrac{7x^2+14x}{7x^2+21x} \Rightarrow \dfrac{(x-8)(x-4)}{(x-8)(x+2)} \cdot \dfrac{7x(x+2)}{7x(x+3)} \Rightarrow \dfrac{x-4}{x+3}$
21. $(10m^2+100m) \cdot \dfrac{18m^3-36m^2}{20m^2-40m} \Rightarrow 10m(m+10) \cdot \dfrac{2 \cdot 9m^2(m-2)}{2 \cdot 10m(m-2)} \Rightarrow$
$9m^2(m+10)$
23. $\dfrac{x^2-1}{2x-4} \cdot \dfrac{x^2-4}{x^2-x-2} \div \dfrac{x^2+x-2}{3x-6} \Rightarrow$
$\dfrac{(x-1)(x+1)}{2(x-2)} \cdot \dfrac{(x+2)(x-2)}{(x-2)(x+1)} \cdot \dfrac{3(x-2)}{(x+2)(x-1)} \Rightarrow \dfrac{3}{2}$

CHAPTER 6.3: LEAST COMMON DENOMINATORS

Finding the least common denominator, or LCD, is very important to working with rational expressions. The process used depends on finding what is common to each rational expression and identifying what is not common. These common and not common factors are then combined to form the LCD.

Example 1

Find the LCD of the numbers 12, 8, and 6.

First, break these three numbers into primes:

$$\begin{aligned} 12 &= 2\cdot 2\cdot 3 \text{ or } 2^2\cdot 3 \\ 8 &= 2\cdot 2\cdot 2 \text{ or } 2^3 \\ 6 &= 2\cdot 3 \end{aligned}$$

Then write out the primes for the first number, 12, and set the LCD to $2^2 \cdot 3$.

Notice the factorization of 8 includes 2^3, yet the LCD currently only has 2^2, so you add one 2.

Now the LCD = $2^3 \cdot 3$.

Checking $6 = 2 \cdot 3$, there already is a $2 \cdot 3$ in the LCD, so we need not add any more primes.

The LCD = $2^3 \cdot 3$ or 24.

This process can be duplicated with variables.

Example 2

Find the LCD of $4x^2y^5$ and $6x^4y^3z^6$.

First, break both terms into primes:

$$\begin{array}{rcl} 4x^2y^5 & = & 2^2 \cdot x^2 \cdot y^5 \\ 6x^4y^3z^6 & = & 2 \cdot 3 \cdot x^4 \cdot y^3 \cdot z^6 \end{array}$$

Then write out the primes for the first term, $4x^2y^5$, and set the LCD to $2^2 \cdot x^2 \cdot y^5$.

The LCD for $6x^4y^3z^6 = 2 \cdot 3 \cdot x^4 \cdot y^3 \cdot z^6$ has an extra 3, x^2, and z^6, which we add to the LCD that we are constructing.

This yields LCD = $2^2 \cdot 3 \cdot x^{2+2} \cdot y^5 \cdot z^6$, or LCD = $12x^4y^5z^6$.

This process can also be duplicated with polynomials.

Example 3

Find the LCD of $x^2 + 2x - 3$ and $x^2 - x - 12$.

First, we factor both of these polynomials, much like finding the primes of the above terms:

$$\begin{array}{rcl} x^2 + 2x - 3 & = & (x-1)(x+3) \\ x^2 - x - 12 & = & (x-4)(x+3) \end{array}$$

The LCD is constructed as we did before, except this time, we write out the factored terms from the first polynomial, so the LCD = $(x-1)(x+3)$.

Notice that $x^2 - x - 12 = (x-4)(x+3)$, where the $(x+3)$ is already in the LCD, which means that we only need to add $(x-4)$.

The LCD = $(x-1)(x+3)(x-4)$.

Questions

For Questions 1 to 10, find each Least Common Denominator.

1. $2a^3, 6a^4b^2, 4a^3b^5$
2. $5x^2y, 25x^3y^5z$
3. $x^2-3x, x-3, x$
4. $4x-8, x-2, 4$
5. $x+2, x-4$
6. $x, x-7, x+1$
7. $x^2-25, x+5$
8. x^2-9, x^2-6x+9
9. x^2+3x+2, x^2+5x+6
10. $x^2-7x+10, x^2-2x-15, x^2+x-6$

For Questions 11 to 20, find the LCD of each fraction and place each expression over the same common denominator.

11. $\dfrac{3a}{5b^2}, \dfrac{2}{10a^3b}$
12. $\dfrac{3x}{x-4}, \dfrac{2}{x+2}$
13. $\dfrac{x+2}{x-3}, \dfrac{x-3}{x+2}$
14. $\dfrac{5}{x^2-6x}, \dfrac{2}{x}, \dfrac{-3}{x-6}$
15. $\dfrac{x}{x^2-16}, \dfrac{3x}{x^2-8x+16}$
16. $\dfrac{5x+1}{x^2-3x-10}, \dfrac{4}{x-5}$
17. $\dfrac{x+1}{x^2-36}, \dfrac{2x+3}{x^2+12x+36}$
18. $\dfrac{3x+1}{x^2-x-12}, \dfrac{2x}{x^2+4x+3}$
19. $\dfrac{4x}{x^2-x-6}, \dfrac{x+2}{x-3}$
20. $\dfrac{3x}{x^2-6x+8}, \dfrac{x-2}{x^2+x-20}, \dfrac{5}{x^2+3x-10}$

Answers to odd questions

1. $12a^4b^5$
3. $x(x-3)$
5. $(x+2)(x-4)$
7. $(x+5)(x-5)$
9. $(x+1)(x+2)(x+3)$

11. $$\text{LCD} = 10a^3b^2$$

$$\frac{3a}{5b^2}\cdot\frac{2a^3}{2a^3} \Rightarrow \frac{6a^4}{10a^3b^2}$$

$$\frac{2}{10a^3b}\cdot\frac{b}{b} \Rightarrow \frac{2b}{10a^3b^2}$$

13. $$\text{LCD} = (x-3)(x+2)$$

$$\frac{(x+2)}{(x-3)}\cdot\frac{(x+2)}{(x+2)} \Rightarrow \frac{x^2+4x+4}{(x-3)(x+2)}$$

$$\frac{(x-3)}{(x+2)}\cdot\frac{(x-3)}{(x-3)} \Rightarrow \frac{x^2-6x+9}{(x-3)(x+2)}$$

15. $$\text{LCD} = (x-4)^2(x+4)$$

$$\frac{x}{x^2-16}\cdot\frac{(x-4)}{(x-4)} \Rightarrow \frac{x^2-4x}{(x-4)^2(x+4)}$$

$$\frac{3x}{(x^2-8x+16)}\cdot\frac{(x+4)}{(x+4)} \Rightarrow \frac{3x^2+12}{(x-4)^2(x+4)}$$

17. $$\text{LCD} = (x+6)^2(x-6)$$

$$\frac{x+1}{x^2-36}\cdot\frac{(x+6)}{(x+6)} \Rightarrow \frac{x^2+7x+6}{(x+6)^2(x-6)}$$

$$\frac{(2x+3)}{(x^2+12x+36)}\cdot\frac{(x-6)}{(x-6)} \Rightarrow \frac{2x^2-9x-18}{(x+6)^2(x-6)}$$

19. $$\text{LCD} = (x-3)(x+2)$$

$$\frac{4x}{x^2-x-6} \Rightarrow \frac{4x}{(x-3)(x+2)}$$

$$\frac{(x+2)}{(x-3)}\cdot\frac{(x+2)}{(x+2)} \Rightarrow \frac{x^2+4x+4}{(x-3)(x+2)}$$

CHAPTER 6.4: ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS

Adding and subtracting rational expressions is identical to adding and subtracting integers. Recall that, when adding fractions with a common denominator, you add the numerators and keep the denominator. This is the same process used with rational expressions. Remember to reduce the final answer if possible.

Example 1

Add the following rational expressions:

$$\frac{x-4}{x^2-2x-8}+\frac{x+8}{x^2-2x-8} \quad \text{Same denominator, so you add the numerators and combine like terms.}$$

$$\frac{2x+4}{x^2-2x-8} \quad \text{Factor the numerator and the denominator.}$$

$$\frac{2(x+2)}{(x+2)(x-4)} \quad \text{Divide out } (x+2).$$

$$\frac{2}{x-4} \quad \text{Solution.}$$

Subtraction of rational expressions with a common denominator follows the same pattern, though the subtraction can cause problems if you are not careful with it. To avoid sign errors, first distribute the subtraction throughout the numerator. Then treat it like an addition problem. This process is the same as "add the opposite," which was seen when subtracting with negatives.

Example 2

Subtract the following rational expressions:

$$\frac{6x-12}{3x-6}-\frac{15x-6}{3x-6}$$ Add the opposite of the second fraction (distribute the negative).

$$\frac{6x-12}{3x-6}+\frac{-15x+6}{3x-6}$$ Add the numerators and combine like terms.

$$\frac{-9x-6}{3x-6}$$ Factor the numerator and the denominator.

$$\frac{-3(3x+2)}{3(x-2)}$$ Divide out the common factor of 3.

$$\frac{-(3x+2)}{x-2}$$ Solution.

When there is not a common denominator, first find the least common denominator (LCD) and alter each fraction so the denominators match.

Example 3

Add the following rational expressions:

$$\frac{7a}{3a^2b}+\frac{4b}{6ab^4} \quad \text{The LCD is } 6a^2b^4.$$

$$\frac{2b^3}{2b^3}\cdot\frac{7a}{3a^2b}+\frac{4b}{6ab^4}\cdot\frac{a}{a} \quad \text{Multiply the first fraction by } 2b^3 \text{ and the second by } a.$$

$$\frac{14ab^3}{6a^2b^4}+\frac{4ab}{6a^2b^4} \quad \text{Add the numerators. No like terms to combine.}$$

$$\frac{14ab^3+4ab}{6a^2b^4} \quad \text{Factor the numerator.}$$

$$\frac{2ab(7b^2+2)}{6a^2b^4} \quad \text{Reduce, dividing out factors } 2, a, \text{ and } b.$$

$$\frac{7b^2+2}{3ab^3} \quad \text{Solution.}$$

Example 4

Subtract the following rational expressions:

$\dfrac{x+1}{x-4}-\dfrac{x+1}{x^2-7x+12}$	Add the opposite of the second fraction (distribute the negative).
$\dfrac{x+1}{x-4}+\dfrac{-x-1}{x^2-7x+12}$	Factor the denominators to find the LCD $=(x-4)(x-3)$.
$\dfrac{(x-3)(x+1)}{(x-3)(x-4)}+\dfrac{-x-1}{(x-3)(x-4)}$	Only the first fraction needs to be multplied by $(x-3)$.
$\dfrac{x^2-2x-3}{(x-3)(x-4)}+\dfrac{-x-1}{(x-3)(x-4)}$	Add the numerators and combine like terms.
$\dfrac{x^2-3x-4}{(x-3)(x-4)}$	Factor the numerator.
$\dfrac{(x-4)(x+1)}{(x-3)(x-4)}$	Divide out the common factor of $(x-4)$.
$\dfrac{x+1}{x-3}$	Solution.

Questions

Add or subtract the rational expressions. Simplify your answers whenever possible.

1. $\dfrac{2}{a+3}+\dfrac{4}{a+3}$
2. $\dfrac{x^2}{x-2}-\dfrac{6x-8}{x-2}$
3. $\dfrac{t^2+4t}{t-1}+\dfrac{2t-7}{t-1}$
4. $\dfrac{a^2+3a}{a^2+5a-6}-\dfrac{4}{a^2+5a-6}$
5. $\dfrac{5}{6r}-\dfrac{5}{8r}$
6. $\dfrac{7}{xy^2}+\dfrac{3}{x^2y}$
7. $\dfrac{8}{9t^3}+\dfrac{5}{6t^2}$
8. $\dfrac{x+5}{8}+\dfrac{x-3}{12}$
9. $\dfrac{x-1}{4x}-\dfrac{2x+3}{x}$
10. $\dfrac{2c-d}{c^2d}-\dfrac{c+d}{cd^2}$

11. $\dfrac{5x+3y}{2x^2y} - \dfrac{3x+4y}{xy^2}$
12. $\dfrac{2}{x-1} + \dfrac{2}{x+1}$
13. $\dfrac{x}{x^2+5x+6} - \dfrac{2}{x^2+3x+2}$
14. $\dfrac{2x}{x^2-1} - \dfrac{3}{x^2+5x+4}$
15. $\dfrac{x}{x^2+15x+56} - \dfrac{7}{x^2+13x+42}$
16. $\dfrac{2x}{x^2-9} + \dfrac{5}{x^2+x-6}$
17. $\dfrac{5x}{x^2-x-6} - \dfrac{18}{x^2-9}$
18. $\dfrac{4x}{x^2-2x-3} - \dfrac{3}{x^2-5x+6}$

Answers to odd questions

1. $\dfrac{2+4}{a+3} = \dfrac{6}{a+3}$
3. $\dfrac{t^2+4t+2t-7}{t-1} \Rightarrow \dfrac{t^2+6t-7}{t-1} \Rightarrow \dfrac{(t+7)(t-1)}{(t-1)} \Rightarrow t+7$
5. $\text{LCD} = 24r$ $\quad \dfrac{5}{6r}\cdot\dfrac{4}{4} - \dfrac{5}{8r}\cdot\dfrac{3}{3} \Rightarrow \dfrac{20}{24r} - \dfrac{15}{24r} \Rightarrow \dfrac{5}{24r}$
7. $\text{LCD} = 18t^3$ $\quad \dfrac{8}{9t^3}\cdot\dfrac{2}{2} + \dfrac{5}{6t^2}\cdot\dfrac{3t}{3t} \Rightarrow \dfrac{15t+16}{18t^3}$
9. $\text{LCD} = 4x$ $\quad \dfrac{x-1}{4x} - \dfrac{4(2x+3)}{4\cdot x} \Rightarrow \dfrac{x-1-8x-12}{4x} \Rightarrow \dfrac{-7x-13}{4x}$
11. $\text{LCD} = 2x^2y^2$ $\quad \dfrac{(5x+3y)(y)}{(2x^2y)(y)} - \dfrac{(3x+4y)(2x)}{(xy^2)(2x)} \Rightarrow \dfrac{5xy+3y^2-6x^2-8xy}{2x^2y^2} \Rightarrow$
$\dfrac{3y^2-3xy-6x^2}{2x^2y^2}$
13. $\text{LCD} = (x+3)(x+2)(x+1)$ $\quad \dfrac{x(x+1)}{(x+3)(x+2)(x+1)} - \dfrac{2(x+3)}{(x+3)(x+2)(x+1)} \Rightarrow$
$\dfrac{x^2+x-2x-6}{(x+3)(x+2)(x+1)} \Rightarrow \dfrac{x^2-x-6}{(x+3)(x+2)(x+1)} \Rightarrow \dfrac{(x-3)(x+2)}{(x+3)(x+2)(x+1)} \Rightarrow$

$\dfrac{x-3}{(x+3)(x+1)}$
15. $\text{LCD} = (x+7)(x+8)(x+6)$ $\quad \dfrac{x(x+6)}{(x+7)(x+8)(x+6)} - \dfrac{7(x+8)}{(x+7)(x+8)(x+6)} \Rightarrow$

$$\frac{x^2+6x-7x-56}{(x+7)(x+8)(x+6)} \Rightarrow \frac{x^2-x-56}{(x+7)(x+8)(x+6)} \Rightarrow \frac{(x-8)(x+7)}{(x+7)(x+8)(x+6)} \Rightarrow$$

$$\frac{x-8}{(x+8)(x+6)}$$

17. $\text{LCD} = (x{-}3)(x{+}2)(x{+}3)$ $\quad \dfrac{5x(x+3)}{(x-3)(x+2)(x+3)} - \dfrac{18(x+2)}{(x-3)(x+2)(x+3)} \Rightarrow$

$$\frac{5x^2+15x-18x-36}{(x-3)(x+2)(x+3)} \Rightarrow \frac{5x^2-3x-36}{(x-3)(x+2)(x+3)} \Rightarrow \frac{(x-3)(5x+12)}{(x-3)(x+2)(x+3)} \Rightarrow$$

$$\frac{5x+12}{(x+2)(x+3)}$$

CHAPTER 6.5: REDUCING COMPLEX FRACTIONS

Complex fractions will have fractions in either the numerator, the denominator, or both. These fractions are generally simplified by multiplying the fractions in the numerator and denominator by the LCD. This process allows you to reduce a complex fraction to a simpler one in one step.

Example 1

Reduce $\dfrac{1-\dfrac{1}{x^2}}{1-\dfrac{1}{x}}$.

For this fraction, the LCD is x^2. To simplify this complex fraction, multiply each term in the numerator and the denominator by the LCD.

$$\frac{1(x^2)-\dfrac{1}{x^2}(x^2)}{1(x^2)-\dfrac{1}{x}(x^2)}$$

This will result in:

$$\frac{x^2-1}{x^2-x}$$

Now, factor both the numerator and denominator, which results in:

$$\frac{(x-1)(x+1)}{x(x-1)},\text{ which reduces to }\frac{x+1}{x}$$

It matters not how complex these fractions are: simply find the LCD to reduce the complex fraction to one that is simpler.

The more fractions there are in a problem, the more times the process is repeated.

Example 2

Reduce the following complex fraction:

$$\frac{\dfrac{x-3}{x+3}-\dfrac{x+3}{x-3}}{\dfrac{x-3}{x+3}+\dfrac{x+3}{x-3}}$$

For this fraction, the LCD is $(x-3)(x+3)$. To simplify the above complex fraction, multiply both the numerator and denominator by the LCD. This looks like:

$$\frac{(x-3)(x+3)\dfrac{x-3}{x+3}-\dfrac{x+3}{x-3}(x-3)(x+3)}{(x-3)(x+3)\dfrac{x-3}{x+3}+\dfrac{x+3}{x-3}(x-3)(x+3)}$$

Which reduces to:

$$\frac{(x-3)(x-3)-(x+3)(x+3)}{(x-3)(x-3)+(x+3)(x+3)}$$

Now multiply out the numerator and denominator and add like terms:

$$\frac{(x^2-6x+9)-(x^2+6x+9)}{(x^2-6x+9)+(x^2+6x+9)}$$

Dropping the brackets leaves:

$$\frac{x^2-6x+9-x^2-6x-9}{x^2-6x+9+x^2+6x+9}$$

Adding these terms together yields:

$$\frac{-12x}{2x^2+18}$$

You will notice there is a common factor of 2 in each of the terms that can be factored out. This results in:

$$\frac{-6x}{x^2+9}$$

Questions

Simplify each of the following complex fractions.

1. $$\frac{1+\frac{1}{x}}{1-\frac{1}{x^2}}$$
2. $$\frac{1-\frac{1}{y^2}}{1+\frac{1}{y}}$$
3. $$\frac{\frac{a}{b}+2}{\frac{a^2}{b^2}-4}$$
4. $$\frac{\frac{1}{y^2}-9}{\frac{1}{y}+3}$$
5. $$\frac{\frac{1}{a^2}-\frac{1}{a}}{\frac{1}{a^2}+\frac{1}{a}}$$
6. $$\frac{\frac{1}{b}+\frac{1}{2}}{\frac{4}{b^2-1}}$$
7. $$\frac{x+2-\frac{9}{x+2}}{x+1+\frac{x-7}{x+2}}$$
8. $$\frac{a-3+\frac{a-3}{a+2}}{a+4-\frac{4a+5}{a+2}}$$
9. $$\frac{\frac{x+y}{y}+\frac{y}{x-y}}{\frac{y}{x-y}}$$
10. $$\frac{\frac{a-b}{a}-\frac{a}{a+b}}{\frac{b^2}{a+b}}$$

11. $$\dfrac{\dfrac{x-y}{y}+\dfrac{x+y}{x-y}}{\dfrac{y}{x-y}}$$

12. $$\dfrac{\dfrac{x-2}{x+2}-\dfrac{x+2}{x-2}}{\dfrac{x-2}{x+2}+\dfrac{x+2}{x-2}}$$

Answers to odd questions

1. $$\dfrac{\left(1+\dfrac{1}{x}\right)x^2}{\left(1-\dfrac{1}{x^2}\right)x^2} \Rightarrow \dfrac{x^2+x}{x^2-1} \Rightarrow \dfrac{x(x+1)}{(x+1)(x-1)} \Rightarrow \dfrac{x}{x-1}$$

3. $$\dfrac{\left(\dfrac{a}{b}+2\right)b^2}{\left(\dfrac{a^2}{b^2}-4\right)b^2} \Rightarrow \dfrac{ab+2b^2}{a^2-4b^2} \Rightarrow \dfrac{b(a+2b)}{(a+2b)(a-2b)} \Rightarrow \dfrac{b}{a-2b}$$

5. $$\dfrac{\left(\dfrac{1}{a^2}-\dfrac{1}{a}\right)a^2}{\left(\dfrac{1}{a^2}+\dfrac{1}{a}\right)a^2} \Rightarrow \dfrac{1-a}{1+a}$$

7. $$\dfrac{\left(x+2-\dfrac{9}{x+2}\right)(x+2)}{\left(x+1+\dfrac{x-7}{x+2}\right)(x+2)} \Rightarrow \dfrac{(x+2)(x+2)-9}{(x+1)(x+2)+x-7} \Rightarrow \dfrac{x^2+4x+4-9}{x^2+3x+2+x-7} \Rightarrow$$
$$\dfrac{x^2+4x-5}{x^2+4x-5} \Rightarrow 1$$

9. $$\dfrac{\left(\dfrac{x+y}{y}+\dfrac{y}{x-y}\right)y(x-y)}{\left(\dfrac{y}{x-y}\right)y(x-y)} \Rightarrow \dfrac{(x+y)(x-y)+y(y)}{y(y)} \Rightarrow$$
$$\dfrac{x^2-y^2+y^2}{y^2} \Rightarrow \dfrac{x^2}{y^2}$$

11. $$\frac{\left(\frac{x-y}{y}+\frac{x+y}{x-y}\right)y(x-y)}{\left(\frac{y}{x-y}\right)y(x-y)} \Rightarrow \frac{(x-y)(x-y)+(x+y)(y)}{y(y)} \Rightarrow$$

$$\frac{x^2-2xy+y^2+xy+y^2}{y^2} \Rightarrow \frac{x^2-xy+2y^2}{y^2}$$

CHAPTER 6.6: SOLVING COMPLEX FRACTIONS

When solving two or more equated fractions, the easiest solution is to first remove all fractions by multiplying both sides of the equations by the LCD. This strategy is shown in the next examples.

Example 1

Solve $\dfrac{x+3}{4}=\dfrac{2}{3}$.

For these two fractions, the LCD is 3 × 4 = 12. Therefore, we multiply both sides of the equation by 12:

$$12\left(\frac{x+3}{4}\right)=\left(\frac{2}{3}\right)12$$

This reduces the complex fraction to:

$$3(x+3)=2(4)$$

Multiplying this out yields:

$$3x+9=8$$

Now just isolate and solve for x:

$$\begin{array}{rrrrr} 3x & + & 9 & = & 8 \\ & - & 9 & & -9 \\ & & 3x & = & -1 \\ \\ & & x & = & -\dfrac{1}{3} \end{array}$$

Example 2

Solve $\dfrac{2x-3}{3x+4}=\dfrac{2}{5}$.

For these two fractions, the LCD is $5(3x+4)$. Therefore, both sides of the equation are multiplied by $5(3x+4)$:

$$5(3x+4)\left(\frac{2x-3}{3x+4}\right)=\left(\frac{2}{5}\right)5(3x+4)$$

This reduces the complex fraction to:

$$5(2x-3)=2(3x+4)$$

Multiplying this out yields:

$$10x-15=6x+8$$

Now isolate and solve for x:

$$\begin{array}{rrrcrrr} 10x & - & 15 & = & 6x & + & 8 \\ -6x & + & 15 & & -6x & + & 15 \\ & & 4x & = & 23 & & \\ \\ & & x & = & \dfrac{23}{4} & & \end{array}$$

Example 3

Solve $\dfrac{k+3}{3}=\dfrac{8}{k-2}$.

For these two fractions, the LCD is $3(k-2)$. Therefore, multiply both sides of the equation by $3(k-2)$:

$$3(k-2)\left(\frac{k+3}{3}\right)=\left(\frac{8}{k-2}\right)3(k-2)$$

This reduces the complex fraction to:

$$(k-2)(k+3)=8(3)$$

This multiplies out to:

$$k^2+k-6=24$$

Now subtract 24 from both sides of the equation to turn this into an equation that can be easily factored:

$$\begin{array}{rcrcrcr} k^2 & + & k & - & 6 & = & 24 \\ & & & - & 24 & & -24 \\ k^2 & + & k & - & 30 & = & 0 \end{array}$$

This equation factors to:

$$(k+6)(k-5)=0$$

The solutions are:

$k=-6$ and $k=5$

Questions

Solve each of the following complex fractions.

1. $\dfrac{m-1}{5}=\dfrac{8}{2}$
2. $\dfrac{8}{2}=\dfrac{8}{x-8}$
3. $\dfrac{2}{9}=\dfrac{10}{p-4}$
4. $\dfrac{9}{n+2}=\dfrac{3}{9}$
5. $\dfrac{3}{10}=\dfrac{a}{a+2}$
6. $\dfrac{x+1}{3}=\dfrac{x+3}{4}$
7. $\dfrac{2}{p+4}=\dfrac{p+5}{3}$
8. $\dfrac{5}{n+1}=\dfrac{n-4}{10}$

9. $\dfrac{x+5}{5}=\dfrac{6}{x-2}$
10. $\dfrac{4}{x-3}=\dfrac{x+5}{5}$
11. $\dfrac{m+3}{4}=\dfrac{11}{m-4}$
12. $\dfrac{x-5}{8}=\dfrac{4}{x-1}$

Answers to odd questions

1. $$\begin{array}{rrrrl} \text{LCD} & = & 5(2) & & \\ \\ 2(m & - & 1) & = & 5(8) \\ 2m & - & 2 & = & 40 \\ & + & 2 & & +2 \\ & & \dfrac{2m}{2} & = & \dfrac{42}{2} \\ \\ & & m & = & 21 \end{array}$$

3. $$\begin{array}{rrrrrl} \text{LCD} & = & 9(p & - & 4) & \\ \\ 2(p & - & 4) & = & 9(10) & \\ 2p & - & 8 & = & 90 & \\ & + & 8 & & +8 & \\ & & \dfrac{2p}{2} & = & \dfrac{98}{2} & \\ \\ & & p & = & 49 & \end{array}$$

5. $$\begin{array}{rrrrrl} \text{LCD} & = & 10(a & + & 2) & \\ \\ 3(a & + & 2) & = & 10(a) & \\ 3a & + & 6 & = & 10a & \\ -3a & & & & -3a & \\ & & \dfrac{6}{7} & = & \dfrac{7a}{7} & \\ \\ & & a & = & \dfrac{6}{7} & \end{array}$$

7. $$\begin{aligned} \text{LCD} &= 3(p+4) \\ \\ 2(3) &= (p+4)(p+5) \\ 6 &= p^2+9p+20 \\ -6 & \quad\quad\quad\quad\quad\quad -6 \\ 0 &= p^2+9p+14 \\ 0 &= (p+7)(p+2) \\ \\ p &= -2,\ -7 \end{aligned}$$

9. $$\begin{aligned} \text{LCD} &= 5(x-2) \\ \\ (x+5)(x-2) &= 5(6) \\ x^2+3x-10 &= 30 \\ -30 &\quad -30 \\ x^2+3x-40 &= 0 \\ (x-5)(x+8) &= 0 \\ \\ x &= 5, -7 \end{aligned}$$

11. $$\begin{aligned} \text{LCD} &= 4(m-4) \\ \\ (m+3)(m-4) &= 4(11) \\ m^2-m-12 &= 44 \\ -44 &\quad -44 \\ (m^2-m-56) &= 0 \\ (m-8)(m+7) &= 0 \\ \\ m &= 8, -7 \end{aligned}$$

CHAPTER 6.7: SOLVING RATIONAL EQUATIONS

When solving equations that are made up of rational expressions, we use the same strategy we used to solve complex fractions, where the easiest solution involved multiplying by the LCD to remove the fractions. Consider the following examples.

Example 1

Solve the following:

$$\frac{2x}{3} - \frac{5}{6} = \frac{3}{4}$$

For these three fractions, the LCD is 12. Therefore, multiply all parts of the equation by 12:

$$12\left(\frac{2x}{3} - \frac{5}{6}\right) = \left(\frac{3}{4}\right)12$$

This reduces the rational equation to:

$$4(2x) - 2(5) = 3(3)$$

Multiplying this out yields:

$$8x - 10 = 9$$

Now isolate and solve for x:

$$\begin{array}{rrrrr} 8x & - & 10 & = & 9 \\ & + & 10 & & +10 \\ & & 8x & = & 19 \\ \\ & & x & = & \dfrac{19}{8} \end{array}$$

Example 2

Solve the following:

$$\frac{x}{x+2}+\frac{1}{x+1}=\frac{5}{(x+1)(x+2)}$$

For these three fractions, the LCD is $(x+1)(x+2)$. Therefore, multiply all parts of the equation by $(x+1)(x+2)$:

$$(x+1)(x+2)\left(\frac{x}{x+2}+\frac{1}{x+1}\right)=\left(\frac{5}{(x+1)(x+2)}\right)(x+1)(x+2)$$

This reduces the rational equation to:

$$x(x+1)+1(x+2)=5$$

Multiplying this out yields:

$$x^2+x+x+2=5$$

Which reduces to:

$$x^2+2x+2=5$$

Now subtract 5 from both sides of the equation to turn this into an equation that can be easily factored:

$$\begin{array}{ccccccccc} x^2 & + & 2x & + & 2 & = & 5 \\ & & & - & 5 & & -5 \\ x^2 & + & 2x & - & 3 & = & 0 \end{array}$$

This equation factors to:

$$(x+3)(x-1)=0$$

The solutions are:

$$x=-3 \text{ and } 1$$

Questions

Solve each rational equation.

1. $3x - \dfrac{1}{2} - \dfrac{1}{x} = 0$
2. $x + 1 = \dfrac{4}{x+1}$
3. $x + \dfrac{20}{x-4} = \dfrac{5x}{x-4} - 2$
4. $\dfrac{x^2+6}{x-1} + \dfrac{x-2}{x-1} = 2x$
5. $x + \dfrac{6}{x-3} = \dfrac{2x}{x-3}$
6. $\dfrac{x-4}{x-1} = \dfrac{12}{3-x} + 1$
7. $\dfrac{3m}{2m-5} - \dfrac{7}{3m+1} = \dfrac{3}{2}$
8. $\dfrac{4-x}{1-x} = \dfrac{12}{3-x}$
9. $\dfrac{7}{y-3} - \dfrac{1}{2} = \dfrac{y-2}{y-4}$
10. $\dfrac{1}{x+2} + \dfrac{1}{x-2} = \dfrac{3x+8}{x^2-4}$
11. $\dfrac{x+1}{x-1} - \dfrac{x-1}{x+1} = \dfrac{5}{6}$
12. $\dfrac{x-2}{x+3} - \dfrac{1}{x-2} = \dfrac{1}{x^2+x-6}$
13. $\dfrac{x}{x-1} - \dfrac{2}{x+1} = \dfrac{4x^2}{x^2-1}$
14. $\dfrac{2x}{x+2} + \dfrac{2}{x-4} = \dfrac{3x}{x^2-2x-8}$
15. $\dfrac{2x}{x+1} - \dfrac{3}{x+5} = \dfrac{-8x^2}{x^2+6x+5}$

Answers to odd questions

1. $\text{LCD} = 2(x)$

$$\begin{array}{rrrrrrl} 3x(2x) & - & x & - & 2 & = & 0 \\ 6x^2 & - & x & - & 2 & = & 0 \\ (3x & - & 2)(2x & + & 1) & = & 0 \\ \\ & & & & x & = & \dfrac{2}{3}, -\dfrac{1}{2} \end{array}$$

3.
$$\begin{array}{rrrrrrrrrrr}
\text{LCD} & = & x & - & 4 & & & & & & \\
\\
x(x & - & 4) & + & 20 & = & 5x & - & 2(x & - & 4) \\
x^2 & - & 4x & + & 20 & = & 5x & - & 2x & + & 8 \\
 & - & 3x & - & 8 & & & - & 3x & - & 8 \\
x^2 & - & 7x & + & 12 & = & 0 & & & & \\
(x & - & 4)(x & - & 3) & = & 0 & & & & \\
\\
 & & & & x & = & 3, & 4 & & &
\end{array}$$

5.
$$\begin{array}{rrrrrrr}
\text{LCD} & = & x & - & 3 & & \\
\\
x(x & - & 3) & + & 6 & = & 2x \\
x^2 & - & 3x & + & 6 & = & 2x \\
 & - & 2x & & & & -2x \\
x^2 & - & 5x & + & 6 & = & 0 \\
(x & - & 3)(x & - & 2) & = & 0 \\
\\
 & & & & x & = & 2,3
\end{array}$$

7.
$$\begin{array}{rrrrrrrrrrrrr}
\text{LCD} & = & (2m & - & 5)(3m & + & 1)(2) & & & & & & \\
\\
3m(3m & + & 1)(2) & - & 7(2m & - & 5)(2) & = & 3(2m & - & 5)(3m & + & 1) \\
18m^2 & + & 6m & - & 28m & + & 70 & = & 18m^2 & - & 39m & - & 15 \\
-18m^2 & & & + & 39m & + & 15 & & -18m^2 & + & 39m & + & 15 \\
 & & & & 17m & + & 85 & = & 0 & & & & \\
 & & & & & - & 85 & & -85 & & & & \\
 & & & & & & \dfrac{17m}{17} & = & \dfrac{-85}{17} & & & & \\
\\
 & & & & & & m & = & -5 & & & &
\end{array}$$

9.
$$\text{LCD} = 2(y-3)(y-4)$$

$$\begin{array}{rrrrrrrrrrrrr}
7(2)(y & - & 4) & - & 1(y & - & 3)(y & - & 4) & = & (y & - & 2)(2)(y & - & 3) \\
14y & - & 56 & - & y^2 & + & 7y & - & 12 & = & 2y^2 & - & 10y & + & 12 \\
 & & & & -\ y^2 & + & 21y & - & 68 & = & 2y^2 & - & 10y & + & 12 \\
 & & & & -2y^2 & + & 10y & - & 12 & & -2y^2 & + & 10y & - & 12 \\
 & & & & -3y^2 & + & 31y & - & 80 & = & 0 \\
 & & & & 3y^2 & - & 31y & + & 80 & = & 0 \\
 & & & & (y & - & 5)(3y & - & 16) & = & 0
\end{array}$$

$$y = 5, \ \frac{16}{3}$$

11.
$$\text{LCD} = (x+1)(x-1)(6)$$

$$\begin{array}{rrrrrrrrrrrrrrr}
(x & + & 1)(x & + & 1)(6) & - & (x & - & 1)(x & - & 1)(6) & = & 5(x & + & 1)(x & - & 1) \\
6(x^2 & + & 2x & + & 1) & - & 6(x^2 & - & 2x & + & 1) & = & 5(x^2 & & & - & 1) \\
6x^2 & + & 12x & + & 6 & - & 6x^2 & + & 12x & - & 6 & = & 5x^2 & & & - & 5 \\
 & & & & & & & & & & 24x & = & 5x^2 & & & - & 5 \\
 & & & & & & & & & & -24x & & & - & 24x \\
 & & & & & & & & & & 0 & = & 5x^2 & - & 24x & - & 5 \\
 & & & & & & & & & & 0 & = & (5x & + & 1)(x & - & 5)
\end{array}$$

$$x = 5, \ -\frac{1}{5}$$

13.
$$\text{LCD} = (x-1)(x+1)$$

$$\begin{array}{rrrrrrrrrrrrr}
x(x & + & 1) & - & 2(x & - & 1) & = & 4x^2 \\
x^2 & + & x & - & 2x & + & 2 & = & 4x^2 \\
-x^2 & & & + & x & - & 2 & & -x^2 & + & x & - & 2 \\
 & & & & & & 0 & = & 3x^2 & + & x & - & 2 \\
 & & & & & & 0 & = & (3x & - & 2)(x & + & 1)
\end{array}$$

$$0 = \frac{2}{3}, \ -1$$

15. $$\text{LCD} = (x+1)(x+5)$$

$$\begin{array}{rrrrrrrrrrr}
2x(x & + & 5) & - & 3(x & + & 1) & = & -8x^2 \\
2x^2 & + & 10x & - & 3x & - & 3 & = & -8x^2 \\
+8x^2 & & & & & & & & +8x^2 \\
& & 10x^2 & + & 7x & - & 3 & = & 0 \\
& & (10x & - & 3)(x & + & 1) & = & 0 \\
& & & & & & x & = & \dfrac{3}{10}, -1
\end{array}$$

CHAPTER 6.8: RATE WORD PROBLEMS: SPEED, DISTANCE AND TIME

Distance, rate and time problems are a standard application of linear equations. When solving these problems, use the relationship **rate** (speed or velocity) times **time** equals **distance**.

$$r \cdot t = d$$

For example, suppose a person were to travel 30 km/h for 4 h. To find the total distance, multiply rate times time or (30km/h)(4h) = 120 km.

The problems to be solved here will have a few more steps than described above. So to keep the information in the problem organized, use a table. An example of the basic structure of the table is below:

Example of a Distance, Rate and Time Chart

Who or What	Rate	Time	Distance

The third column, distance, will always be filled in by multiplying the rate and time columns together. If given a total distance of both persons or trips, put this information in the distance column. Now use this table to set up and solve the following examples.

Example 1

Joey and Natasha start from the same point and walk in opposite directions. Joey walks 2 km/h faster than Natasha. After 3 hours, they are 30 kilometres apart. How fast did each walk?

Who or What	Rate	Time	Distance
Natasha	r	3 h	$3\text{ h}(r)$
Joey	$r+2$	3 h	$3\text{ h}(r+2)$

The distance travelled by both is 30 km. Therefore, the equation to be solved is:

$$\begin{array}{rcrcrcl} 3r & + & 3(r & + & 2) & = & 30 \\ 3r & + & 3r & + & 6 & = & 30 \\ & & & - & 6 & & -6 \\ & & & & \dfrac{6r}{6} & = & \dfrac{24}{6} \\ & & & & r & = & 4 \text{ km/h} \end{array}$$

This means that Natasha walks at 4 km/h and Joey walks at 6 km/h.

Example 2

Nick and Chloe left their campsite by canoe and paddled downstream at an average speed of 12 km/h. They turned around and paddled back upstream at an average rate of 4 km/h. The total trip took 1 hour. After how much time did the campers turn around downstream?

Who or What	Rate	Time	Distance
Downstream	12 km/h	t	$12 \text{ km/h } (t)$
Upstream	4 km/h	$(1-t)$	$4 \text{ km/h } (1-t)$

The distance travelled downstream is the same distance that they travelled upstream. Therefore, the equation to be solved is:

$$\begin{array}{rcrcl} 12(t) & = & 4(1 & - & t) \\ 12t & = & 4 & - & 4t \\ +4t & & & + & 4t \\ \dfrac{16t}{16} & = & \dfrac{4}{16} & & \\ t & = & 0.25 & & \end{array}$$

This means the campers paddled downstream for 0.25 h and spent 0.75 h paddling back.

Example 3

Terry leaves his house riding a bike at 20 km/h. Sally leaves 6 h later on a scooter to catch up with him travelling at 80 km/h. How long will it take her to catch up with him?

Who or What	Rate	Time	Distance
Terry	20 km/h	t	$20 \text{ km/h } (t)$
Sally	80 km/h	$(t - 6 \text{ h})$	$80 \text{ km/h } (t - 6 \text{ h})$

The distance travelled by both is the same. Therefore, the equation to be solved is:

$$\begin{array}{rcrcr} 20(t) & = & 80(t & - & 6) \\ 20t & = & 80t & - & 480 \\ -80t & & -80t & & \\ \dfrac{-60t}{-60} & = & \dfrac{-480}{-60} & & \\ & & & & \\ t & = & 8 & & \end{array}$$

This means that Terry travels for 8 h and Sally only needs 2 h to catch up to him.

Example 4

On a 130-kilometre trip, a car travelled at an average speed of 55 km/h and then reduced its speed to 40 km/h for the remainder of the trip. The trip took 2.5 h. For how long did the car travel 40 km/h?

Who or What	Rate	Time	Distance
Fifty-five	55 km/h	t	$55\text{ km/h }(t)$
Forty	40 km/h	$(2.5\text{ h} - t)$	$40\text{ km/h }(2.5\text{ h} - t)$

The distance travelled by both is 30 km. Therefore, the equation to be solved is:

$$\begin{array}{rrrrrr}
55(t) & + & 40(2.5 & - & t) & = & 130 \\
55t & + & 100 & - & 40t & = & 130 \\
 & - & 100 & & & & -100 \\
 & & & & \dfrac{15t}{15} & = & \dfrac{30}{15} \\
 & & & & & & \\
 & & & & t & = & 2
\end{array}$$

This means that the time spent travelling at 40 km/h was 0.5 h.

Distance, time and rate problems have a few variations that mix the unknowns between distance, rate and time. They generally involve solving a problem that uses the combined distance travelled to equal some distance or a problem in which the distances travelled by both parties is the same. These distance, rate and time problems will be revisited later on in this textbook where quadratic solutions are required to solve them.

Questions

For Questions 1 to 8, find the equations needed to solve the problems. Do not solve.

1. A is 60 kilometres from B. An automobile at A starts for B at the rate of 20 km/h at the same time that an automobile at B starts for A at the rate of 25 km/h. How long will it be before the automobiles meet?
2. Two automobiles are 276 kilometres apart and start to travel toward each other at the same time. They travel at rates differing by 5 km/h. If they meet after 6 h, find the rate of each.
3. Two trains starting at the same station head in opposite directions. They travel at the rates of 25 and 40 km/h, respectively. If they start at the same time, how soon will they be 195 kilometres apart?
4. Two bike messengers, Jerry and Susan, ride in opposite directions. If Jerry rides at the rate of 20 km/h, at what rate must Susan ride if they are 150 kilometres apart in 5 hours?
5. A passenger and a freight train start toward each other at the same time from two points 300 kilometres apart. If the rate of the passenger train exceeds the rate of the freight train by 15 km/h, and they meet after 4 hours, what must the rate of each be?

6. Two automobiles started travelling in opposite directions at the same time from the same point. Their rates were 25 and 35 km/h, respectively. After how many hours were they 180 kilometres apart?
7. A man having ten hours at his disposal made an excursion by bike, riding out at the rate of 10 km/h and returning on foot at the rate of 3 km/h. Find the distance he rode.
8. A man walks at the rate of 4 km/h. How far can he walk into the country and ride back on a trolley that travels at the rate of 20 km/h, if he must be back home 3 hours from the time he started?

Solve Questions 9 to 22.

9. A boy rides away from home in an automobile at the rate of 28 km/h and walks back at the rate of 4 km/h. The round trip requires 2 hours. How far does he ride?
10. A motorboat leaves a harbour and travels at an average speed of 15 km/h toward an island. The average speed on the return trip was 10 km/h. How far was the island from the harbour if the trip took a total of 5 hours?
11. A family drove to a resort at an average speed of 30 km/h and later returned over the same road at an average speed of 50 km/h. Find the distance to the resort if the total driving time was 8 hours.
12. As part of his flight training, a student pilot was required to fly to an airport and then return. The average speed to the airport was 90 km/h, and the average speed returning was 120 km/h. Find the distance between the two airports if the total flying time was 7 hours.
13. Sam starts travelling at 4 km/h from a campsite 2 hours ahead of Sue, who travels 6 km/h in the same direction. How many hours will it take for Sue to catch up to Sam?
14. A man travels 5 km/h. After travelling for 6 hours, another man starts at the same place as the first man did, following at the rate of 8 km/h. When will the second man overtake the first?
15. A motorboat leaves a harbour and travels at an average speed of 8 km/h toward a small island. Two hours later, a cabin cruiser leaves the same harbour and travels at an average speed of 16 km/h toward the same island. How many hours after the cabin cruiser leaves will it be alongside the motorboat?
16. A long distance runner started on a course, running at an average speed of 6 km/h. One hour later, a second runner began the same course at an average speed of 8 km/h. How long after the second runner started will they overtake the first runner?
17. Two men are travelling in opposite directions at the rate of 20 and 30 km/h at the same time and from the same place. In how many hours will they be 300 kilometres apart?
18. Two trains start at the same time from the same place and travel in opposite directions. If the rate of one is 6 km/h more than the rate of the other and they are 168 kilometres apart at the end of 4 hours, what is the rate of each?
19. Two cyclists start from the same point and ride in opposite directions. One cyclist rides twice as fast as the other. In three hours, they are 72 kilometres apart. Find the rate of each cyclist.
20. Two small planes start from the same point and fly in opposite directions. The first plane is flying 25

km/h slower than the second plane. In two hours, the planes are 430 kilometres apart. Find the rate of each plane.

21. On a 130-kilometre trip, a car travelled at an average speed of 55 km/h and then reduced its speed to 40 km/h for the remainder of the trip. The trip took a total of 2.5 hours. For how long did the car travel at 40 km/h?
22. Running at an average rate of 8 m/s, a sprinter ran to the end of a track and then jogged back to the starting point at an average of 3 m/s. The sprinter took 55 s to run to the end of the track and jog back. Find the length of the track.

Answers to odd questions

1.

Who or What	Rate	Time	Equation
A	20	t	$20t$
B	25	t	$25t$

$$20t + 25t = 60$$

3.

Who or What	Rate	Time	Equation
T_1	25	t	$25t$
T_2	40	t	$40t$

$$25t + 40t = 195$$

5.

Who or What	Rate	Time	Equation
P	$r + 15$	4	$4(r + 15)$
F	r	4	$4r$

$$4(r + 15) + 4r = 300$$

7.

Who or What	Rate	Time	Equation
Away	10	t	$10(t)$
Return	3	$10-t$	$3(10-t)$

$10t = 3(10-t)$

9.

Who or What	Rate	Time	Equation
Away	28	t	$28t$
Back	4	$2-t$	$4(2-t)$

$$\begin{array}{rcll} 28t & = & 4(2 & - \quad t) \\ 28t & = & 8 & - \quad 4t \\ +4t & & & + \quad 4t \\ \dfrac{32t}{32} & = & \dfrac{8}{32} & \\ t & = & \dfrac{1}{4} & \\ d & = & rt & \\ d & = & 28(\dfrac{1}{4}) & \\ d & = & 7 \text{ km} & \end{array}$$

11.

Who or What	Rate	Time	Equation
To resort	30	t	$30t$
Return	50	$8-t$	$50(8-t)$

$$\begin{array}{rcrcr} 30t & = & 50(8 & - & t) \\ 30t & = & 400 & - & 50t \\ +50t & & & + & 50t \\ \dfrac{80t}{80} & = & \dfrac{400}{80} & & \\ \\ t & = & 5 & & \\ \\ d & = & rt & & \\ d & = & 30(5) & & \\ d & = & 150 \text{ km} & & \end{array}$$

13.

Who or What	**Rate**	**Time**	**Equation**
Sam	4	t	$4t$
Sue	6	$t-2$	$6(t-2)$

$$\begin{array}{rcrcr} 4t & = & 6(t & - & 2) \\ 4t & = & 6t & - & 12 \\ -6t & & -6t & & \\ \dfrac{-2t}{-2} & = & \dfrac{-12}{-2} & & \\ \\ t & = & 6 & & \\ t-2 & = & 4 & & \end{array}$$

15.

Who or What	**Rate**	**Time**	**Equation**
MB	8	t	$8t$
CC	16	$t-2$	$16(t-2)$

$$\begin{array}{rcrcr} 8t & = & 16(t & - & 2) \\ 8t & = & 16t & - & 32 \\ -16t & & -16t & & \\ \dfrac{-8t}{-8} & = & \dfrac{-32}{-8} & & \\ \\ t & = & 4 & & \\ t-2 & = & 2 & & \end{array}$$

17.

Who or What	Rate	Time	Equation
M₁	20	t	$20t$
M₂	30	t	$30t$

$$\begin{aligned} 20t + 30t &= 300 \\ \frac{50t}{50} &= \frac{300}{50} \\ t &= 6 \end{aligned}$$

19.

Who or What	Rate	Time	Equation
C_1	r	3	$3r$
C_2	$2r$	3	$3(2r)$

$$\begin{aligned} 3r + 3(2r) &= 72 \\ 3r + 6r &= 72 \\ \frac{9r}{9} &= \frac{72}{9} \\ r &= 8 \\ C_1 &= 8 \text{ km/h} \\ C_2 &= 16 \text{ km/h} \end{aligned}$$

21.

Who or What	Rate	Time	Equation
S_1	55	t	$55t$
S_2	40	$2.5-t$	$40(2.5-t)$

$$\begin{aligned} 55t + 40(2.5 - t) &= 130 \\ 55t + 100 - 40t &= 130 \\ -100 \qquad &\quad -100 \\ \frac{15t}{15} &= \frac{30}{15} \\ t &= 2 \end{aligned}$$

CHAPTER 6.9: RATIONAL FUNCTIONS

Learning Objectives

In this section, you will:

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, x, produced. This is given by the equation $C\left(x\right)=15,000x-0.1{x}^{2}+1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x.

The average cost function, which yields the average cost per item for x items produced, is
$f\left(x\right)=\frac{15,000x-0.1{x}^{2}+1000}{x}$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in (Figure), and notice some of their features.

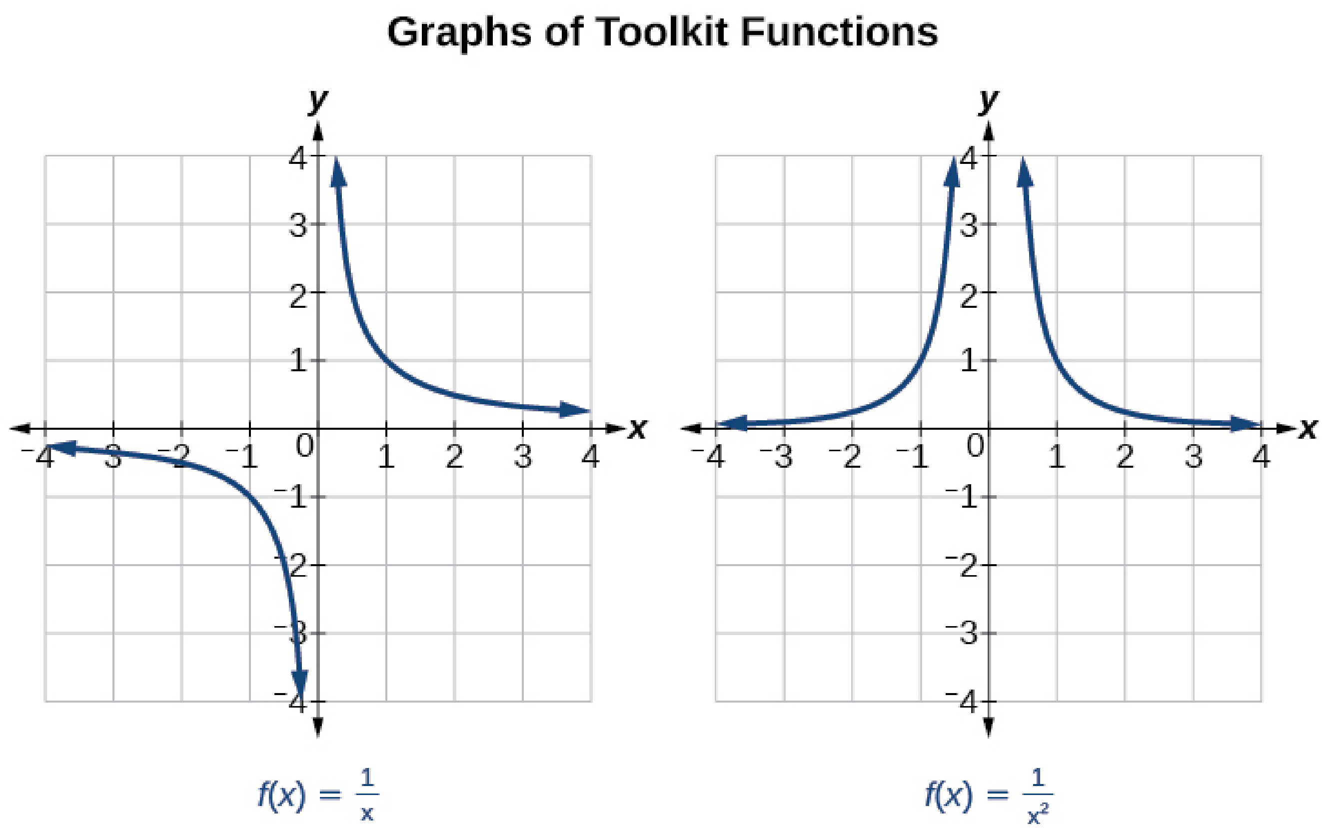


Figure 1.

Several things are apparent if we examine the graph of $f\left(x\right)=\frac{1}{x}$.

1. On the left branch of the graph, the curve approaches the *x*-axis $\left(y=0\right)\text{as}x\to -\infty .$
2. As the graph approaches $x=0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
3. Finally, on the right branch of the graph, the curves approaches the *x*-axis $\left(y=0\right)\text{as}x\to \infty .$

To summarize, we use arrow notation to show that x or $f\left(x\right)$ is approaching a particular value. See (Figure).

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in (Figure).

x	–0.1	–0.01	–0.001	–0.0001
$f(x) = \frac{1}{x}$	–10	–100	–1000	–10,000

We write in arrow notation
as $x \to 0^-, f(x) \to -\infty$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in (Figure).

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

We write in arrow notation
As $x \to 0^+, f(x) \to \infty$.

See (Figure).

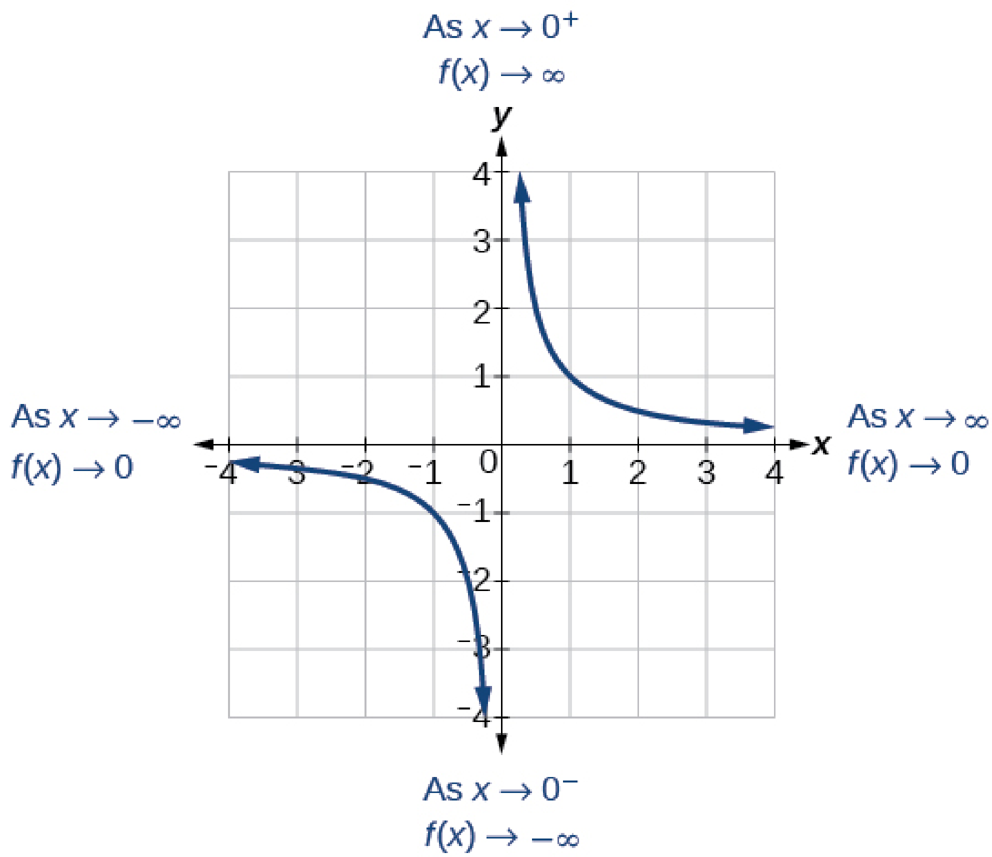


Figure 2.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero. See (Figure).

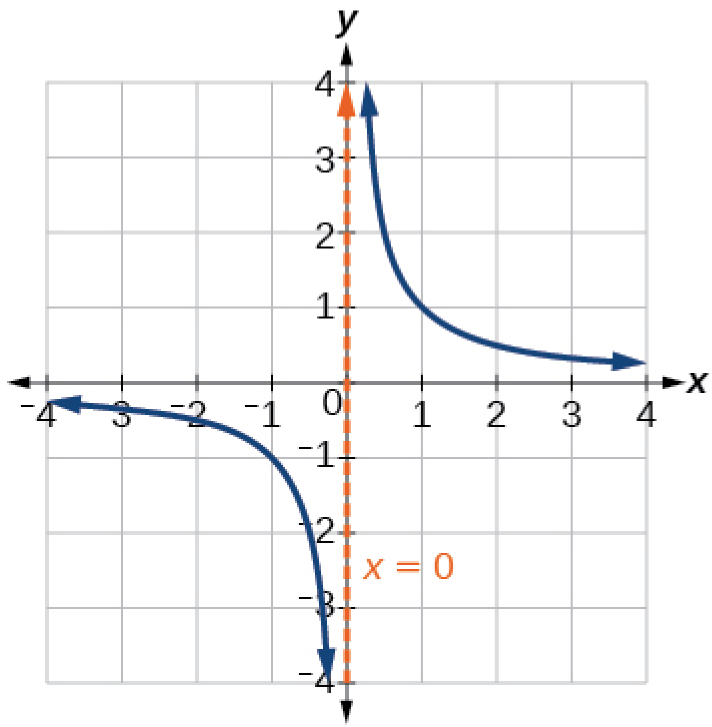


Figure 3.

Vertical Asymptote

A vertical asymptote of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a. We write

$$\text{As } x \to a, f(x) \to \infty, \text{or as } x \to a, f(x) \to -\infty.$$

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0. See (Figure). Symbolically, using arrow notation

$$\text{As } x \to \infty, f(x) \to 0, \text{and as } x \to -\infty, f(x) \to 0.$$

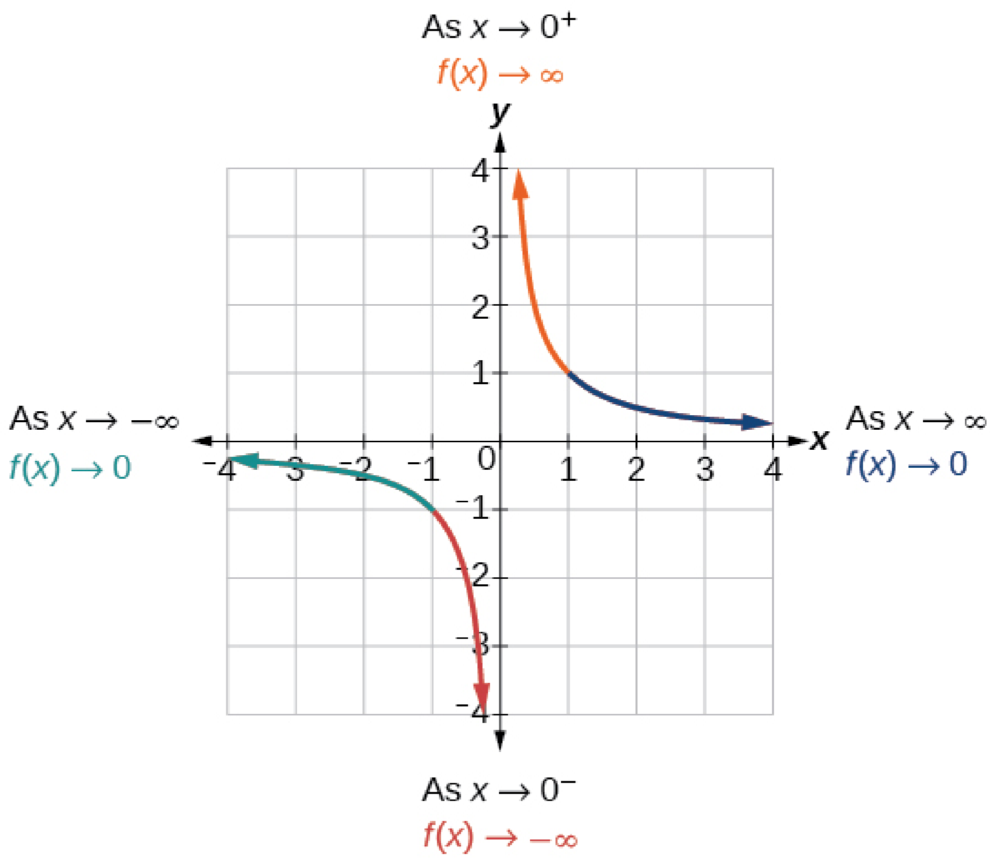


Figure 4.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See (Figure)**.**

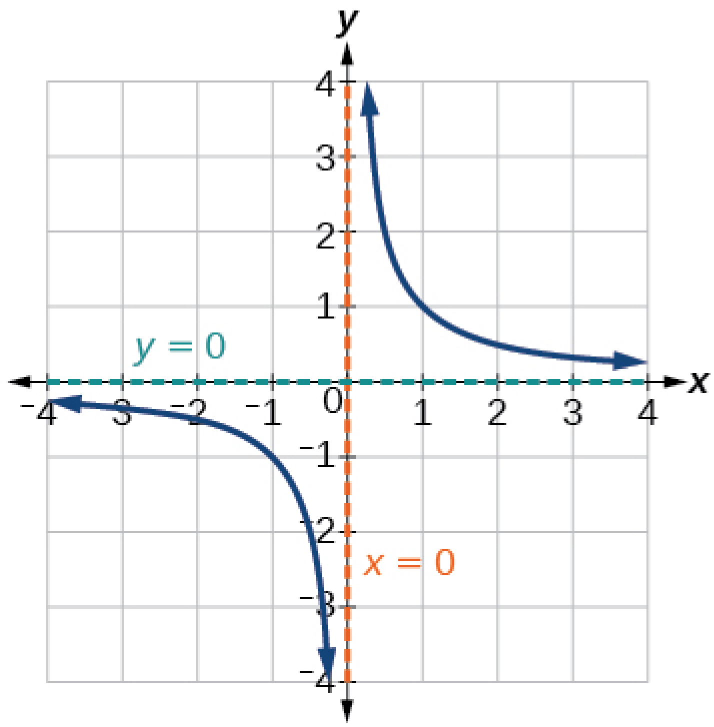


Figure 5.

Horizontal Asymptote

A horizontal asymptote of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty,\ f(x) \to b.$$

Using Arrow Notation

Use arrow notation to describe the end behavior and local behavior of the function graphed in (Figure).

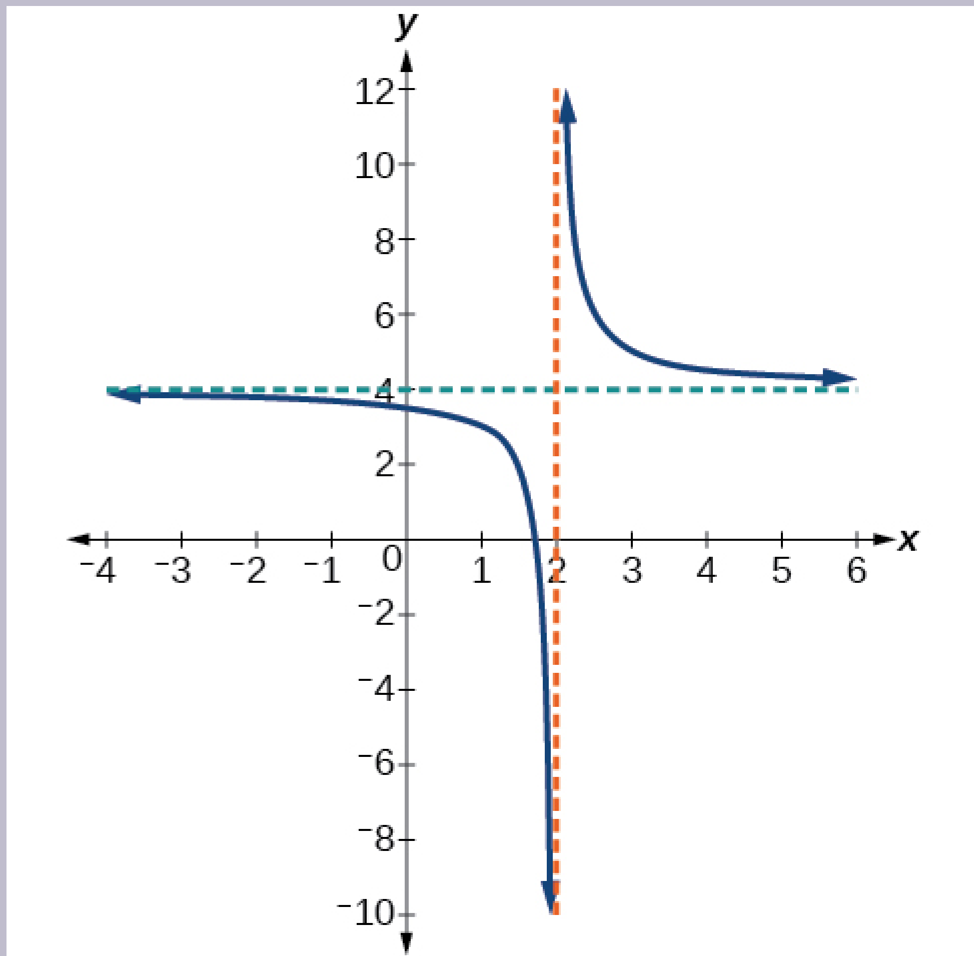


Figure 6.

Show Solution

Notice that the graph is showing a vertical asymptote at $x = 2,$ which tells us that the function is undefined at $x = 2.$

$$\text{As } x \to 2^{-}, f\left(x\right) \to -\infty, \text{ and as } x \to 2^{+}, \ f\left(x\right) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4.$ As the inputs increase without bound, the graph levels off at 4.

$$\text{As } x \to \infty, \ f\left(x\right) \to 4 \text{ and as } x \to -\infty, \ f\left(x\right) \to 4.$$

Try It

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

Show Solution

End behavior: as $x \to \infty, f(x) \to 0;$ Local behavior: as $x \to 0, f(x) \to \infty$ (there are no *x*– or *y*-intercepts)

Using Transformations to Graph a Rational Function

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Show Solution

Shifting the graph left 2 and up 3 would result in the function

$f(x) = \frac{1}{x+2} + 3$

or equivalently, by giving the terms a common denominator,

$f(x) = \frac{3x+7}{x+2}$

The graph of the shifted function is displayed in (Figure).

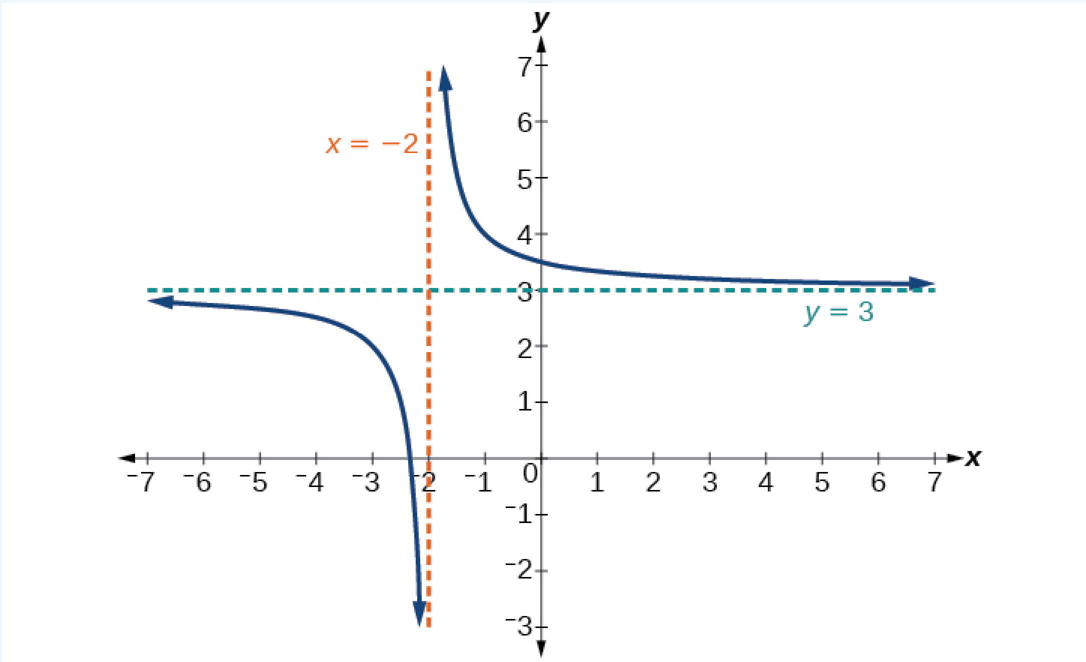


Figure 7.

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^{-}, f\left(x\right) \to -\infty, \text{and as} x \to -2^{+}, f\left(x\right) \to \infty.$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \infty, f\left(x\right) \to 3.$$

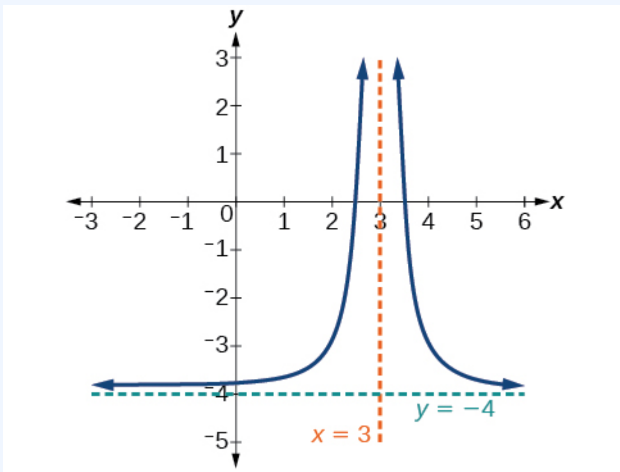
Analysis

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

Try It

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Show Solution



The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3, f\left(x\right) \to \infty$, and as $x \to \infty, f\left(x\right) \to -4.$

The function is $f\left(x\right) = \frac{1}{(x-3)^2} - 4.$

Solving Applied Problems Involving Rational Functions

In (Figure), we shifted a toolkit function in a way that resulted in the function $f\left(x\right) = \frac{3x+7}{x+2}$. This is an example of a rational function. A **rational function** is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Rational Function

A rational function is a function that can be written as the quotient of two polynomial functions $P\left(x\right) \text{and} Q\left(x\right).$

$$f\left(x\right)=\frac{P(x)}{Q(x)}=\frac{a_px^p+a_{p-1}x^{p-1}+...+a_1x+a_0}{b_qx^q+b_{q-1}x^{q-1}+...+b_1x+b_0},Q\left(x\right)\ne 0$$

Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the ratio of sugar to water, in pounds per gallon in the tank after 12 minutes. Is that a greater ratio of sugar to water, in pounds per gallon than at the beginning?

Show Solution

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\begin{array}{rcl}\text{water: }W\left(t\right)&=&100+10t\text{ in gallons}\\ \text{sugar: }S\left(t\right)&=&5+1t\text{ in pounds}\end{array}$$

The ratio of sugar to water, in pounds per gallon, C, will be the ratio of pounds of sugar to gallons of water

$$C\left(t\right)=\frac{5+t}{100+10t}$$

The ratio of sugar to water, in pounds per gallon after 12 minutes is given by evaluating $C\left(t\right)$ at $t=\ 12.$

$$\begin{array}{rcl}C\left(12\right)&=&\frac{5+12}{100+10(12)}\\ &=&\frac{17}{220}\end{array}$$

This means the ratio of sugar to water, in pounds per gallon is 17 pounds of sugar to 220 gallons of water.

At the beginning, the ratio of sugar to water, in pounds per gallon is

$$\begin{aligned} C(0) &= \frac{5+0}{100+10(0)} \\ &= \frac{1}{20} \end{aligned}$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05,$ the ratio of sugar to water, in pounds per gallon is greater after 12 minutes than at the beginning.

Try It

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

Show Solution

$\frac{12}{11}$

Finding the Domains of Rational Functions

A vertical asymptote represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Domain of a Rational Function

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

How To

Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the *x*-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Finding the Domain of a Rational Function

Find the domain of $f\left(x\right)=\frac{x+3}{x^2-9}.$

Show Solution

Begin by setting the denominator equal to zero and solving.

$$\begin{array}{rcl} x^2-9 & = & 0 \\ x^2 & = & 9 \\ x & = & 3 \end{array}$$

The denominator is equal to zero when $x=3.$ The domain of the function is all real numbers except $x=3.$

Analysis

A graph of this function, as shown in (Figure), confirms that the function is not defined when $x = 3.$

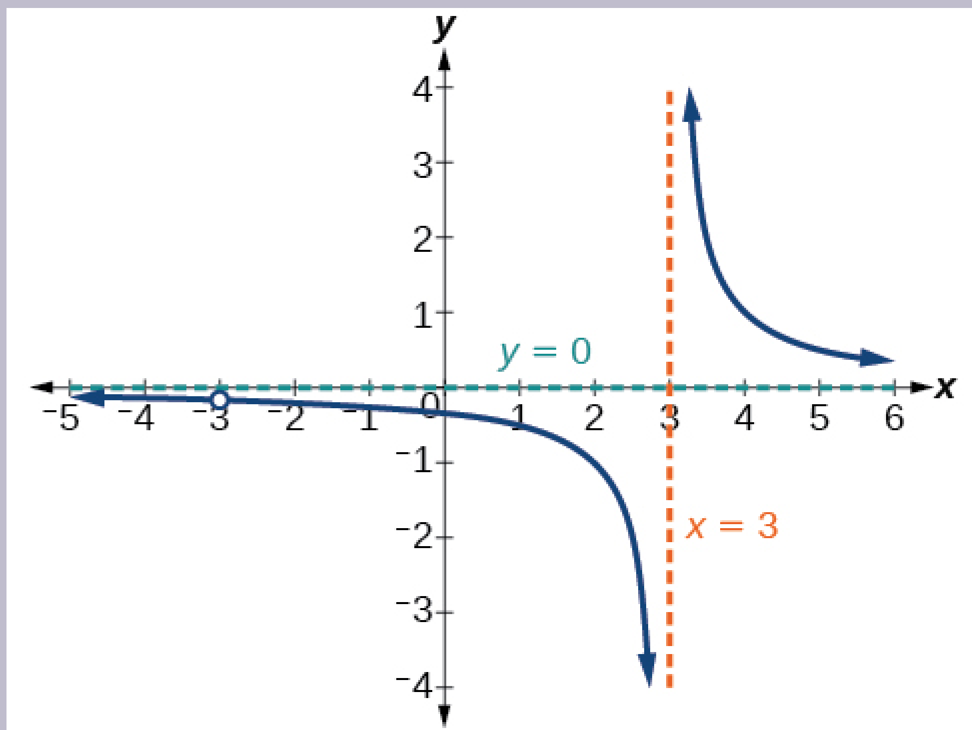


Figure 8.

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3.$ We will discuss these types of holes in greater detail later in this section.

Try It

Find the domain of $f\left(x\right) = \frac{4x}{5(x-1)(x-5)}.$

Show Solution

The domain is all real numbers except $x = 1$ and $x = 5$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

Given a rational function, identify any vertical asymptotes of its graph.

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or “holes.”

Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k\left(x\right) = \frac{5+2x^2}{2-x-x^2}$.

Show Solution

First, factor the numerator and denominator.

$$\begin{aligned} k\left(x\right) &= \frac{5+2x^2}{2-x-x^2} \\ &= \frac{5+2x^2}{(2+x)(1-x)} \end{aligned}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned} \left(2+x\right)\left(1-x\right) &= 0 \\ x &= -2, 1 \end{aligned}$$

Neither $x=-2$ nor $x=1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in (Figure) confirms the location of the two vertical asymptotes.

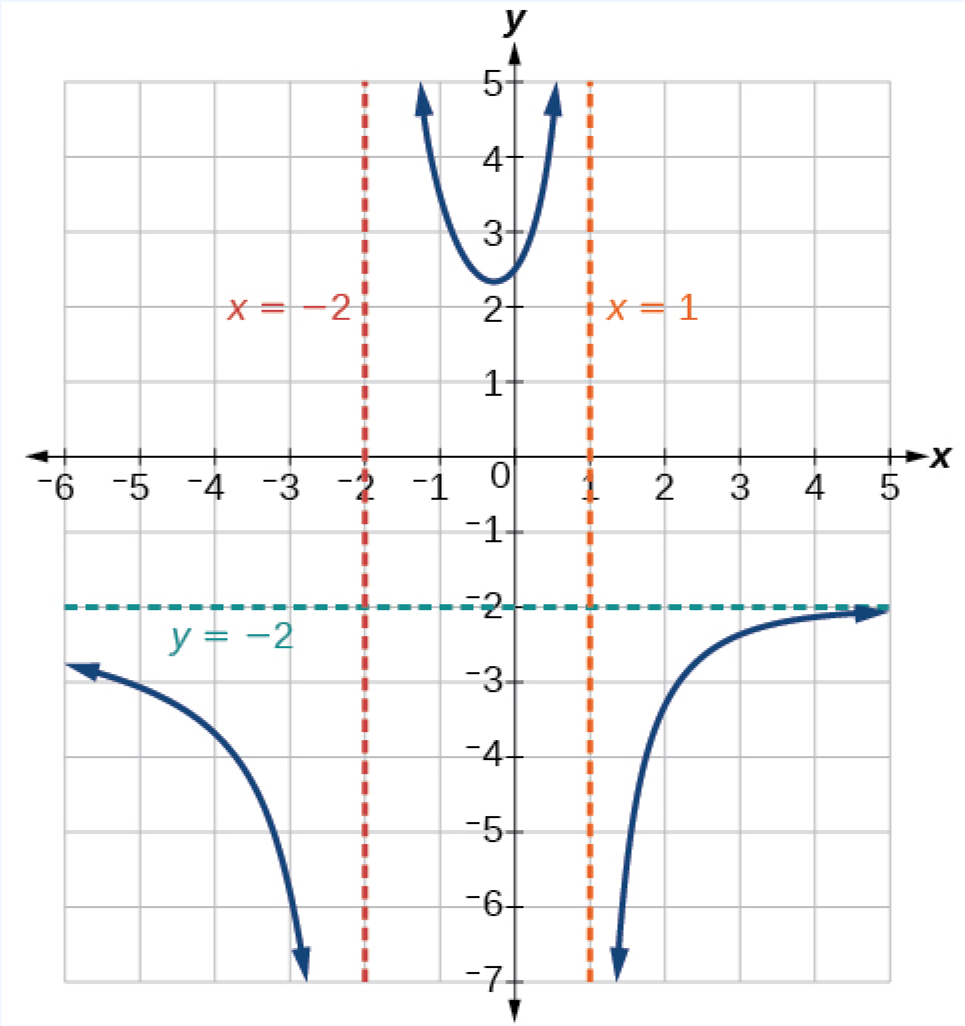


Figure 9.

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a removable discontinuity.

For example, the function $f\left(x\right)=\frac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$f\left(x\right)=\frac{(x+1)(x-1)}{(x+1)(x-3)}$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x=-1$, is the location of the removable discontinuity. Notice also that $x-3$ is not a factor in both the numerator and denominator. The zero of this factor, $x=3$, is the vertical asymptote. See (Figure). [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

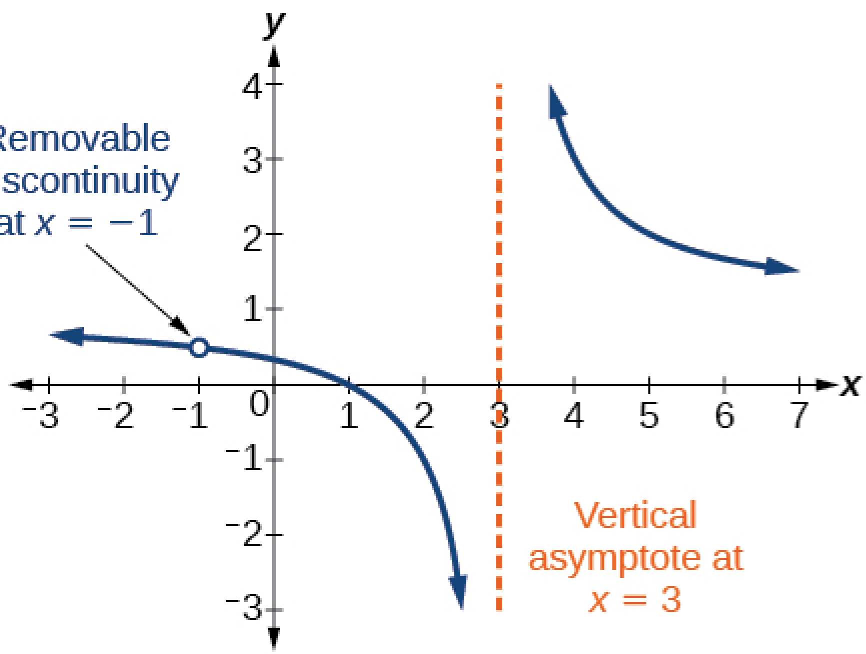


Figure 10.

Removable Discontinuities of Rational Functions

A removable discontinuity occurs in the graph of a rational function at $x=a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k\left(x\right)=\frac{x-2}{x^{2}-4}$.

Show Solution

Factor the numerator and the denominator.

$k\left(x\right)=\frac{x-2}{(x-2)(x+2)}$

Notice that there is a common factor in the numerator and the denominator, $x-2.$ The zero for this factor is $x=2.$ This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2.$ The zero for this factor is $x=-2.$ The vertical asymptote is $x=-2.$ See (Figure).

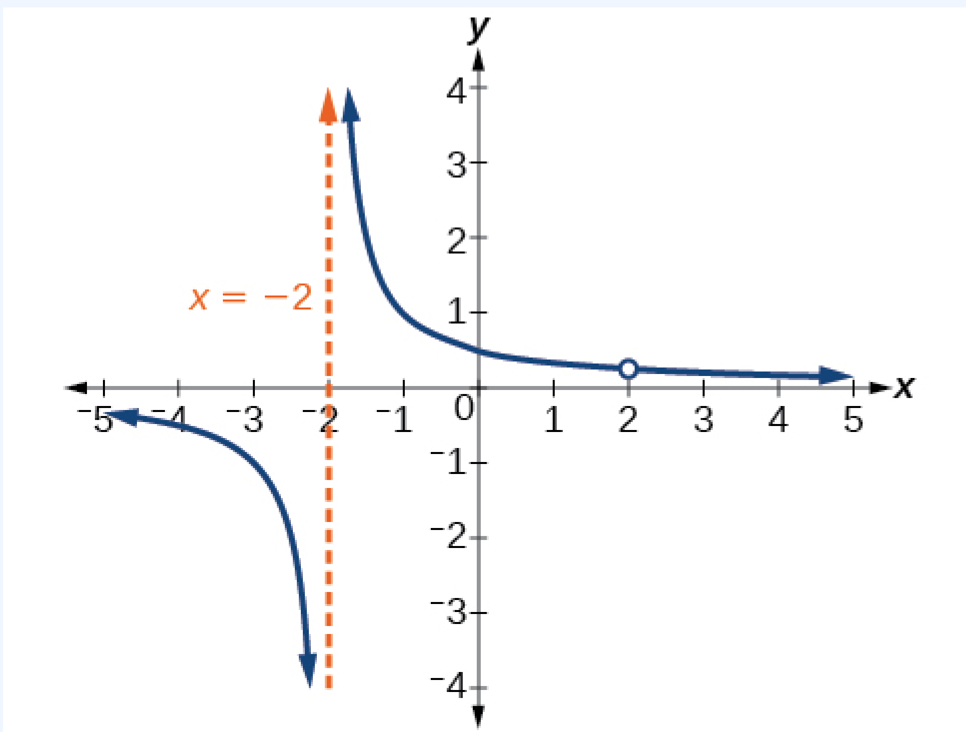


Figure 11.

The graph of this function will have the vertical asymptote at $x=-2,$ but at $x=2$ the graph will have a hole.

Try It

Find the vertical asymptotes and removable discontinuities of the graph of $f\left(x\right)=\frac{x^2-25}{x^3-6x^2+5x}$.

Show Solution

Removable discontinuity at $x=5.$ Vertical asymptotes: $x=0,\ \ x=1.$

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y=0$.

$\text{Example: }\ f\left(x\right)=\frac{4x+2}{x^2+4x-5}$

In this case, the end behavior is $f\left(x\right)\approx\frac{4x}{x^2}=\frac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g\left(x\right)=\frac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y=0$. See (Figure). Note that this graph crosses the horizontal asymptote.

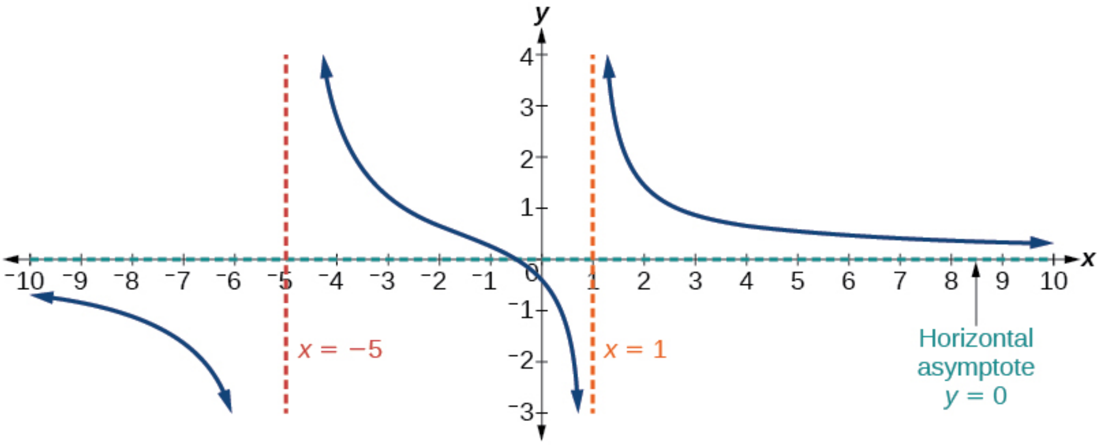


Figure 12. Horizontal asymptote $y = 0$ when
$f(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0$ where degree of $p <$ degree of q.

Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

$$\text{Example: } f(x) = \frac{3x^2-2x+1}{x-1}$$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since f will behave similarly to g, it will approach a line close to $y = 3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\frac{3x^2-2x+1}{x-1}$. The quotient is $3x + 1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x) = 3x + 1$. See (Figure).

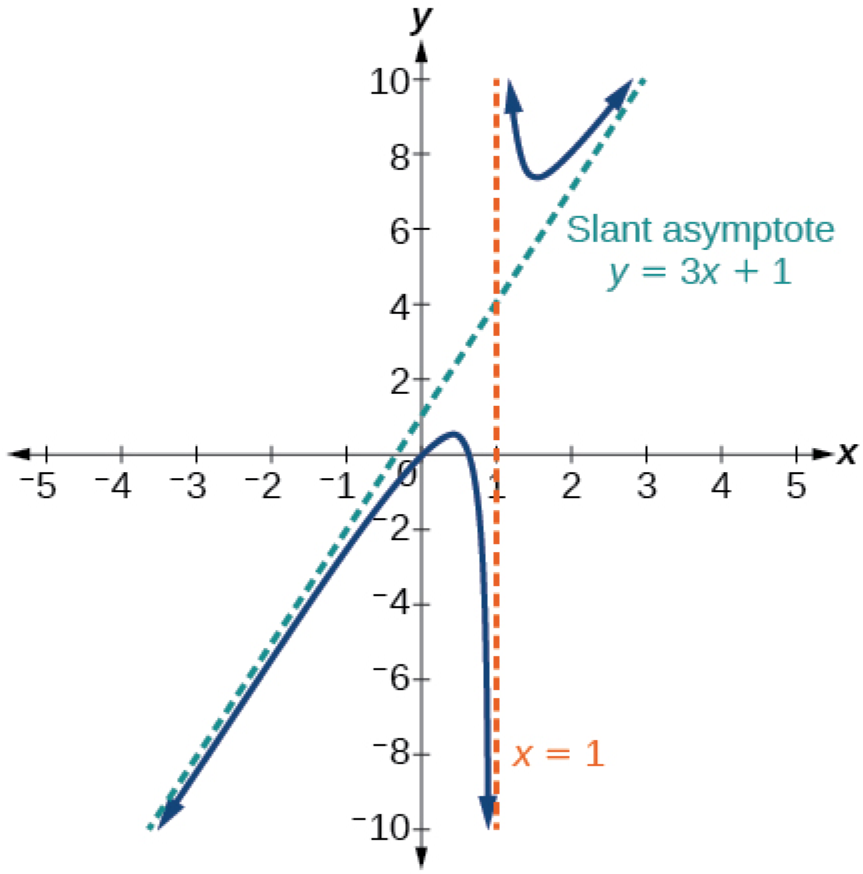


Figure 13. Slant asymptote when $f\left(x\right)=\frac{p(x)}{q(x)},\,q\left(x\right)\ne 0$ where degree of $p>\text{degree of }q\text{ by }1.$

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y=\frac{a_n}{b_n}$, where a_n and b_n are the leading coefficients of $p\left(x\right)$ and $q\left(x\right)$ for $f\left(x\right)=\frac{p(x)}{q(x)},\,q\left(x\right)\ne 0.$

$$\text{Example: }\,f\left(x\right)=\frac{3x^2+2}{x^2+4x-5}$$

In this case, the end behavior is $f\left(x\right)\approx\frac{3x^2}{x^2}=3.$ This tells us that as the inputs grow large, this function will behave like the function $g\left(x\right)=3,$ which is a horizontal line. As $x\to\infty,\,f\left(x\right)\to 3,$ resulting in a horizontal asymptote at $y=3.$ See (Figure). Note that this graph crosses the horizontal asymptote.

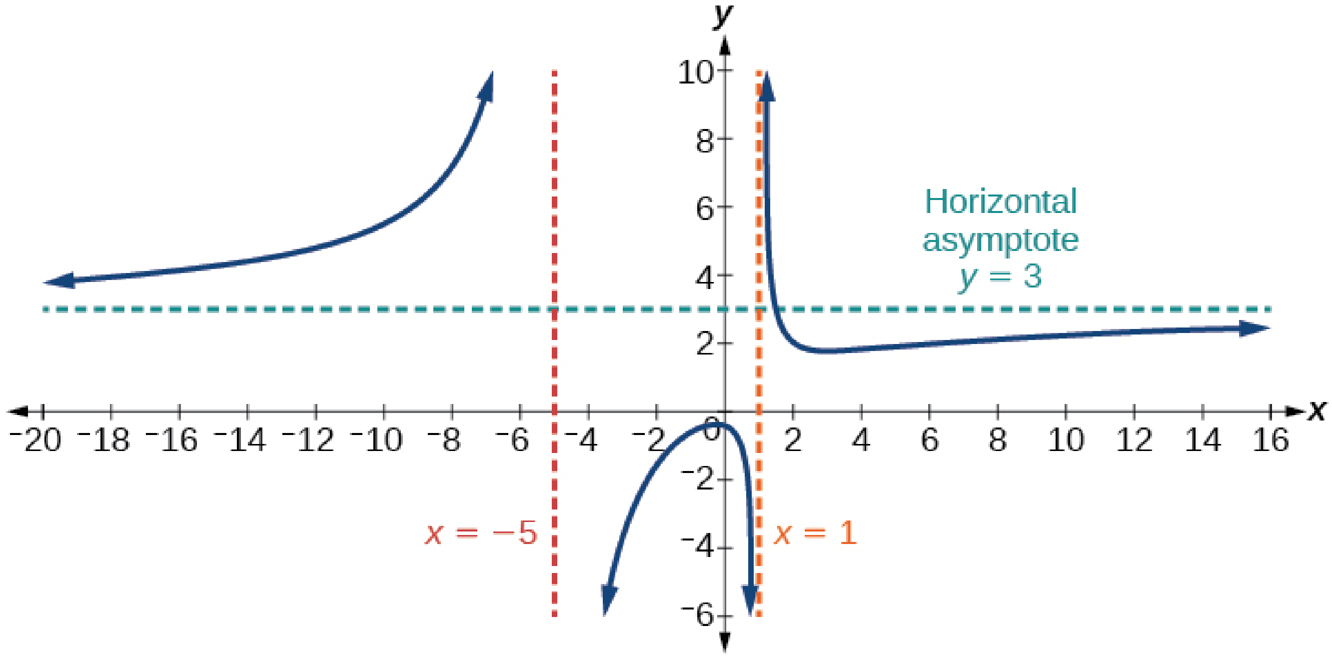


Figure 14. Horizontal asymptote when $f\left(x\right)=\frac{p(x)}{q(x)},\;q\left(x\right)\ne 0$ where degree of $p=$ degree of q.

Notice that, while the graph of a rational function will never cross a vertical asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the end behavior of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f\left(x\right)=\frac{3x^5-x^2}{x+3}$$

with end behavior

$$f\left(x\right)\approx\frac{3x^5}{x}=3x^4,$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x\to\infty, f\left(x\right)\to\infty$$

Horizontal Asymptotes of Rational Functions

The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator *is less than* degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator *is greater than degree of denominator by one*: no horizontal asymptote; slant asymptote.
- Degree of numerator *is equal to* degree of denominator: horizontal asymptote at ratio of leading coefficients.

Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

a. $g\left(x\right) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$
b. $h\left(x\right) = \frac{x^2 - 4x + 1}{x + 2}$
c. $k\left(x\right) = \frac{x^2 + 4x}{x^3 - 8}$

Show Solution

For these solutions, we will use $f\left(x\right) = \frac{p(x)}{q(x)}, q\left(x\right) \neq 0.$

a. $g\left(x\right) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of $p = \text{degree of} q = 3,$ so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3.$
b. $h\left(x\right) = \frac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1.$ Since $p > q$ by 1, there is a slant asymptote found at $\frac{x^2 - 4x + 1}{x + 2}.$

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

The quotient is $x - 6$ and the remainder is 13. There is a slant asymptote at $y = x - 6.$

c. $k\left(x\right)=\frac{x^2+4x}{x^3-8}$: The degree of $p=2$ $<$ degree of $q=3,$ so there is a horizontal asymptote $y=0.$

Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C\left(t\right)=\frac{5+t}{100+10t}.$

Find the horizontal asymptote and interpret it in context of the problem.

Show Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is $t,$ with coefficient 1. In the denominator, the leading term is $10t,$ with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$t\to \infty ,C\left(t\right)\to \frac{1}{10}$

This function will have a horizontal asymptote at $y=\frac{1}{10}.$

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function

$f\left(x\right)=\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Show Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x=1,-2,\text{and }5,$ indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x\to\infty,f\left(x\right)\to 0.$ This function will have a horizontal asymptote at $y=0.$ See (Figure).

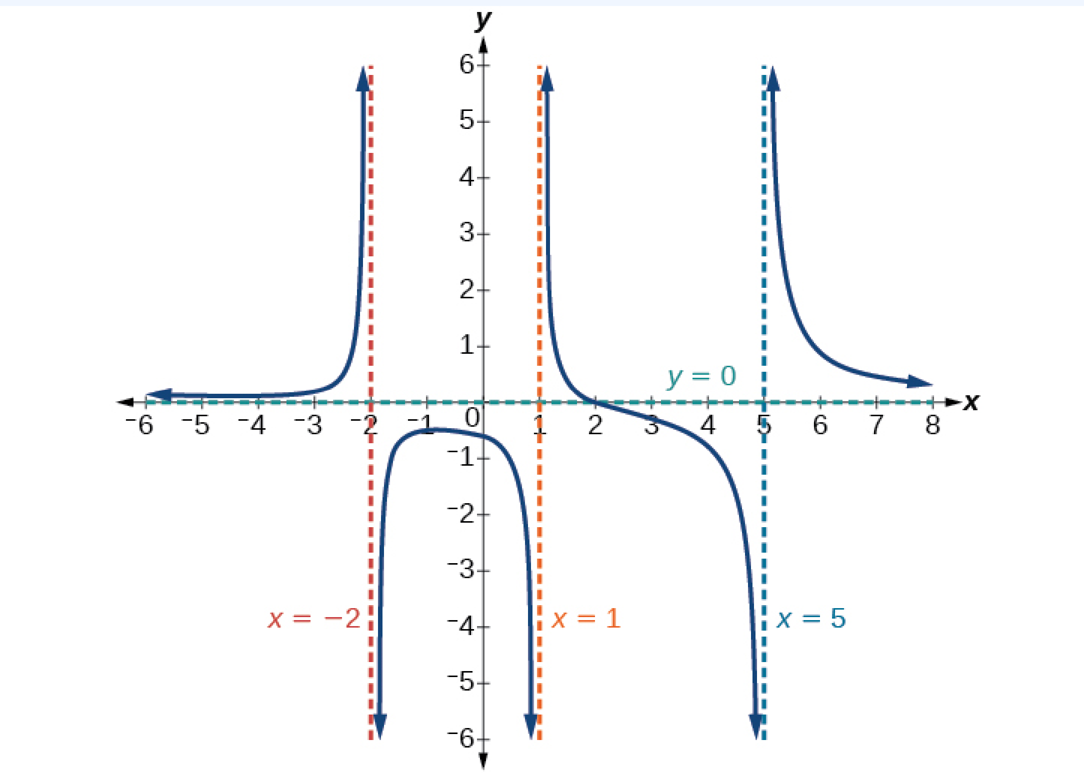


Figure 15.

Try It

Find the vertical and horizontal asymptotes of the function:

$f\left(x\right)=\frac{\left(2x-1\right)\left(2x+1\right)}{\left(x-2\right)\left(x+3\right)}$

Show Solution

Vertical asymptotes at $x=2$ and $x=-3;$ horizontal asymptote at $y=4.$

Intercepts of Rational Functions

A rational function will have a *y*-intercept at $f\left(0\right)$, if the function is defined at zero. A rational function will not have a *y*-intercept if the function is not defined at zero.

Likewise, a rational function will have *x*-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, *x*-intercepts can only occur when the numerator of the rational function is equal to zero.

Finding the Intercepts of a Rational Function

Find the intercepts of $f\left(x\right)=\frac{\left(x-2\right)\left(x+3\right)}{\left(x-1\right)\left(x+2\right)\left(x-5\right)}.$

Show Solution

We can find the *y*-intercept by evaluating the function at zero

$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= \frac{-6}{10} \\ &= -\frac{3}{5} \\ &= -0.6 \end{aligned}$$

The *x*-intercepts will occur when the function is equal to zero:

$$\begin{aligned} 0 &= \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} && \text{This is zero when the numerator is zero.} \\ 0 &= (x-2)(x+3) \\ x &= 2, -3 \end{aligned}$$

The *y*-intercept is $(0, -0.6)$, the *x*-intercepts are $(2, 0)$ and $(-3, 0)$. See (Figure).

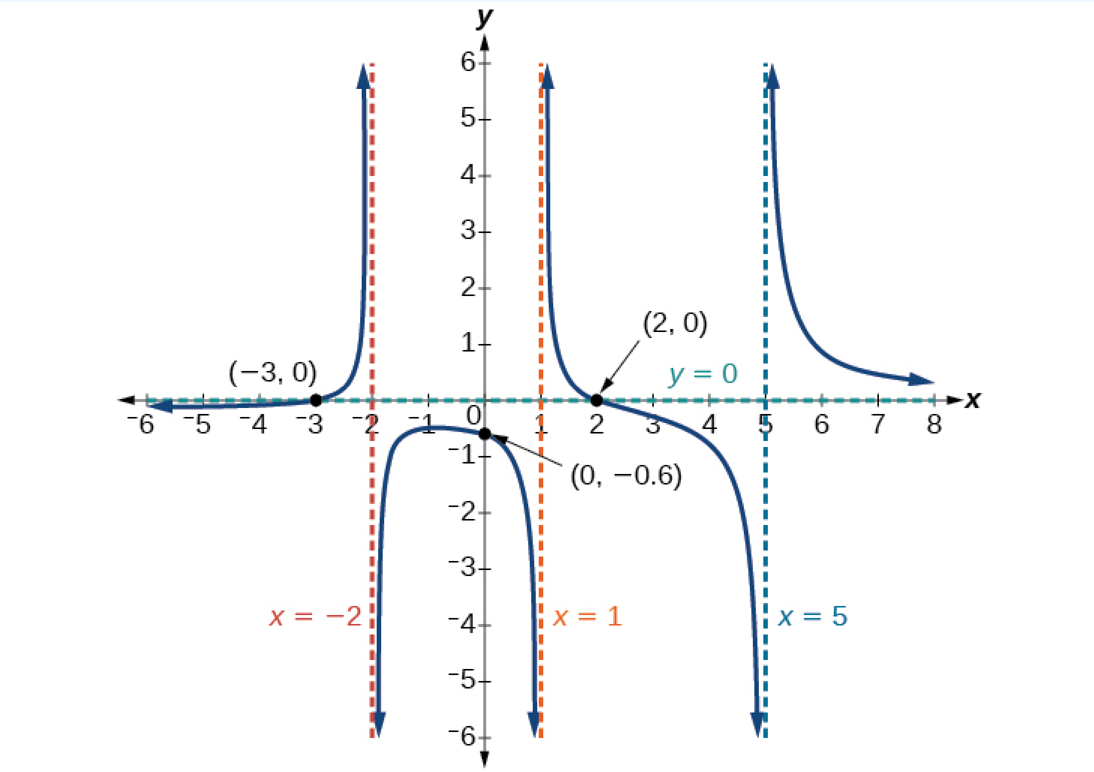


Figure 16.

Try It

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the *x*– and *y*-intercepts and the horizontal and vertical asymptotes.

Show Solution

For the transformed reciprocal squared function, we find the rational form.

$$f\left(x\right)=\frac{1}{{\left(x-3\right)}^{2}}-4=\frac{1-4{\left(x-3\right)}^{2}}{{\left(x-3\right)}^{2}}=\frac{1-4\left({x}^{2}-6x+9\right)}{\left(x-3\right)\left(x-3\right)}=\frac{-4{x}^{2}+24x-35}{{x}^{2}-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x\to \infty ,f\left(x\right)\to -4;\text{so}y=-4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x=3,$ because as $x\to 3,f\left(x\right)\to \infty .$ We then set the numerator equal to 0 and find the *x*-intercepts are at $\left(2.5,0\right)$ and $\left(3.5,0\right)$. Finally, we evaluate the function at 0 and find the *y*-intercept to be at $\left(0,\frac{-35}{9}\right)$.

Graphing Rational Functions

In (Figure), we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See (Figure).

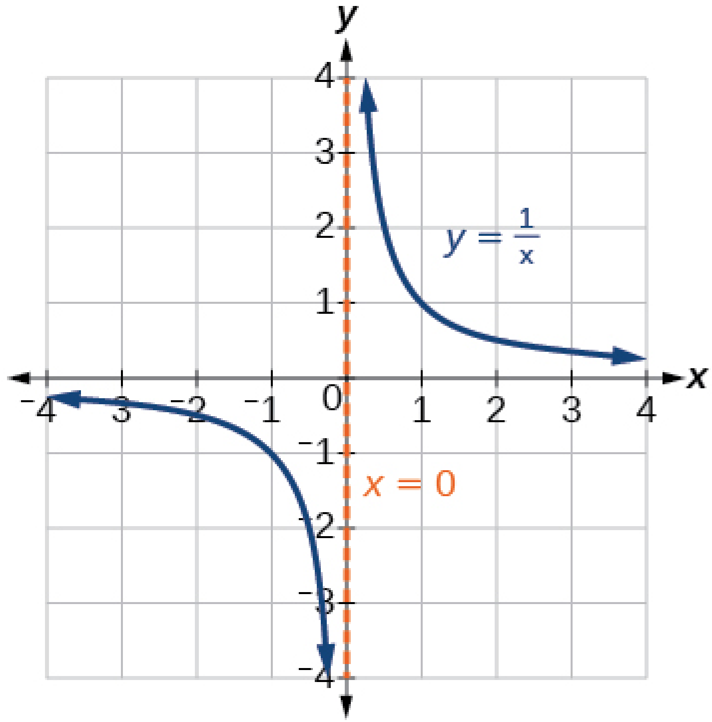


Figure 17.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See (Figure).

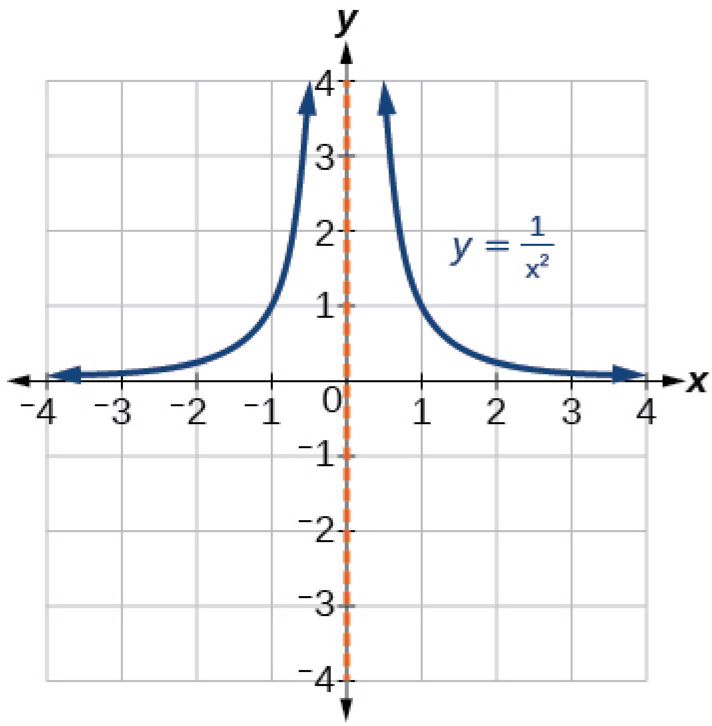


Figure 18.

For example, the graph of $f\left(x\right)=\frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in (Figure).

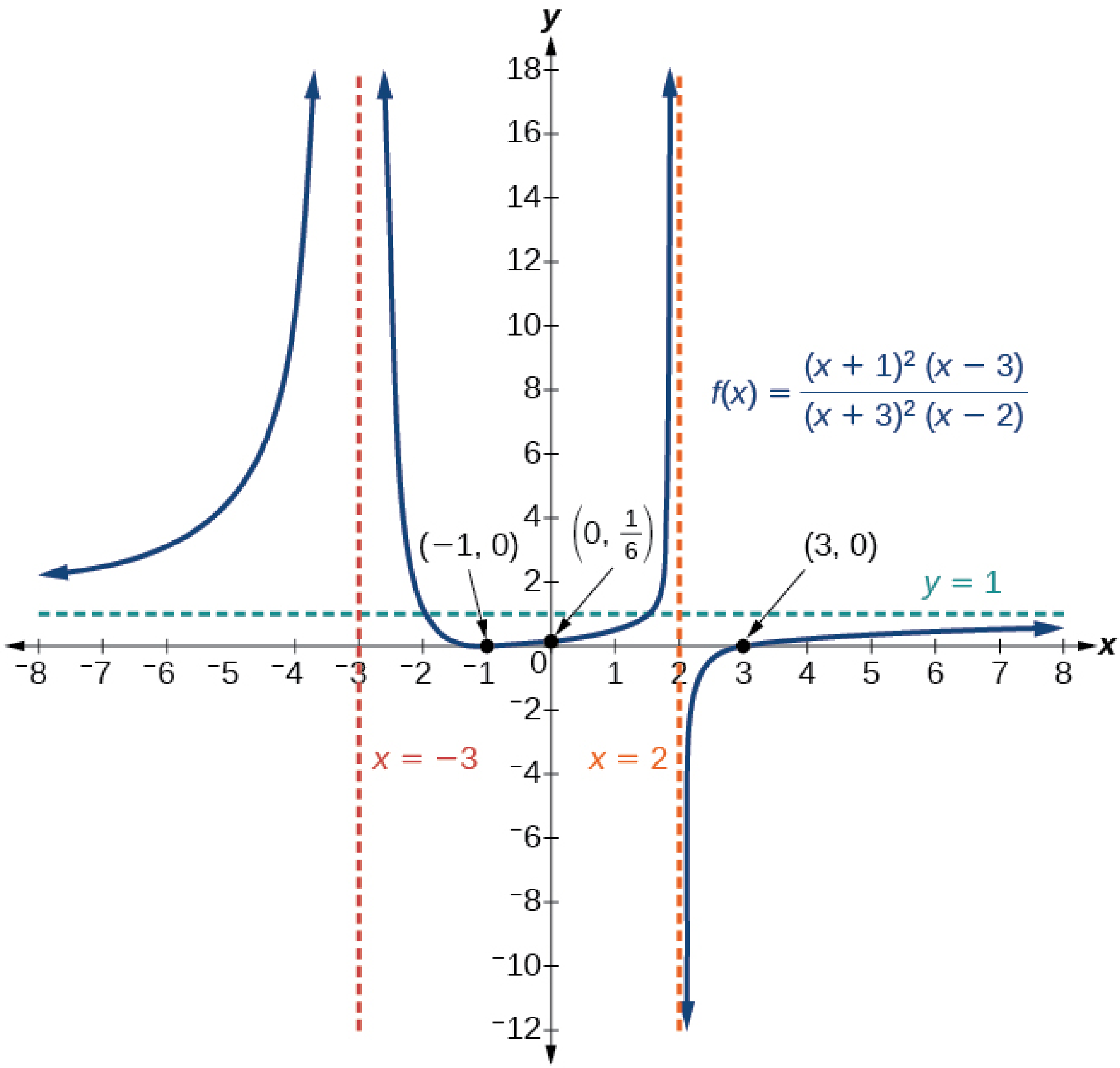


Figure 19.

- At the *x*-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the *x*-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x) = \frac{1}{x}$.

How To

Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the *y*-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the *x*-intercepts.
4. Find the multiplicities of the *x*-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Graphing a Rational Function

Sketch a graph of $f\left(x\right)=\frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Show Solution

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the *y*-intercept:

$$\begin{array}{ccc}f\left(0\right)&=&\frac{(0+2)(0-3)}{(0+1)^2(0-2)}\\ &=&3\end{array}$$

To find the *x*-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find *x*-intercepts at $x = -2$ and $x = 3.$ At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a *y*-intercept at $(0, 3)$ and *x*-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x - 2 = 0,$ giving us vertical asymptotes at $x = -1$ and $x = 2.$

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0.$

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no *x*-intercepts between the vertical asymptotes, and the *y*-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in (Figure).

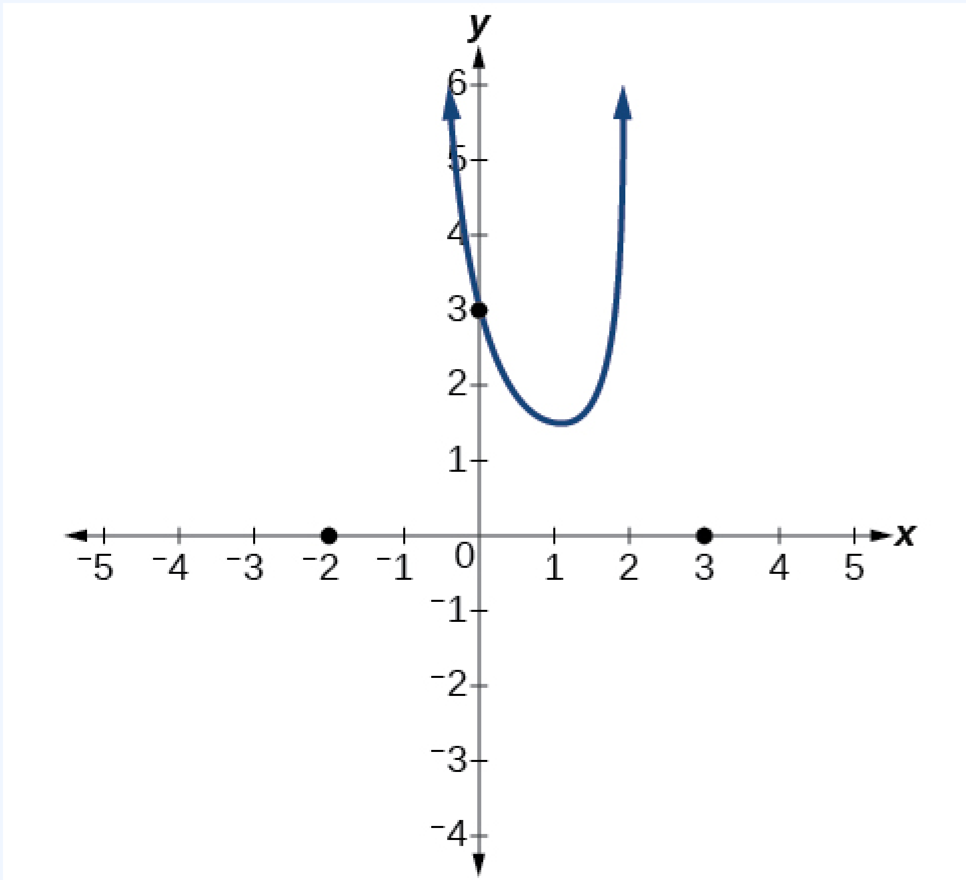


Figure 20.

The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2,$ the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See (Figure). After passing through the *x*-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

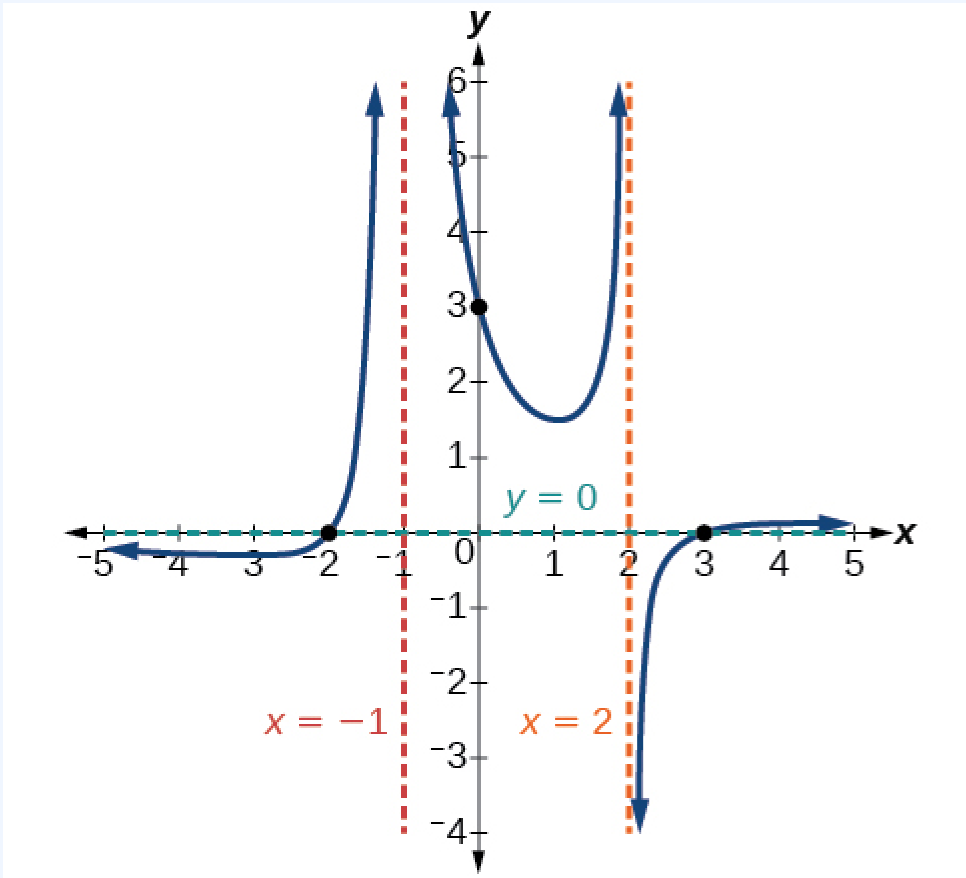


Figure 21.

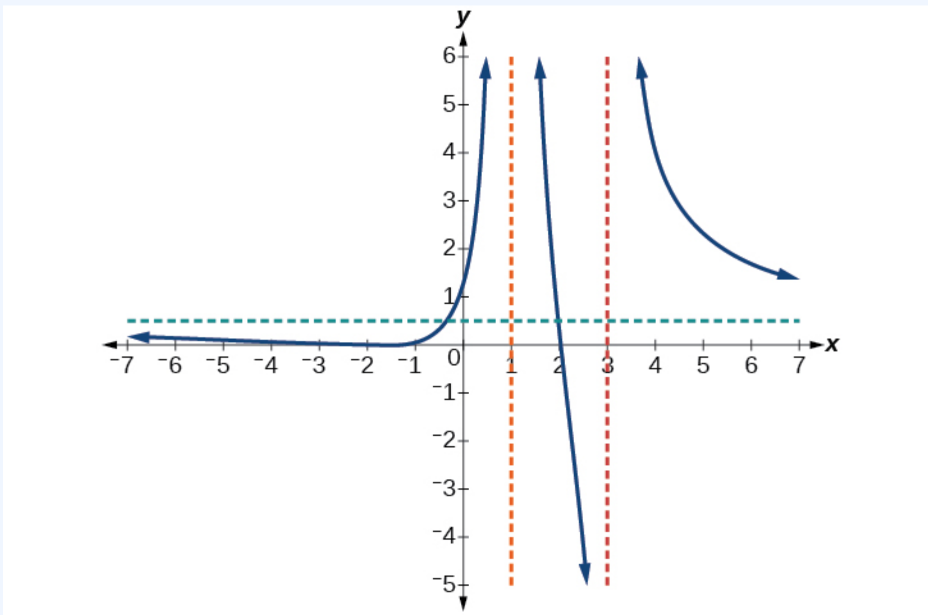
Try It

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Show Solution

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1 \text{and} x = 3$. y-intercept at $(0, \frac{4}{3}.)$

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.



Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an x-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of x-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

If a rational function has *x*-intercepts at $x=x_1,x_2,...,x_n$, vertical asymptotes at $x=v_1,v_2,\ldots,v_m$, and no $x_i=\text{any } v_j$, then the function can be written in the form:

$$f\left(x\right)=a\frac{\left(x-x_1\right)^{p_1}\left(x-x_2\right)^{p_2}\cdots\left(x-x_n\right)^{p_n}}{\left(x-v_1\right)^{q_1}\left(x-v_2\right)^{q_2}\cdots\left(x-v_m\right)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the *x*-intercept or by the horizontal asymptote if it is nonzero.

How To

Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the “simplest” function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in (Figure).

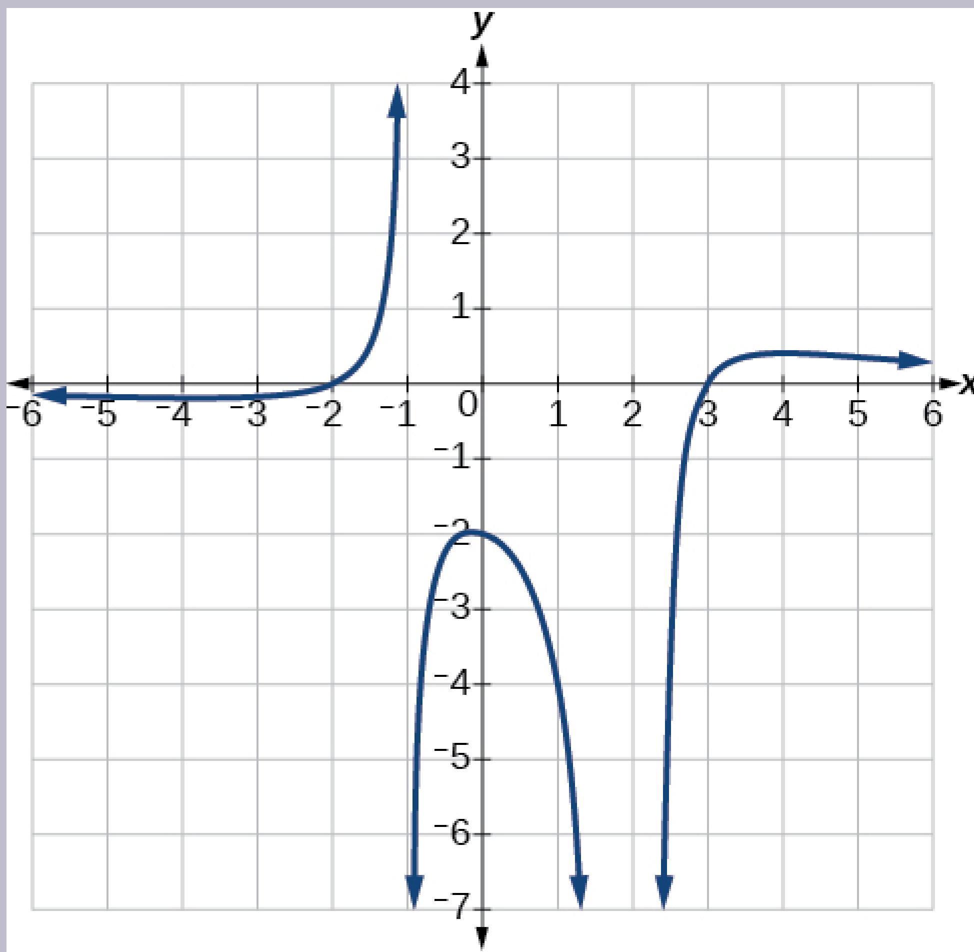


Figure 22.

Show Solution

The graph appears to have *x*-intercepts at $x = -2$ and $x = 3.$ At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See (Figure).

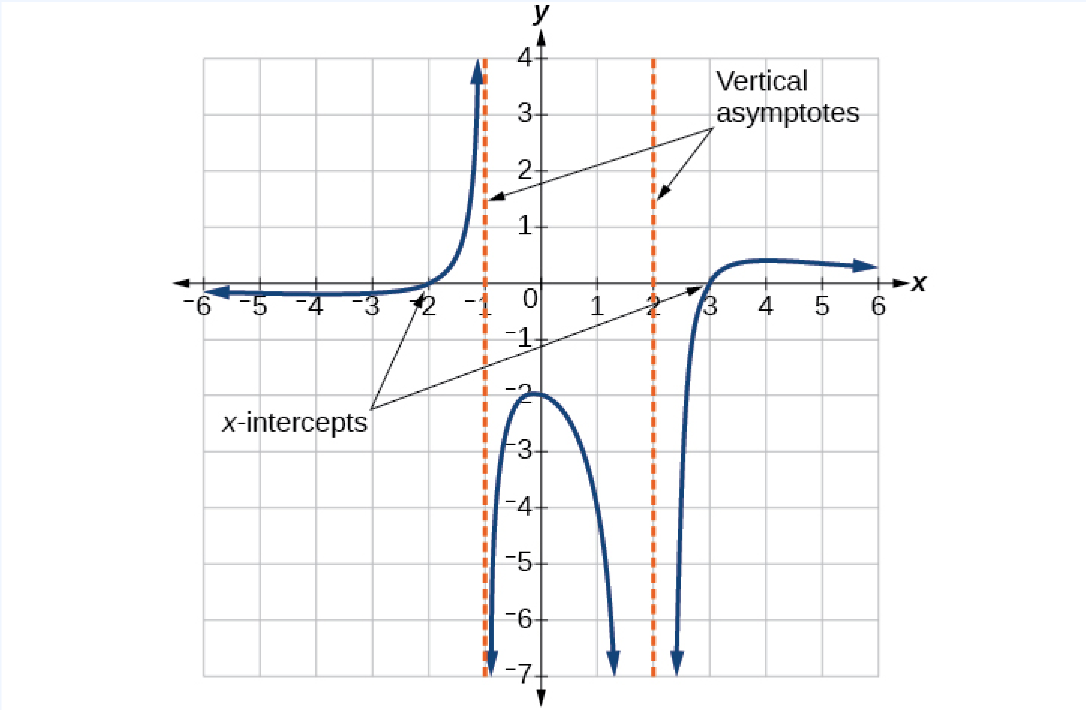


Figure 23.

We can use this information to write a function of the form

$$f\left(x\right)=a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $\left(0,-2\right)$.

$$\begin{array}{ccc}-2&=&a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}\\ -2&=&a\frac{-6}{4}\\ a&=&\frac{-8}{-6}=\frac{4}{3}\end{array}$$

This gives us a final function of $f\left(x\right)=\frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

Access these online resources for additional instruction and practice with rational functions.

- Graphing Rational Functions
- Find the Equation of a Rational Function

- Determining Vertical and Horizontal Asymptotes
- Find the Intercepts, Asymptotes, and Hole of a Rational Function

Key Equations

Rational Function	$f\left(x\right)=\frac{P(x)}{Q(x)}=\frac{a_p x^p+a_{p-1}x^{p-1}+...+a_1 x+a_0}{b_q x^q+b_{q-1}x^{q-1}+...+b_1 x+b_0}, Q\left(x\right)\ne 0$

Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f\left(x\right)=\frac{1}{x}$ and $f\left(x\right)=\frac{1}{x^2}$. See (Figure).
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See (Figure).
- Application problems involving rates and concentrations often involve rational functions. See (Figure).
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See (Figure).
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See (Figure).
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See (Figure).
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See (Figure), (Figure), (Figure), and (Figure).
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See (Figure).
- If a rational function has *x*-intercepts at $x={x}_{1},{x}_{2},...,{x}_{n}$, vertical asymptotes at $x={v}_{1},{v}_{2},\dots ,{v}_{m}$, and no ${x}_{i}=\text{any }{v}_{j}$, then the function can be written in the form

$f\left(x\right)=a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$
See (Figure).

Section Exercises

Verbal

1. What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?

Show Solution
The rational function will be represented by a quotient of polynomial functions.

2. What is the fundamental difference in the graphs of polynomial functions and rational functions?

3. If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?

Show Solution
The numerator and denominator must have a common factor.

4. Can a graph of a rational function have no vertical asymptote? If so, how?

5. Can a graph of a rational function have no x-intercepts? If so, how?

Show Solution
Yes. The numerator of the formula of the functions would have only complex roots and/or factors common to both the numerator and denominator.

Algebraic

For the following exercises, find the domain of the rational functions.

6. $f\left(x\right)=\frac{x-1}{x+2}$

7. $f\left(x\right)=\frac{x+1}{x^2-1}$

Show Solution

$\text{All reals } x\ne -1,1$

8. $f\left(x\right)=\frac{x^2+4}{x^2-2x-8}$

9. $f\left(x\right)=\frac{x^2+4x-3}{x^4-5x^2+4}$

Show Solution

$\text{All reals } x\ne -1,-2,1,2$

For the following exercises, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

10. $f\left(x\right)=\frac{4}{x-1}$

11. $f\left(x\right)=\frac{2}{5x+2}$

Show Solution

V.A. at $x=-\frac{2}{5}$; H.A. at $y=0$; Domain is all reals $x\ne -\frac{2}{5}$

12. $f\left(x\right)=\frac{x}{x^2-9}$

13. $f\left(x\right)=\frac{x}{x^2+5x-36}$

Show Solution

V.A. at $x = 4, -9$; H.A. at $y = 0$; Domain is all reals $x \neq 4, -9$

14. $f\left(x\right) = \frac{3+x}{x^3-27}$

15. $f\left(x\right) = \frac{3x-4}{x^3-16x}$

Show Solution

V.A. at $x = 0, 4, -4$; H.A. at $y = 0$; Domain is all reals $x \neq 0, 4, -4$

16. $f\left(x\right) = \frac{x^2-1}{x^3+9x^2+14x}$

17. $f\left(x\right) = \frac{x+5}{x^2-25}$

Show Solution

V.A. at $x = -5$; H.A. at $y = 0$; Domain is all reals $x \neq 5, -5$

18. $f\left(x\right) = \frac{x-4}{x-6}$

19. $f\left(x\right) = \frac{4-2x}{3x-1}$

Show Solution

V.A. at $x = \frac{1}{3}$; H.A. at $y = -\frac{2}{3}$; Domain is all reals $x \neq \frac{1}{3}$.

For the following exercises, find the *x*– and *y*-intercepts for the functions.

20. $f\left(x\right) = \frac{x+5}{x^2+4}$

21. $f\left(x\right)=\frac{x}{x^2-x}$

Show Solution

none

22. $f\left(x\right)=\frac{x^2+8x+7}{x^2+11x+30}$

23. $f\left(x\right)=\frac{x^2+x+6}{x^2-10x+24}$

Show Solution

$x\text{-intercepts none, }y\text{-intercept }\left(0,\frac{1}{4}\right)$

24. $f\left(x\right)=\frac{94-2x^2}{3x^2-12}$

For the following exercises, describe the local and end behavior of the functions.

25. $f\left(x\right)=\frac{x}{2x+1}$

Show Solution

Local behavior: $x\to-\frac{1}{2}^{+},f\left(x\right)\to-\infty,x\to-\frac{1}{2}^{-},f\left(x\right)\to\infty$

End behavior: $x\to\infty,f\left(x\right)\to\frac{1}{2}$

26. $f\left(x\right)=\frac{2x}{x-6}$

27. $f\left(x\right)=\frac{-2x}{x-6}$

Show Solution

Local behavior: $x \to 6^+, f(x) \to -\infty, x \to 6^-, f(x) \to \infty$, End behavior: $x \to \infty, f(x) \to -2$

28. $f(x) = \frac{x^2-4x+3}{x^2-4x-5}$

29. $f(x) = \frac{2x^2-32}{6x^2+13x-5}$

Show Solution

Local behavior: $x \to -\frac{1}{3}^+, f(x) \to \infty, x \to -\frac{1}{3}^-,$
$f(x) \to -\infty, x \to \frac{5}{2}^-, f(x) \to \infty, x \to \frac{5}{2}^+, f(x) \to -\infty$
End behavior: $x \to \infty,\ f(x) \to \frac{1}{3}$

For the following exercises, find the slant asymptote of the functions.

30. $f(x) = \frac{24x^2+6x}{2x+1}$

31. $f(x) = \frac{4x^2-10}{2x-4}$

Show Solution

$y = 2x + 4$

32. $f(x) = \frac{81x^2-18}{3x-2}$

33. $f(x) = \frac{6x^3-5x}{3x^2+4}$

Show Solution

$y = 2x$

34. $f\left(x\right) = \frac{x^2+5x+4}{x-1}$

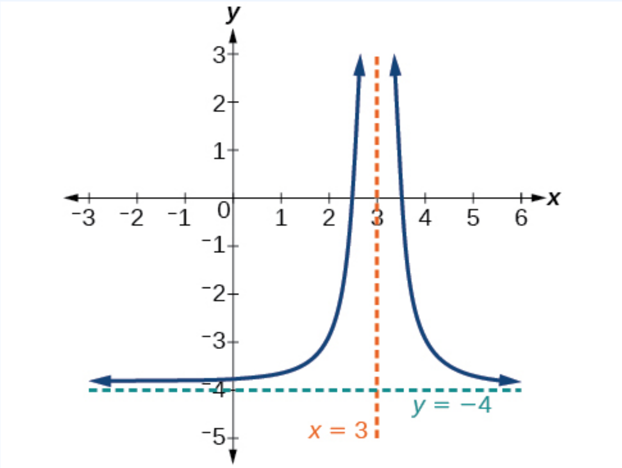
Graphical

For the following exercises, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

35. The reciprocal function shifted up two units.

Show Solution

$V.A.\ x = 0, H.A.\ y = 2$

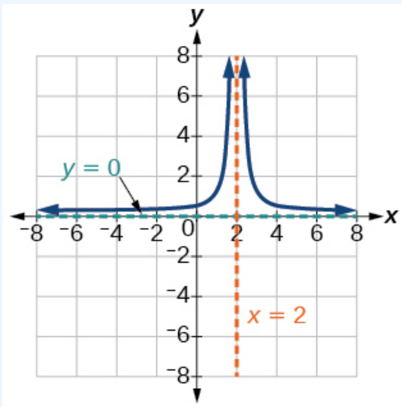


36. The reciprocal function shifted down one unit and left three units.

37. The reciprocal squared function shifted to the right 2 units.

Show Solution

$V.A.\ x=2,\ \ H.A.\ y=0$



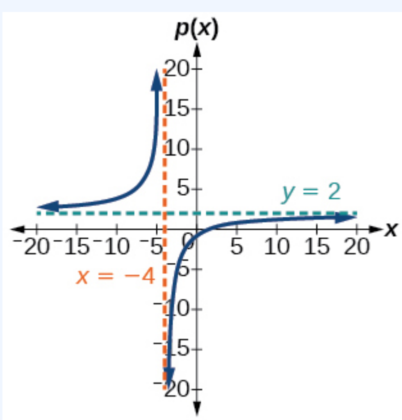
38. The reciprocal squared function shifted down 2 units and right 1 unit.

For the following exercises, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

39. $p\left(x\right)=\frac{2x-3}{x+4}$

Show Solution

$V.A.\ x=-4,\ \ H.A.\ y=2;\left(\frac{3}{2},0\right);\left(0,-\frac{3}{4}\right)$

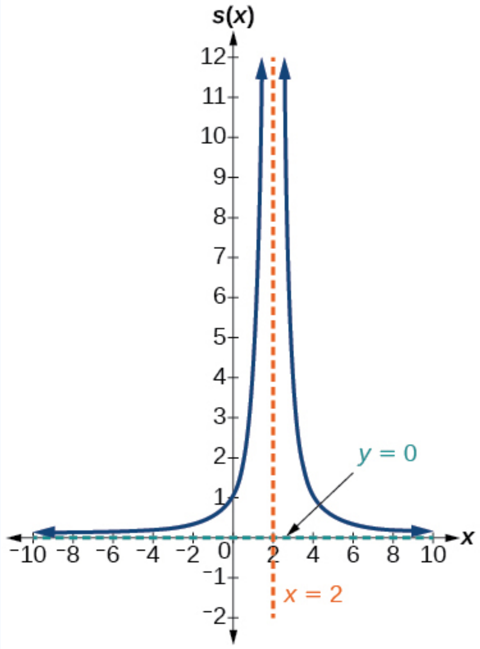


40. $q\left(x\right)=\frac{x-5}{3x-1}$

41. $s\left(x\right)=\frac{4}{(x-2)^2}$

Show Solution

$V.A.\ x=2,\ \ H.A.\ y=0,\ \ (0,1)$

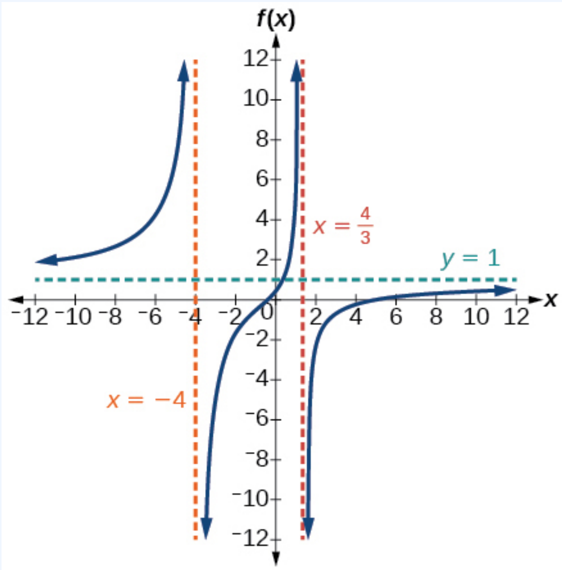


42. $r\left(x\right)=\frac{5}{(x+1)^2}$

43. $f\left(x\right)=\frac{3x^2-14x-5}{3x^2+8x-16}$

Show Solution

$V.A.\ x=-4,\ \ x=\frac{4}{3},\ \ H.A.\ y=1;\left(5,0\right);\left(-\frac{1}{3},0\right);\left(0,\frac{5}{16}\right)$

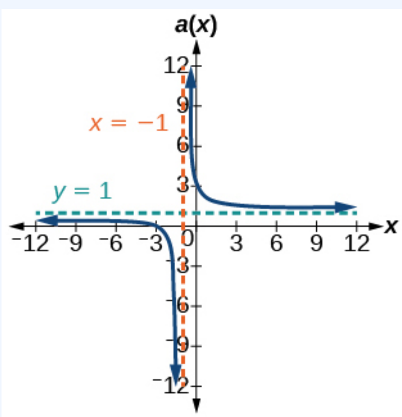


44. $g\left(x\right)=\frac{2x^2+7x-15}{3x^2-14x+15}$

45. $a\left(x\right)=\frac{x^2+2x-3}{x^2-1}$

Show Solution

$V.A.\ x=-1,\ \ H.A.\ y=1;\left(-3,0\right);\left(0,3\right)$

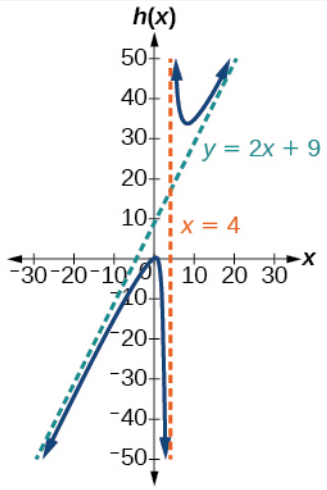


46. $b\left(x\right)=\frac{x^2-x-6}{x^2-4}$

47. $h\left(x\right)=\frac{2x^2+x-1}{x-4}$

Show Solution

$V.A.\ x=4,\ \ S.A.\ y=2x+9;\left(-1,0\right);\left(\frac{1}{2},0\right);\left(0,\frac{1}{4}\right)$

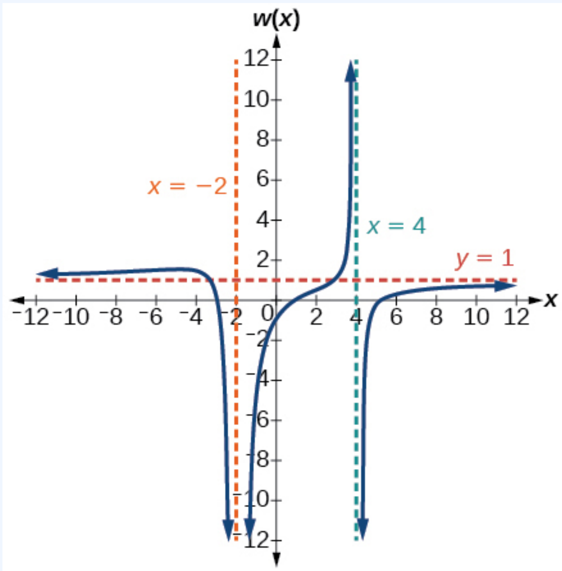


48. $k\left(x\right)=\frac{2x^2-3x-20}{x-5}$

49. $w\left(x\right)=\frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

Show Solution

$V.A.\ x=-2,\ \ x=4,\ \ H.A.\ y=1,\left(1,0\right);\left(5,0\right);\left(-3,0\right);\left(0,-\frac{15}{16}\right)$



50. $z\left(x\right)=\frac{{\left(x+2\right)}^{2}\left(x-5\right)}{\left(x-3\right)\left(x+1\right)\left(x+4\right)}$

For the following exercises, write an equation for a rational function with the given characteristics.

51. Vertical asymptotes at $x=5$ and $x=-5$, *x*-intercepts at $\left(2,0\right)$ and $\left(-1,0\right)$, *y*-intercept at $\left(0,4\right)$

Show Solution

$y=50\frac{{x}^{2}-x-2}{{x}^{2}-25}$

52. Vertical asymptotes at $x=-4$ and $x=-1$, *x*-intercepts at $\left(1,0\right)$ and $\left(5,0\right)$, *y*-intercept at $\left(0,7\right)$

53. Vertical asymptotes at $x=-4$ and $x=-5$, *x*-intercepts at $\left(4,0\right)$ and $\left(-6,0\right)$, Horizontal asymptote at $y=7$

Show Solution

$y=7\frac{x^2+2x-24}{x^2+9x+20}$

54. Vertical asymptotes at $x=-3$ and $x=6$, x-intercepts at $(-2,0)$ and $(1,0)$, Horizontal asymptote at $y=-2$

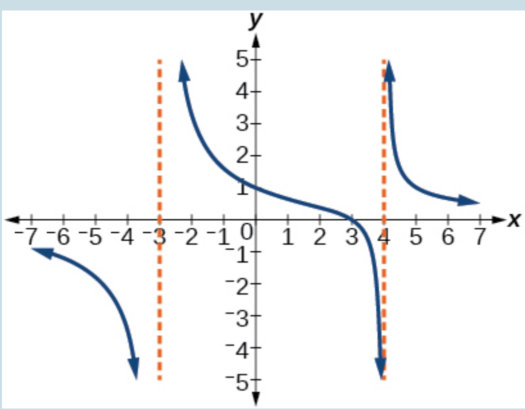
55. Vertical asymptote at $x=-1$, Double zero at $x=2$, y-intercept at $(0,2)$

Show Solution

$y=\frac{1}{2}\frac{x^2-4x+4}{x+1}$

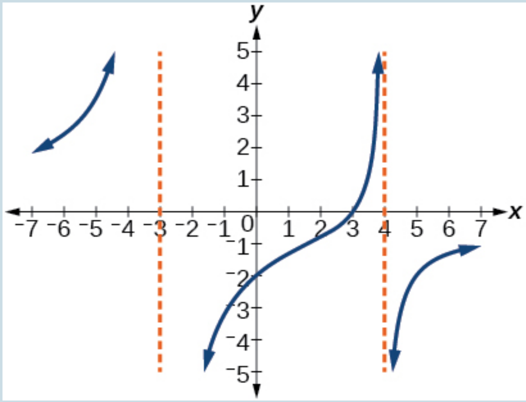
56. Vertical asymptote at $x=3$, Double zero at $x=1$, y-intercept at $(0,4)$

For the following exercises, use the graphs to write an equation for the function.

57. 

Show Solution

$y=4\frac{x-3}{x^2-x-12}$

58. 

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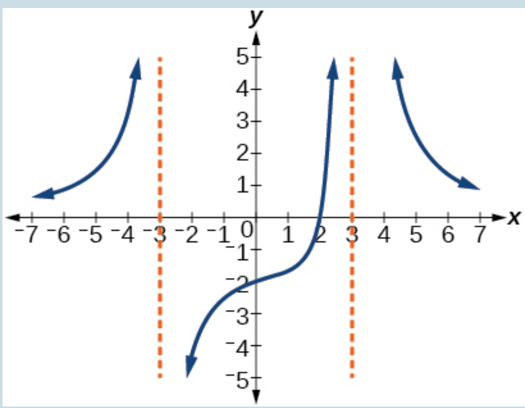
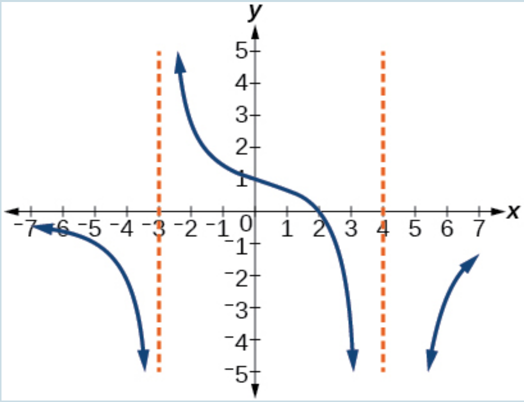
59. 

Show Solution

$y = -9\frac{x-2}{x^2-9}$

60. 

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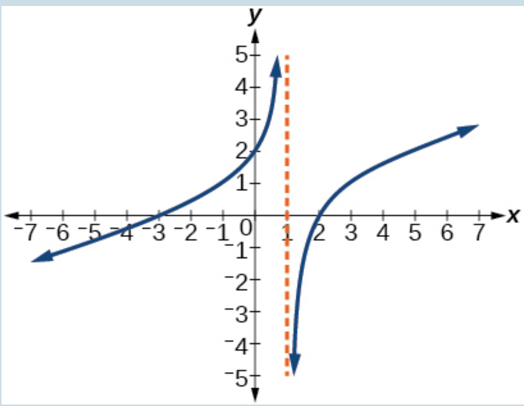
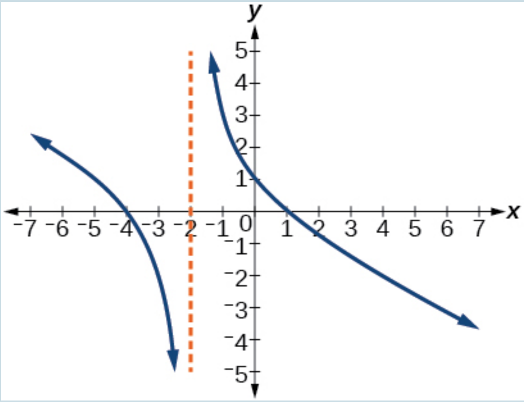
61. 

Show Solution

$y = \frac{1}{3}\frac{x^2+x-6}{x-1}$

62. 

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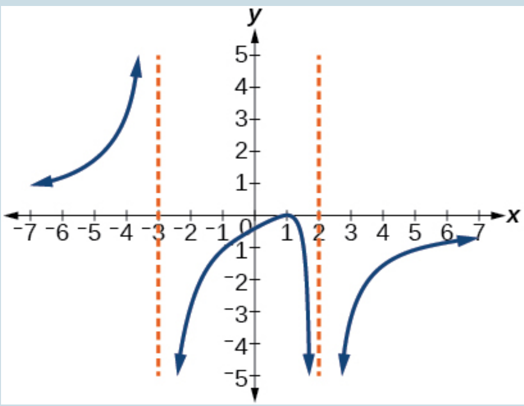
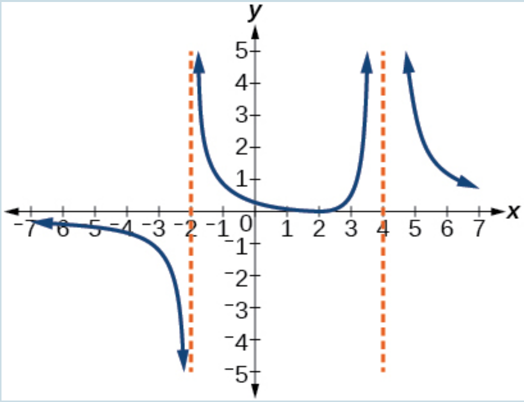
63. 

Show Solution

$y = -6\frac{(x-1)^2}{(x+3)(x-2)^2}$

64. 

Numeric

For the following exercises, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote

65. $f\left(x\right)=\frac{1}{x-2}$

Show Solution

x	2.01	2.001	2.0001	1.99	1.999
y	100	1,000	10,000	–100	–1,000

x	**10**	**100**	**1,000**	**10,000**	**100,000**
y	.125	.0102	.001	.0001	.00001

Vertical asymptote $x=2,$ Horizontal asymptote $y=0$

66. $f\left(x\right)=\frac{x}{x-3}$

67. $f\left(x\right)=\frac{2x}{x+4}$

Show Solution

x	–4.1	–4.01	–4.001	–3.99	–3.999
y	82	802	8,002	–798	–7998

x	10	100	1,000	10,000	100,000
y	1.4286	1.9331	1.992	1.9992	1.999992

Vertical asymptote $x = -4,$ Horizontal asymptote $y = 2$

68. $f\left(x\right) = \frac{2x}{\left(x-3\right)^2}$

69. $f\left(x\right) = \frac{x^2}{x^2+2x+1}$

Show Solution

x	–.9	–.99	–.999	–1.1	–1.01
y	81	9,801	998,001	121	10,201

x	10	100	1,000	10,000	100,000
y	.82645	.9803	.998	.9998	

Vertical asymptote $x = -1,$ Horizontal asymptote $y = 1$

Technology

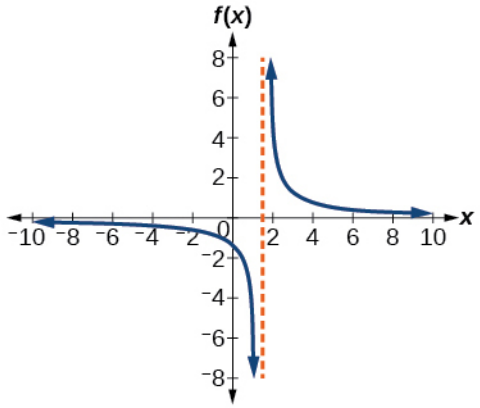
For the following exercises, use a calculator to graph $f\left(x\right)$. Use the graph to solve $f\left(x\right) > 0.$

70. $f\left(x\right) = \frac{2}{x+1}$

71. $f\left(x\right)=\frac{4}{2x-3}$

Show Solution

$\left(\frac{3}{2},\infty\right)$

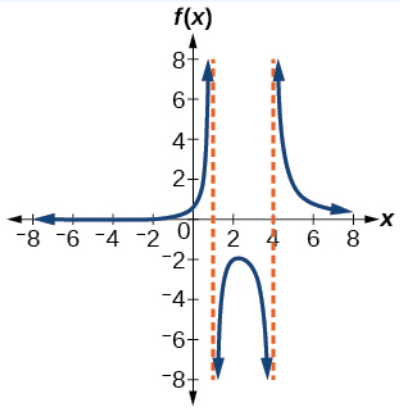


72. $f\left(x\right)=\frac{2}{\left(x-1\right)\left(x+2\right)}$

73. $f\left(x\right)=\frac{x+2}{\left(x-1\right)\left(x-4\right)}$

Show Solution

$\left(-2,1\right)\cup\left(4,\infty\right)$



74. $f\left(x\right)=\frac{{\left(x+3\right)}^{2}}{{\left(x-1\right)}^{2}\left(x+1\right)}$

Extensions

For the following exercises, identify the removable discontinuity.

75. $f\left(x\right)=\frac{{x}^{2}-4}{x-2}$

Show Solution
$\left(2,4\right)$

76. $f\left(x\right)=\frac{{x}^{3}+1}{x+1}$

77. $f\left(x\right)=\frac{{x}^{2}+x-6}{x-2}$

Show Solution
$\left(2,5\right)$

78. $f\left(x\right)=\frac{2{x}^{2}+5x-3}{x+3}$

79. $f\left(x\right)=\frac{{x}^{3}+{x}^{2}}{x+1}$

Show Solution
$\left(-1,1\right)$

Real-World Applications

For the following exercises, express a rational function that describes the situation.

80. A large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar

have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

81. A large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Show Solution
$C\left(t\right)=\frac{8+2t}{300+20t}$

For the following exercises, use the given rational function to answer the question.

82. The concentration C of a drug in a patient's bloodstream t hours after injection in given by $C\left(t\right)=\frac{2t}{3+t^2}$. What happens to the concentration of the drug as t increases?

83. The concentration C of a drug in a patient's bloodstream t hours after injection is given by $C\left(t\right)=\frac{100t}{2t^2+75}$. Use a calculator to approximate the time when the concentration is highest.

Show Solution
After about 6.12 hours.

For the following exercises, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

84. An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let x = length of the side of the base.

85. A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/ square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let x = length of the side of the base.

Show Solution

$A\left(x\right)=50{x}^{2}+\frac{800}{x}$. 2 by 2 by 5 feet.

86. A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let x = radius.

87. A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let x = radius.

Show Solution

$A\left(x\right)=\pi {x}^{2}+\frac{100}{x}$. Radius = 2.52 meters.

88. A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let x = radius.

Glossary

arrow notation
: a way to represent symbolically the local and end behavior of a function by using arrows to indicate that an input or output approaches a value

horizontal asymptote
: a horizontal line $y=b$ where the graph approaches the line as the inputs increase or decrease without bound.

rational function
: a function that can be written as the ratio of two polynomials

removable discontinuity
: a single point at which a function is undefined that, if filled in, would make the function continuous; it appears as a hole on the graph of a function

vertical asymptote
: a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a

CHAPTER 7: PERIODIC FUNCTIONS

CHAPTER 7.1: INTRODUCTION TO PERIODIC FUNCTIONS

Figure 1. (credit: "Maxxer_", Flickr)

Each day, the sun rises in an easterly direction, approaches some maximum height relative to the celestial equator, and sets in a westerly direction. The celestial equator is an imaginary line that divides the visible universe into two halves in much the same way Earth's equator is an imaginary line that divides the planet into two halves. The exact path the sun appears to follow depends on the exact location on Earth, but each location observes a predictable pattern over time.

The pattern of the sun's motion throughout the course of a year is a periodic function. Creating a visual representation of a periodic function in the form of a graph can help us analyze the properties of the function. In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

CHAPTER 7.2: GRAPHS OF THE SINE AND COSINE FUNCTIONS

Learning Objectives

In this section, you will:

- Graph variations of $y=\sin(x)$ and $y=\cos(x)$.
- Use phase shifts of sine and cosine curves.

Figure 1. Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.

Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the *x*– and *y*-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. (Figure) lists some of the values for the sine function on a unit circle.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the *x*-axis gives the shape of the sine function. See (Figure).

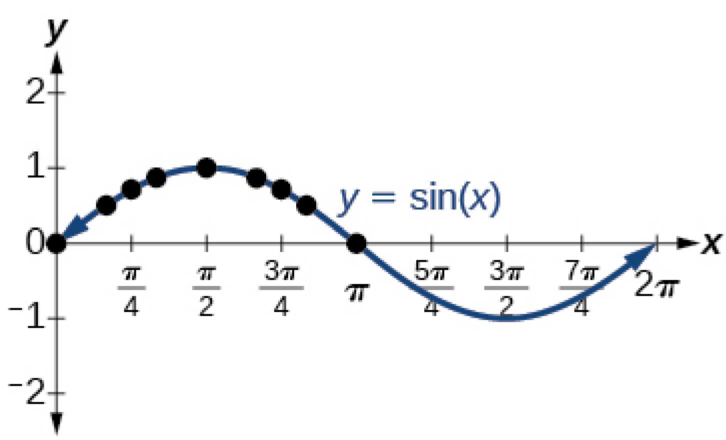


Figure 2. The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See (Figure).

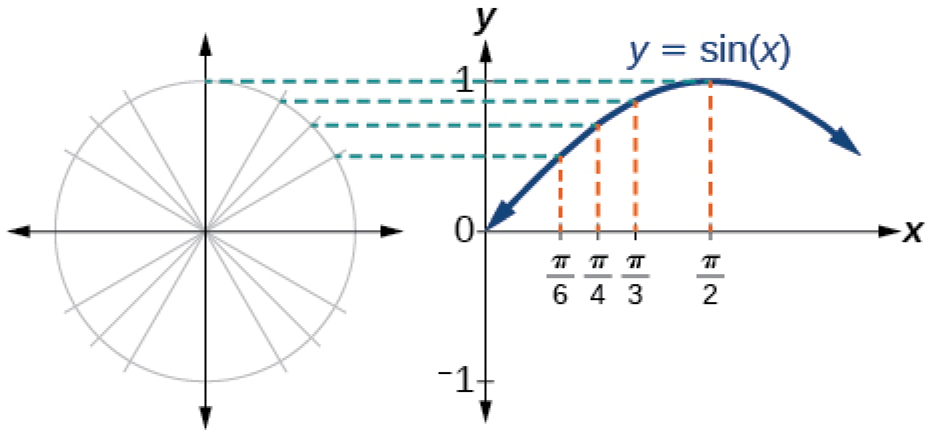


Figure 3. Plotting values of the sine function

Now let's take a similar look at the cosine function. Again, we can create a table of values and use them to sketch a graph. (Figure) lists some of the values for the cosine function on a unit circle.

$\mathbf{x}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\mathbf{cos}\left(\mathbf{x}\right)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plots points to create a graph of the cosine function as in (Figure).

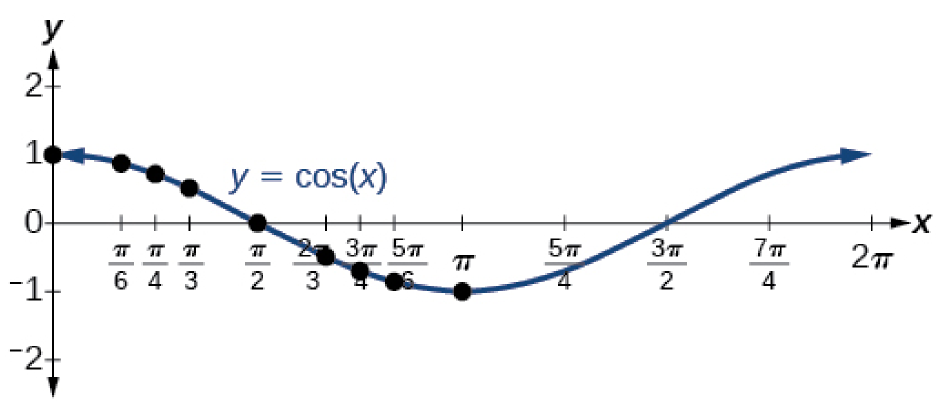


Figure 4. The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $\left[-1, 1\right]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A periodic function is a function for which a specific horizontal shift, P, results in a function equal to the original function: $f\left(x+P\right)=f\left(x\right)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the period of the function. (Figure) shows several periods of the sine and cosine functions.

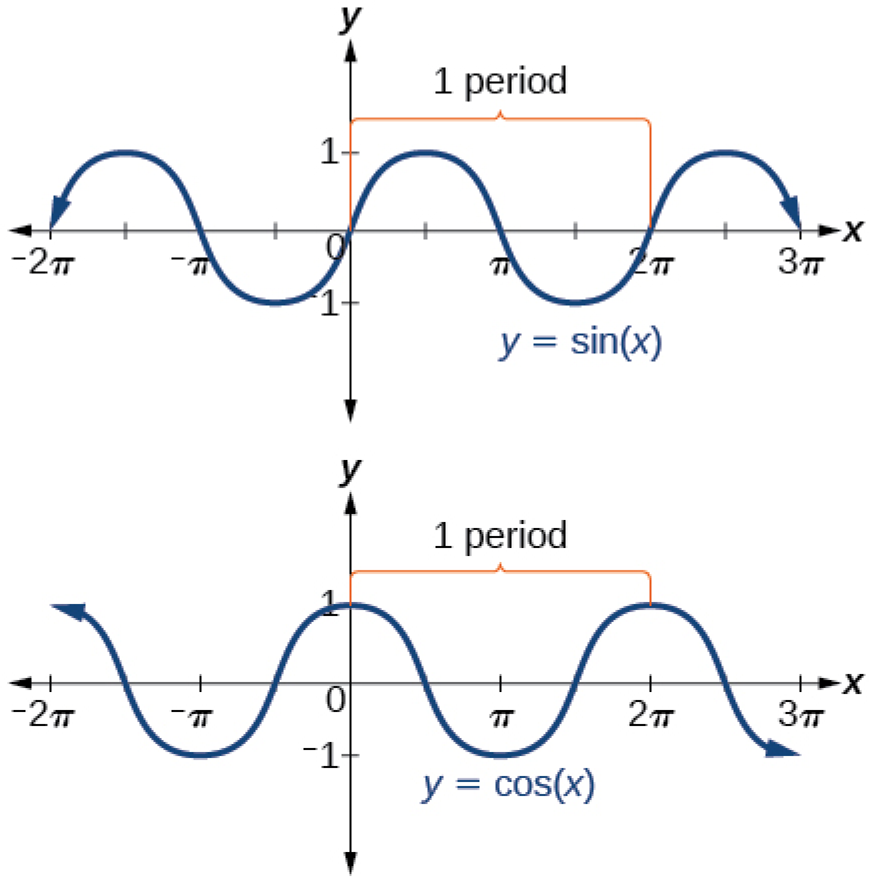


Figure 5.

Looking again at the sine and cosine functions on a domain centered at the *y*-axis helps reveal symmetries. As we can see in (Figure), the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin x$.

Now we can clearly see this property from the graph.

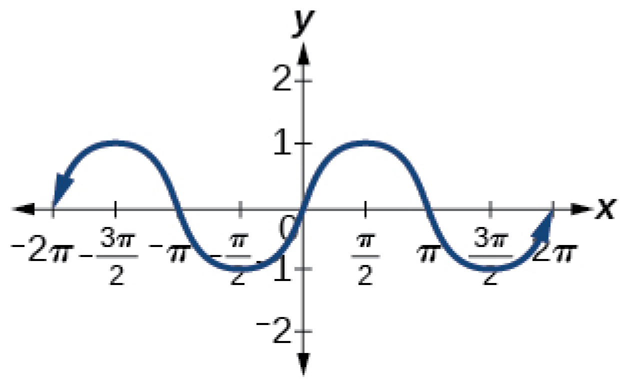


Figure 6. Odd symmetry of the sine function

(Figure) shows that the cosine function is symmetric about the *y*-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos\left(-x\right)=\cos\ x$.

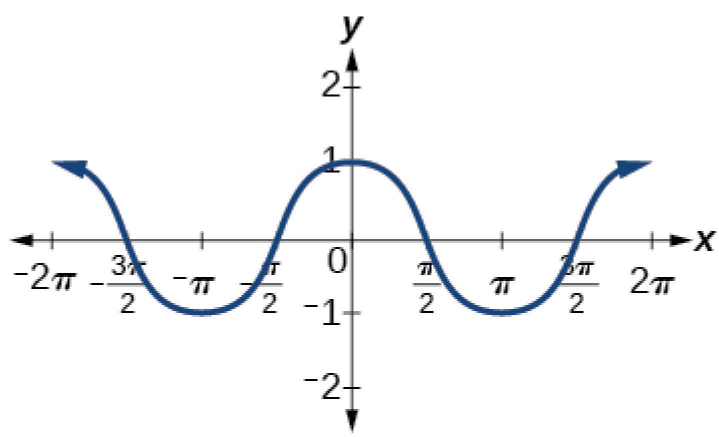


Figure 7. Even symmetry of the cosine function

Characteristics of Sine and Cosine Functions

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $\left(-\infty,\infty\right)$ and the range is $\left[-1,1\right]$.
- The graph of $y=\sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y=\cos\ x$ is symmetric about the y- axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or cosine function is known as a sinusoidal function. The general forms of sinusoidal functions are

$y = A\sin\left(Bx - C\right) + D$

and

$y = A\cos\left(Bx - C\right) + D$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \frac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f\left(x\right) = \sin\left(x\right)$, $B = 1$, so the period is 2π, which we knew. If $f\left(x\right) = \sin\left(2x\right)$, then $B = 2$, so the period is π and the graph is compressed. If $f\left(x\right) = \sin\left(\frac{x}{2}\right)$, then $B = \frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in (Figure) how the period is indirectly related to $|B|$.

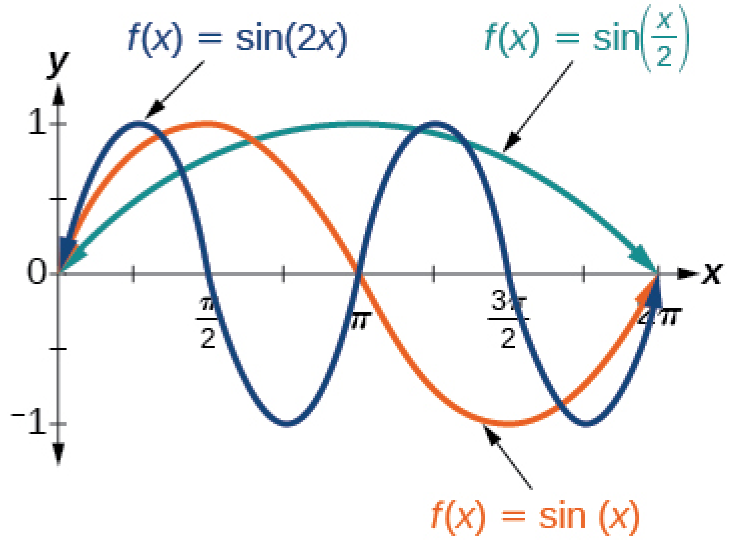


Figure 8.

Period of Sinusoidal Functions

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$y = A\sin\left(Bx\right)$

$y = A\cos\left(Bx\right)$

The period is $\frac{2\pi}{|B|}$.

Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f\left(x\right)=\sin\left(\frac{\pi}{6}x\right)$.

Show Solution

Let's begin by comparing the equation to the general form $y=A\sin\left(Bx\right)$.

In the given equation, $B=\frac{\pi}{6}$, so the period will be

$$\begin{aligned}P&=\frac{2\pi}{|B|}\\&=\frac{2\pi}{\frac{\pi}{6}}\\&=2\pi\cdot\frac{6}{\pi}\\&=12\end{aligned}$$

Try It

Determine the period of the function $g\left(x\right)=\cos\left(\frac{x}{3}\right)$.

Show Solution

6π

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the horizontal **midline** of the graph, which is the line $y = D$; because $D = 0$ in this case, the midline is the *x*-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4\sin x$ is twice the amplitude of $f(x) = 2\sin x$. If $|A| < 1$, the function is compressed. (Figure) compares several sine functions with different amplitudes.

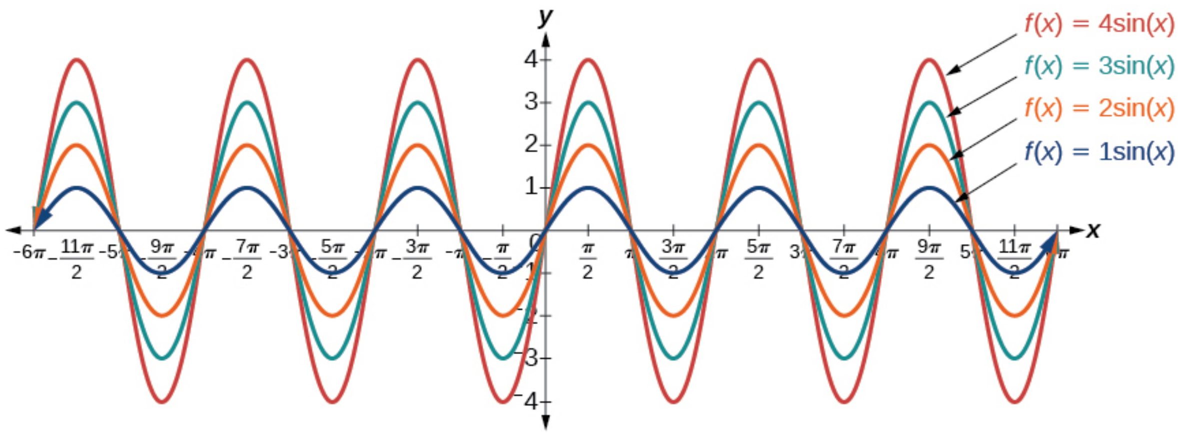


Figure 9.

Amplitude of Sinusoidal Functions

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx) \text{ and } y = A\cos(Bx)$$

The amplitude is A, and the vertical height from the midline is $|A|$. In addition, notice in the example that

$$|A| = \text{amplitude} = \frac{1}{2}|\text{maximum} - \text{minimum}|$$

Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Show Solution

Let's begin by comparing the function to the simplified form $y = A\sin\left(Bx\right).$

In the given function, $A = -4,$ so the amplitude is $|A| = |-4| = 4.$ The function is stretched.

Analysis

The negative value of A results in a reflection across the *x*-axis of the sine function, as shown in (Figure).

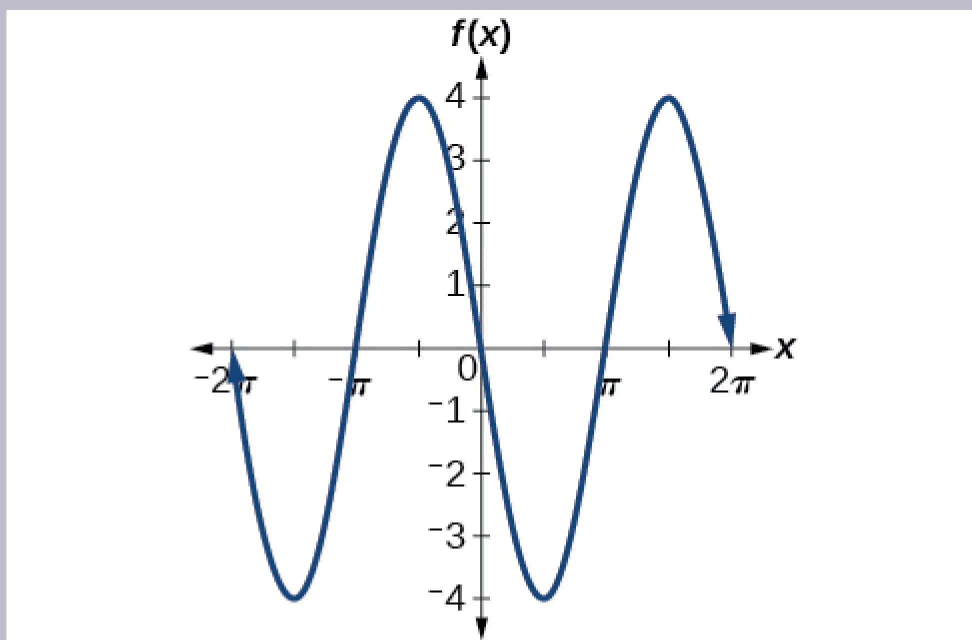


Figure 10.

Try It

What is the amplitude of the sinusoidal function $f\left(x\right)=\frac{1}{2}\sin\left(x\right)$? Is the function stretched or compressed vertically?

compressed

$\frac{1}{2}$

Analyzing Graphs of Variations of *y* = sin *x* and *y* = cos *x*

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y=A\sin\left(Bx-C\right)+D\text{ and }y=A\cos\left(Bx-C\right)+D$$
$$or$$
$$y=A\sin\left(B\left(x-\frac{C}{B}\right)\right)+D\text{ and }y=A\cos\left(B\left(x-\frac{C}{B}\right)\right)+D$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C>0$, the graph shifts to the right. If $C<0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. (Figure) shows that the graph of $f\left(x\right)=\sin\left(x-\pi\right)$ shifts to the right by π units, which is more than we see in the graph of $f\left(x\right)=\sin\left(x-\frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

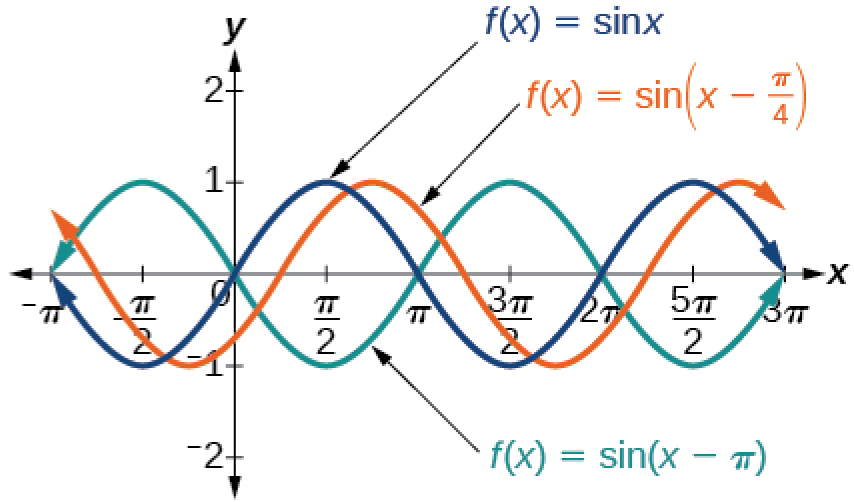


Figure 11.

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See (Figure). The function $y = \cos(x) + D$ has its midline at $y = D$.

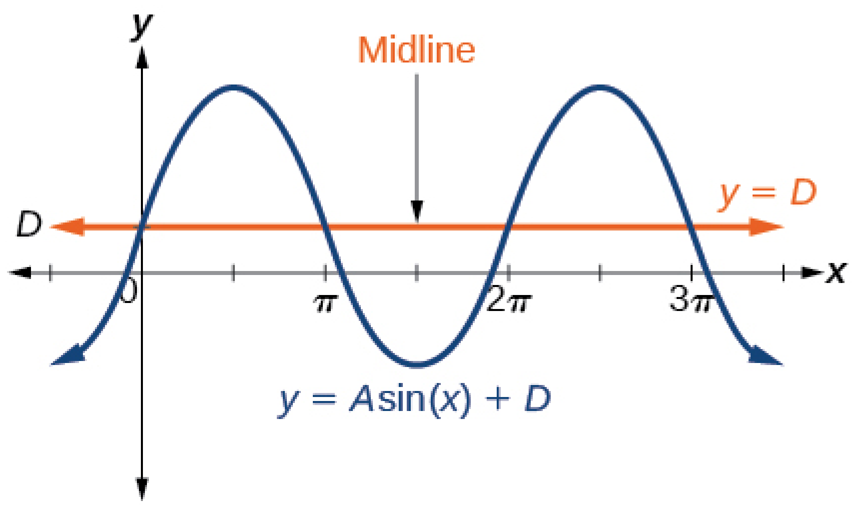


Figure 12.

Any value of D other than zero shifts the graph up or down. (Figure) compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

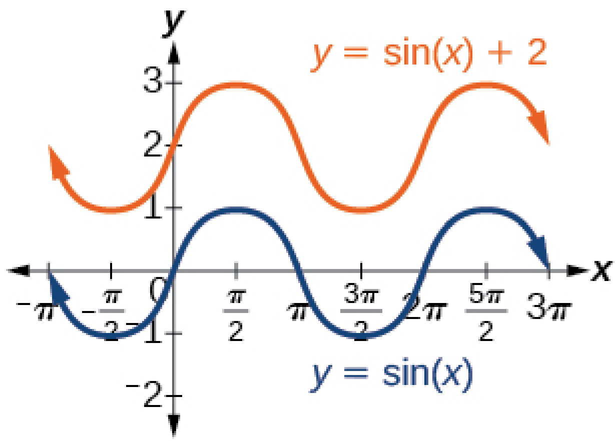


Figure 13.

Variations of Sine and Cosine Functions

Given an equation in the form $f\left(x\right)=A\sin\left(Bx-C\right)+D$ or $f\left(x\right)=A\cos\left(Bx-C\right)+D,\frac{C}{B}$ is the phase shift and D is the vertical shift.

Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f\left(x\right)=\sin\left(x+\frac{\pi}{6}\right)-2.$

Show Solution

Let's begin by comparing the equation to the general form $y=A\sin\left(Bx-C\right)+D.$

In the given equation, notice that $B=1$ and $C=-\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned}\frac{C}{B}&=-\frac{\frac{\pi}{6}}{1}\\&=-\frac{\pi}{6}\end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f\left(x\right)=\sin\left(x+\frac{\pi}{6}\right)-2$ can be rewritten as $f\left(x\right)=\sin\left(x-\left(-\frac{\pi}{6}\right)\right)-2.$ If the value of C is negative, the shift is to the left.

Try It

Determine the direction and magnitude of the phase shift for $f\left(x\right)=3\cos\left(x-\frac{\pi}{2}\right).$

Show Solution

$\frac{\pi}{2}$; right

Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f\left(x\right)=\cos\left(x\right)-3.$

Show Solution

Let's begin by comparing the equation to the general form $y=A\cos\left(Bx-C\right)+D.$

In the given equation, $D=-3$ so the shift is 3 units downward.

Try It

Determine the direction and magnitude of the vertical shift for $f\left(x\right)=3\sin\left(x\right)+2.$

Show Solution

2 units up

Given a sinusoidal function in the form $f\left(x\right)=A\sin\left(Bx-C\right)+D,$ identify the midline, amplitude, period, and phase shift.

1. Determine the amplitude as $|A|.$
2. Determine the period as $P=\frac{2\pi}{|B|}.$
3. Determine the phase shift as $\frac{C}{B}.$
4. Determine the midline as $y=D.$

Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y=3\sin\left(2x\right)+1.$

Show Solution

Let's begin by comparing the equation to the general form $y=A\sin\left(Bx-C\right)+D.$

$A=3,$ so the amplitude is $|A|=3.$

Next, $B=2,$ so the period is $P=\frac{2\pi}{|B|}=\frac{2\pi}{2}=\pi.$

There is no added constant inside the parentheses, so $C=0$ and the phase shift is $\frac{C}{B}=\frac{0}{2}=0.$

Finally, $D = 1,$ so the midline is $y = 1.$

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y = 1$, and the amplitude is 3. See (Figure).

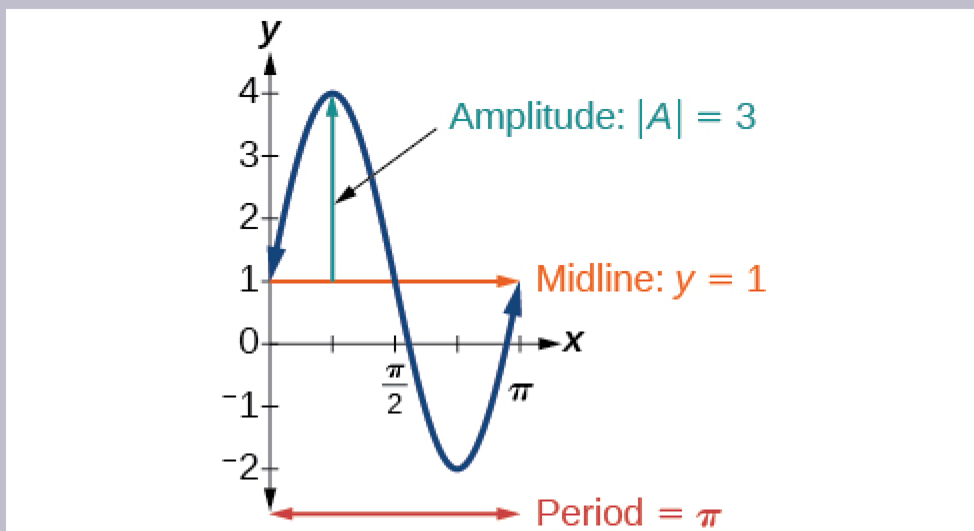


Figure 14.

Try It

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right).$

Show Solution

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{|B|} = 6\pi$; phase shift: $\frac{C}{B} = \pi$

Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in (Figure).

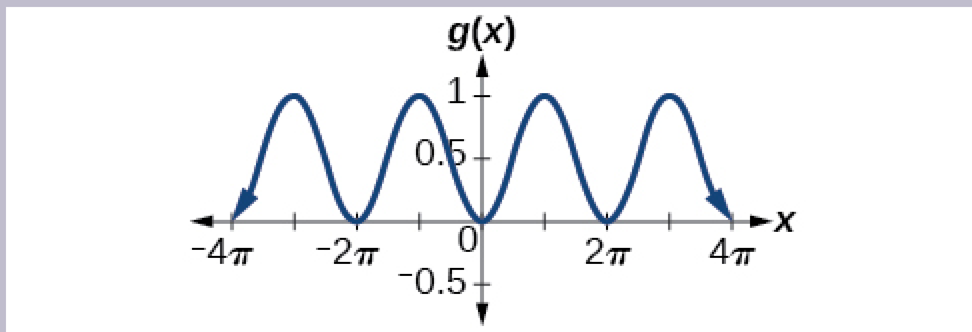


Figure 15.

Show Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin\left(Bx - C\right) + D$$
$$y = A\cos\left(Bx - C\right) + D$$

The graph could represent either a sine or a cosine function that is shifted and/or reflected. When $x = 0,$ the graph has an extreme point, $\left(0, 0\right)$. Since the cosine function has an extreme point for $x = 0,$ let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5.$ This value, which is the midline, is D in the equation, so $D = 0.5.$

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5.$ Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5.$ Also, the graph is reflected about the *x*-axis so that $A = -0.5.$

The graph is not horizontally stretched or compressed, so $B = 1;$ and the graph is not shifted horizontally, so $C = 0.$

Putting this all together,

$$g\left(x\right) = -0.5\cos\left(x\right) + 0.5$$

Try It

Determine the formula for the sine function in (Figure).

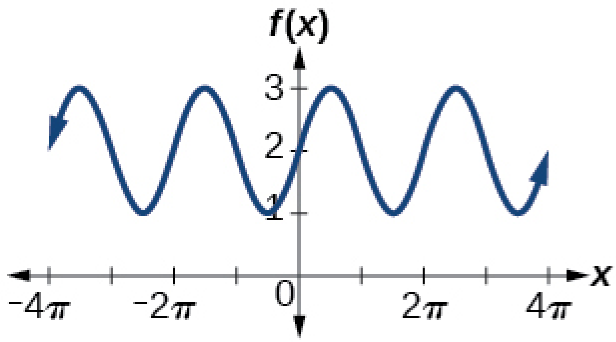


Figure 16.

Show Solution

$f(x) = \sin(x) + 2$

Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in (Figure).

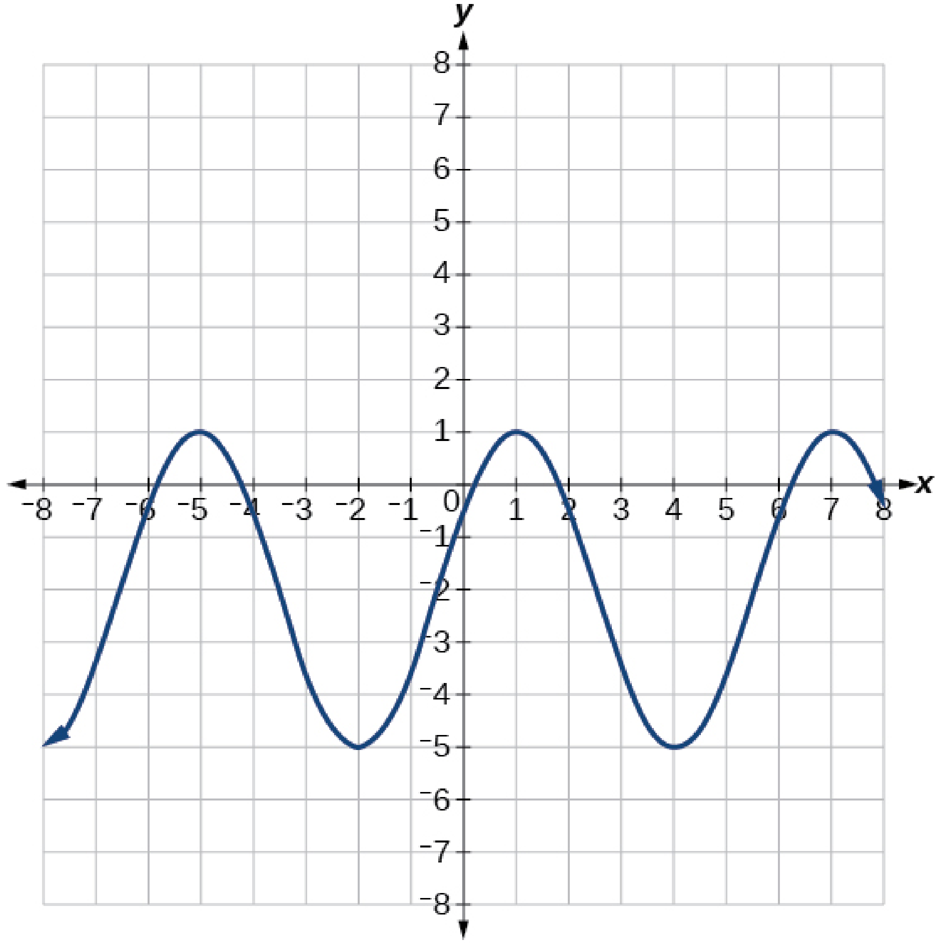


Figure 17.

Show Solution

With the highest value at 1 and the lowest value at $-5,$ the midline will be halfway between at $-2.$ So $D=-2.$

The distance from the midline to the highest or lowest value gives an amplitude of $|A|=3.$

The period of the graph is 6, which can be measured from the peak at $x=1$ to the next peak at $x=7,$ or from the distance between the lowest points. Therefore, $P=\frac{2\pi}{|B|}=6.$ Using the positive value for $B,$ we find that

$B=\frac{2\pi}{P}=\frac{2\pi}{6}=\frac{\pi}{3}$

So far, our equation is either $y=3\sin\left(\frac{\pi}{3}x-C\right)-2$ or $y=3\cos\left(\frac{\pi}{3}x-C\right)-2.$ For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$y = 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2 \text{ or } y = -3\cos\left(\frac{\pi}{3}x + \frac{2\pi}{3}\right) - 2$

Again, these functions are equivalent, so both yield the same graph.

Try It

Write a formula for the function graphed in (Figure).

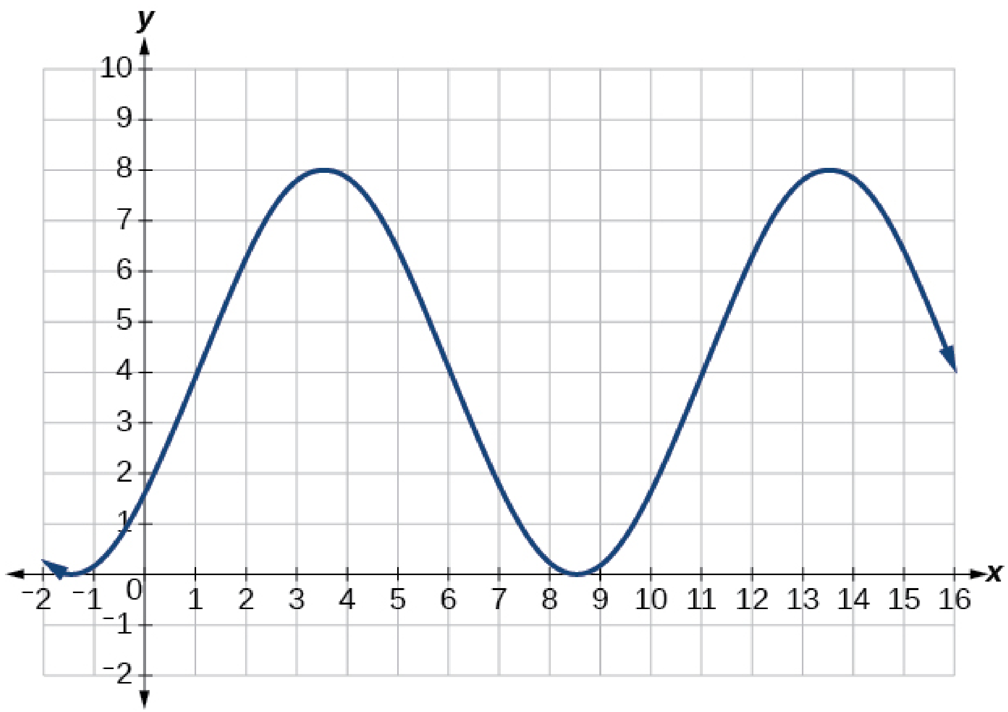


Figure 18.

Show Solution

two possibilities: $y = 4\sin\left(\frac{\pi}{5}x - \frac{\pi}{5}\right) + 4$ or $y = -4\sin\left(\frac{\pi}{5}x + \frac{4\pi}{5}\right) + 4$

Graphing Variations of *y* = sin *x* and *y* = cos *x*

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D,$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$ **, sketch its graph.**

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the *x*-axis at $x = \frac{2\pi}{|B|}$.

Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f\left(x\right) = -2\sin\left(\frac{\pi x}{2}\right)$.

Show Solution

Let's begin by comparing the equation to the form $y = A\sin\left(Bx\right)$.

- *Step 1.* We can see from the equation that $A = -2,$ so the amplitude is 2.
  $|A| = 2$
- *Step 2.* The equation shows that $B = \frac{\pi}{2}$, so the period is
  $$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$
- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4–7.* The *x*-intercepts are at the beginning of one period, $x = 0,$ the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4.$

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3.$ A local minimum will occur 2 units below the midline, at $x = 1,$ and a local maximum will occur at 2 units above the midline, at $x = 3.$ (Figure) shows the graph of the function.

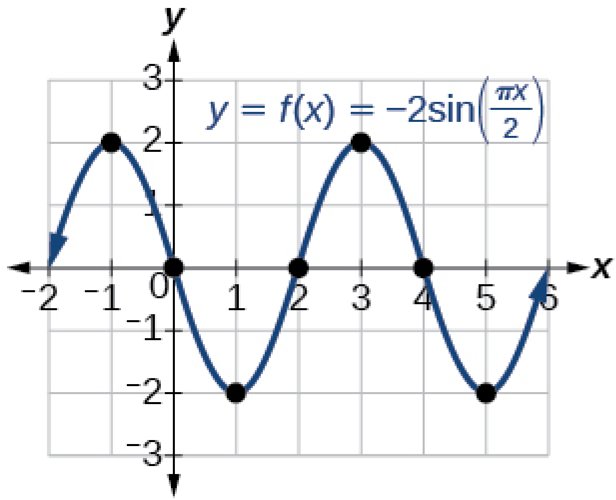
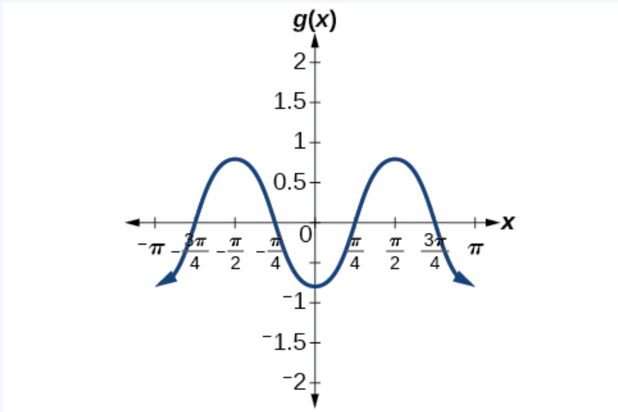


Figure 19.

Try It

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

Show Solution



midline: $y = 0$; amplitude: $|A| = 0.8$; period: $P = \frac{2\pi}{|B|} = \pi$; phase shift: $\frac{C}{B} = 0$ or none

Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph.

1. Express the function in the general form $y = A\sin\left(Bx - C\right) + D$ or $y = A\cos\left(Bx - C\right) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \frac{2\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $f\left(x\right) = A\sin\left(Bx\right)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D.

Graphing a Transformed Sinusoid

Sketch a graph of $f\left(x\right) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

Show Solution

- *Step 1.* The function is already written in general form: $f\left(x\right)=3\text{sin}\left(\frac{\pi}{4}x-\frac{\pi}{4}\right)$. This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A|=|3|=3.$ The amplitude is 3.
- *Step 3.* Since $|B|=|\frac{\pi}{4}|=\frac{\pi}{4},$ we determine the period as follows.
  $P=\frac{2\pi}{|B|}=\frac{2\pi}{\frac{\pi}{4}}=2\pi\cdot\frac{4}{\pi}=8$
  The period is 8.
- *Step 4.* Since $C=\frac{\pi}{4},$ the phase shift is
  $\frac{C}{B}=\frac{\frac{\pi}{4}}{\frac{\pi}{4}}=1.$
  The phase shift is 1 unit.
- *Step 5.*(Figure) shows the graph of the function.

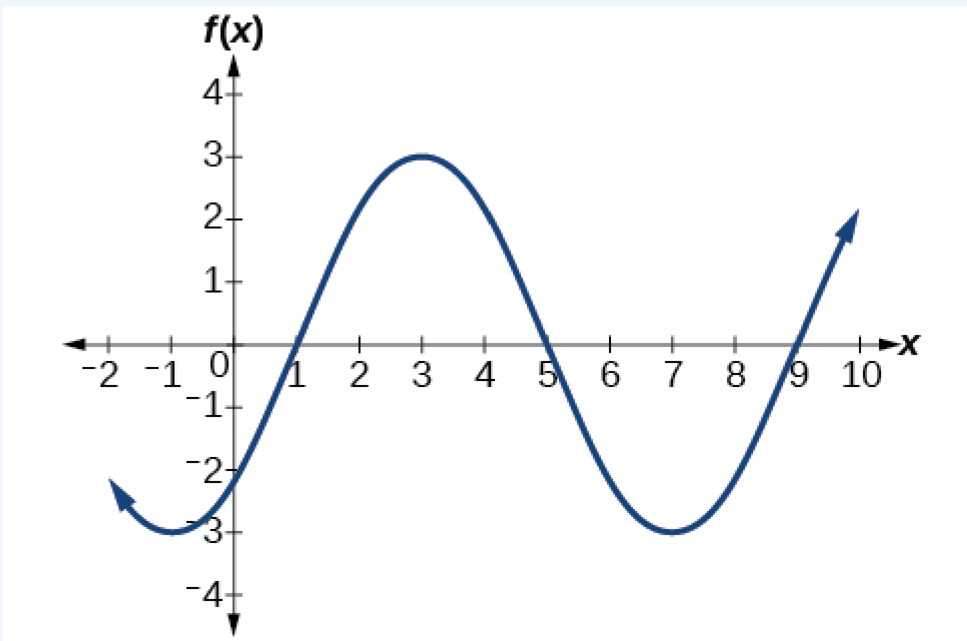
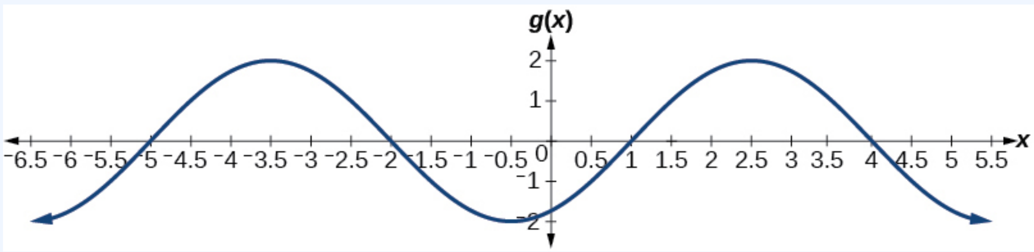


Figure 20. A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

Try It

Draw a graph of $g\left(x\right)=-2\cos\left(\frac{\pi}{3}x+\frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Show Solution



midline: $y=0;$ amplitude: $|A|=2;$ period: $P=\frac{2\pi}{|B|}=6;$ phase shift: $\frac{C}{B}=-\frac{1}{2}$

Identifying the Properties of a Sinusoidal Function

Given $y=-2\cos\left(\frac{\pi}{2}x+\pi\right)+3,$ determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Show Solution

Begin by comparing the equation to the general form and use the steps outlined in (Figure).

$$y=A\cos\left(Bx-C\right)+D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A=-2,$ the amplitude is $|A|=2.$
- *Step 3.* $|B|=\frac{\pi}{2},$ so the period is $P=\frac{2\pi}{|B|}=\frac{2\pi}{\frac{\pi}{2}}=2\pi\cdot\frac{2}{\pi}=4.$ The period is 4.

- *Step 4.* $C = -\pi,$ so we calculate the phase shift as $\frac{C}{B} = \frac{-\pi,}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2.$ The phase shift is $-2.$
- *Step 5.* $D = 3,$ so the midline is $y = 3,$ and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the *x*-axis.

(Figure) shows one cycle of the graph of the function.

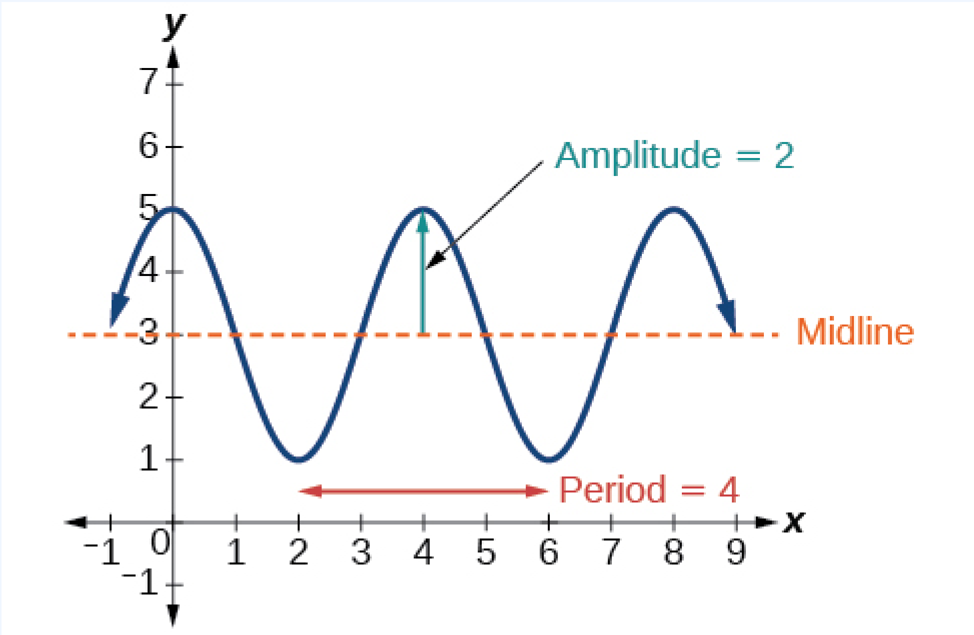


Figure 21.

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter, circular motion can be modeled using either the sine or cosine function.

Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the *y*-coordinate of the point as a function of the angle of rotation.

Show Solution

Recall that, for a point on a circle of radius *r*, the *y*-coordinate of the point is $y = r\sin\left(x\right),$ so in this case, we get the equation $y\left(x\right) = 3\sin\left(x\right).$

The constant 3 causes a vertical stretch of the *y*-values of the function by a factor of 3, which we can see in the graph in (Figure).

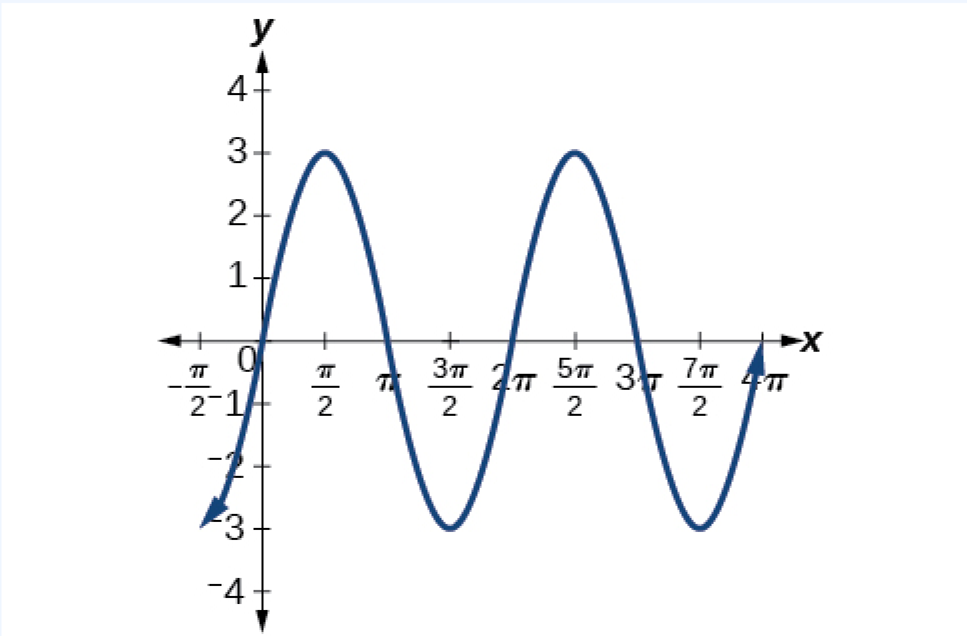


Figure 22.

Analysis

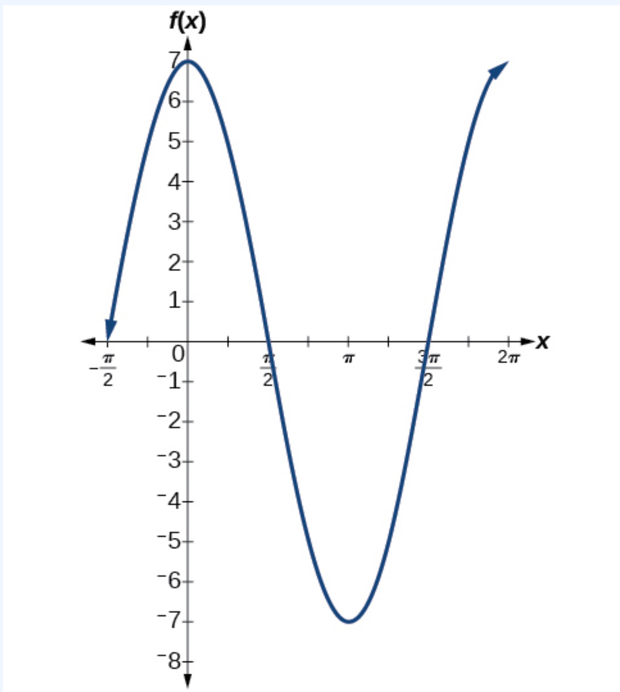
Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi, \ldots$. Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

Try It

What is the amplitude of the function $f\left(x\right)=7\cos\left(x\right)$? Sketch a graph of this function.

Show Solution

7



Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled *P*, as shown in (Figure). Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

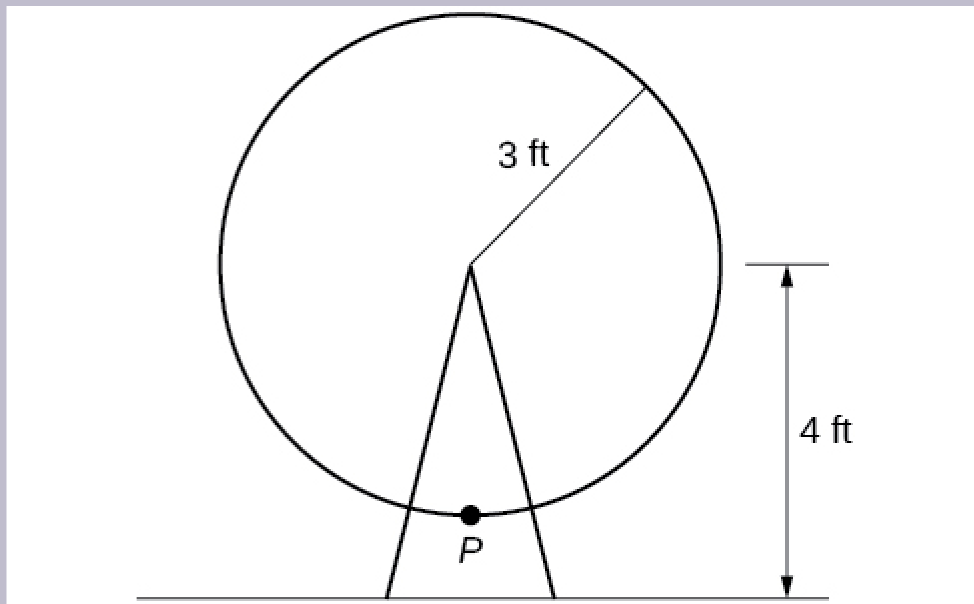


Figure 23.

Show Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in (Figure).

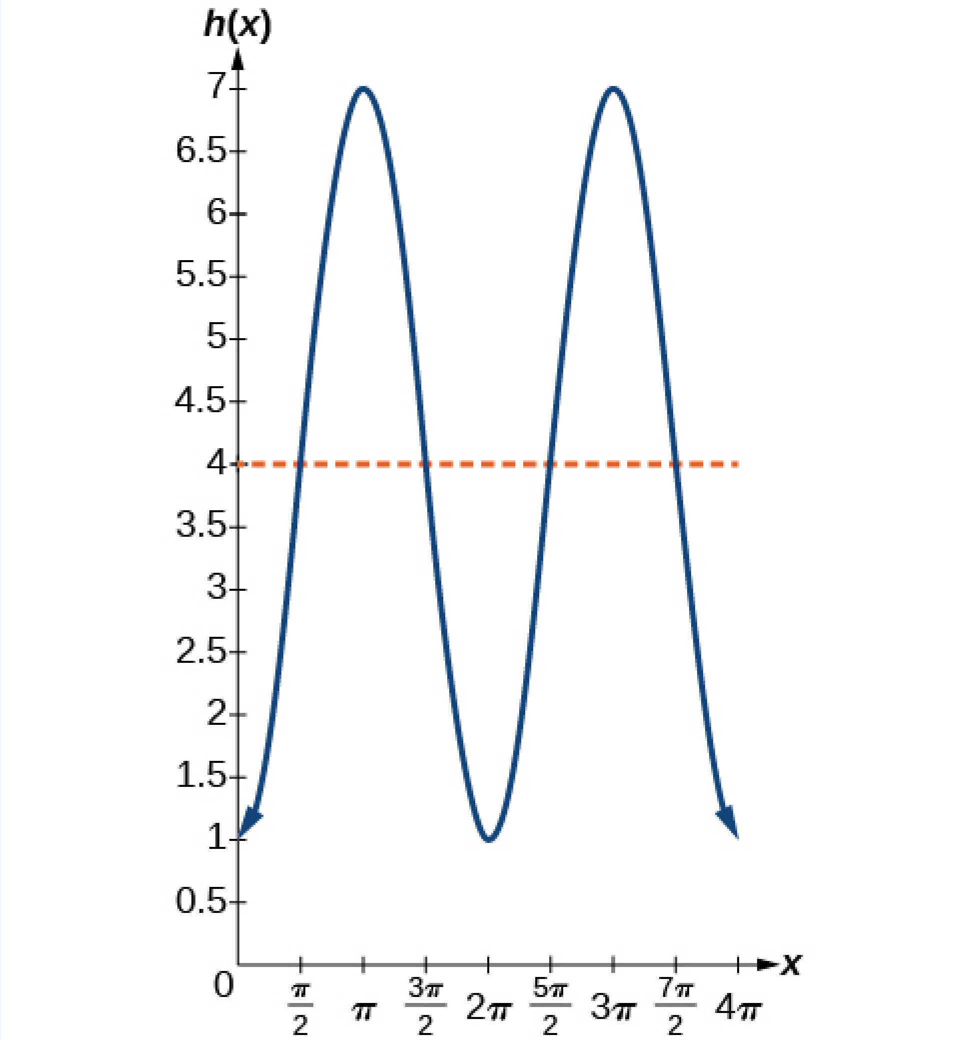


Figure 24.

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

Try It

A weight is attached to a spring that is then hung from a board, as shown in (Figure). As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

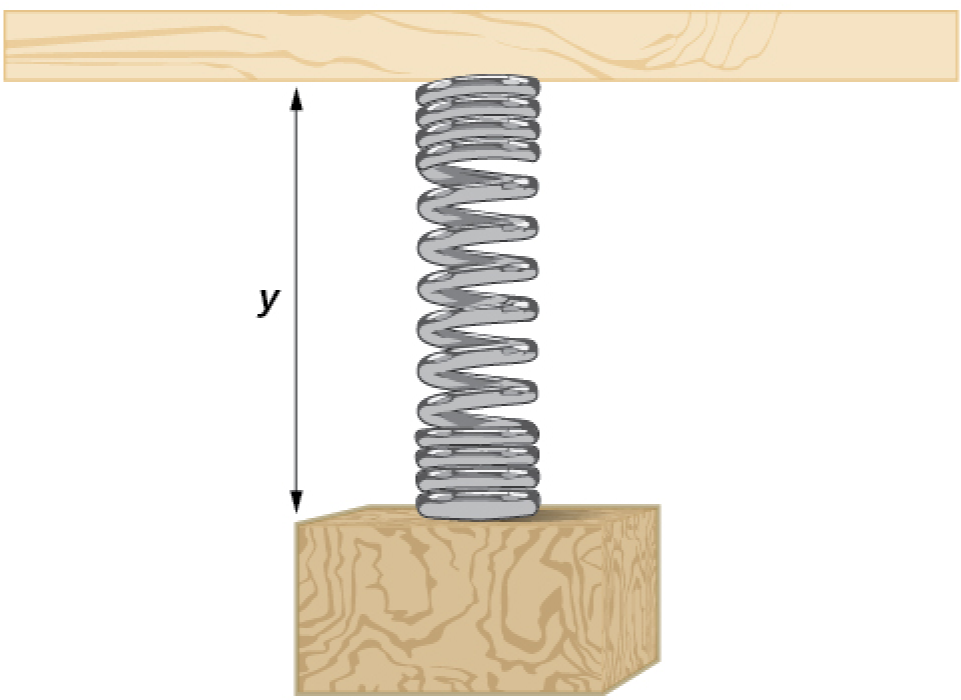
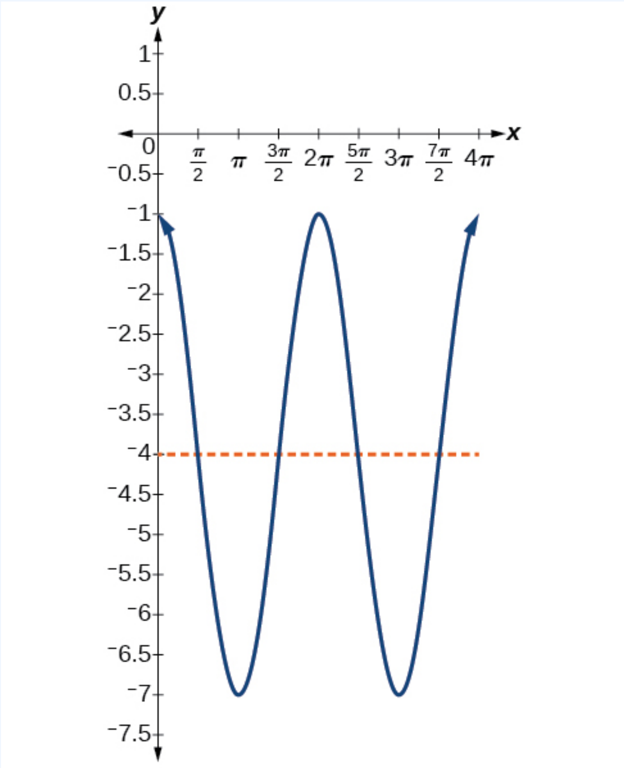


Figure 25.

Show Solution

$y = 3\cos(x) - 4$



Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Show Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period: 30, so $B = \frac{2\pi}{30} = \frac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine
- Translations of Sine and Cosine
- Graphing Sine and Cosine Transformations
- Graphing the Sine Function

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the *y*-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \frac{2\pi}{|B|}$. See (Figure).
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See (Figure).
- The value $\frac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See (Figure).
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See (Figure).
- Combinations of variations of sinusoidal functions can be detected from an equation. See (Figure).
- The equation for a sinusoidal function can be determined from a graph. See (Figure) and (Figure).
- A function can be graphed by identifying its amplitude and period. See (Figure) and (Figure).
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See (Figure).
- Sinusoidal functions can be used to solve real-world problems. See (Figure), (Figure), and (Figure).

Section Exercises

Verbal

1. Why are the sine and cosine functions called periodic functions?

Show Solution

The sine and cosine functions have the property that $f\left(x+P\right)=f\left(x\right)$ for a certain P. This means that the function values repeat for every P units on the *x*-axis.

2. How does the graph of $y=\sin x$ compare with the graph of $y=\cos x$? Explain how you could horizontally translate the graph of $y=\sin x$ to obtain $y=\cos x$.

3. For the equation $A\cos\left(Bx+C\right)+D$, what constants affect the range of the function and how do they affect the range?

Show Solution

The absolute value of the constant A (amplitude) increases the total range and the constant D (vertical shift) shifts the graph vertically.

4. How does the range of a translated sine function relate to the equation $y=A\sin\left(Bx+C\right)+D$?

5. How can the unit circle be used to construct the graph of $f\left(t\right)=\sin t$?

Show Solution

At the point where the terminal side of t intersects the unit circle, you can determine that the $\sin t$ equals the *y*-coordinate of the point.

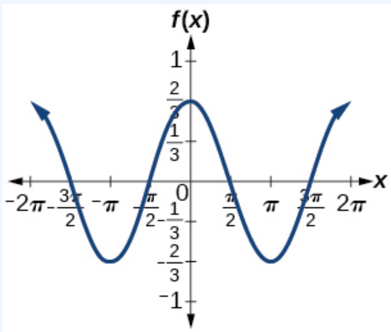
Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum *y*-values and their corresponding *x*-values on one period for $x>0$. Round answers to two decimal places if necessary.

6. $f\left(x\right)=2\sin x$

7. $f\left(x\right)=\frac{2}{3}\cos x$

Show Solution

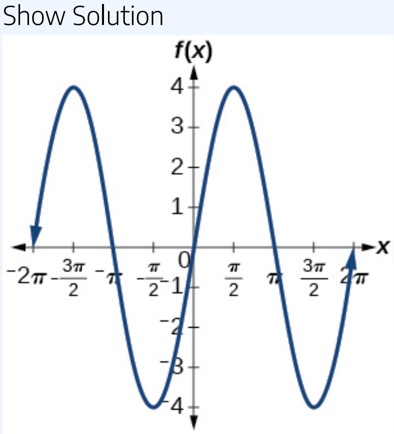


amplitude: $\frac{2}{3}$; period: 2π; midline: $y=0$; maximum: $y=\frac{2}{3}$ occurs at $x=0$; minimum: $y=-\frac{2}{3}$ occurs at $x=\pi$; for one period, the graph starts at 0 and ends at 2π

8. $f\left(x\right)=-3\sin x$

9. $f\left(x\right)=4\sin x$

Show Solution

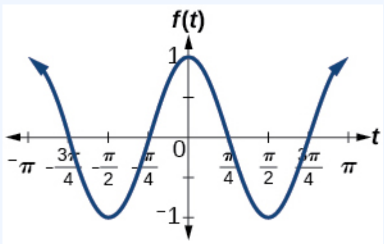


amplitude: 4; period: 2π; midline: $y=0$; maximum $y=4$ occurs at $x=\frac{\pi}{2}$; minimum: $y=-4$ occurs at $x=\frac{3\pi}{2}$; one full period occurs from $x=0$ to $x=2\pi$

10. $f\left(x\right)=2\text{cos}\,x$

11. $f\left(x\right)=\cos\left(2x\right)$

Show Solution

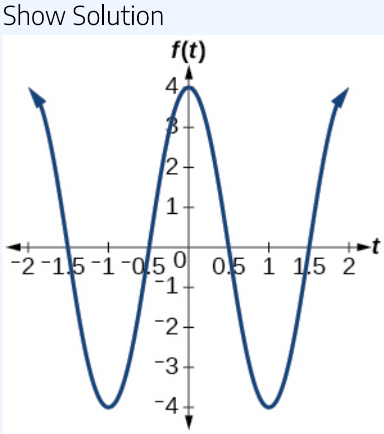


amplitude: 1; period: π; midline: $y=0$; maximum: $y=1$ occurs at $x=\pi$; minimum: $y=-1$ occurs at $x=\frac{\pi}{2}$; one full period is graphed from $x=0$ to $x=\pi$

12. $f\left(x\right)=2\sin\left(\frac{1}{2}x\right)$

13. $f\left(x\right)=4\cos\left(\pi x\right)$

Show Solution

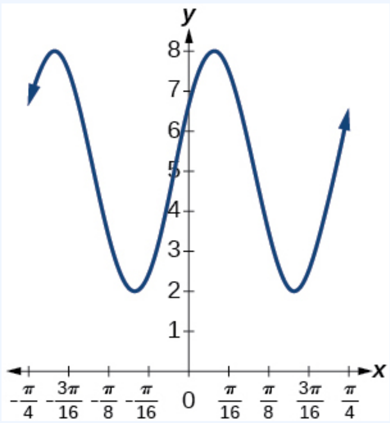


amplitude: 4; period: 2; midline: $y=0$; maximum: $y=4$ occurs at $x=0$; minimum: $y=-4$ occurs at $x=1$

14. $f\left(x\right)=3\cos\left(\frac{6}{5}x\right)$

15. $y = 3\sin\left(8\left(x+4\right)\right)+5$

Show Solution

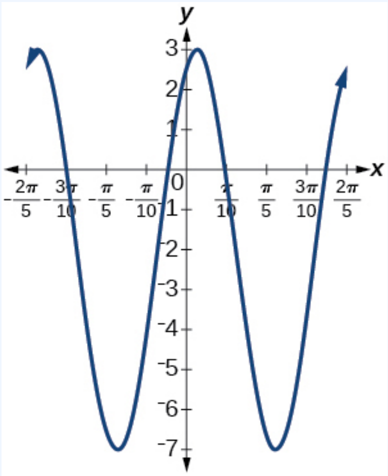


amplitude: 3; period: $\frac{\pi}{4}$; midline: $y=5$; maximum: $y=8$ occurs at $x=0.12$; minimum: $y=2$ occurs at $x=0.516$; horizontal shift: -4; vertical translation 5; one period occurs from $x=0$ to $x=\frac{\pi}{4}$

16. $y = 2\sin\left(3x-21\right)+4$

17. $y = 5\sin\left(5x+20\right)-2$

Show Solution



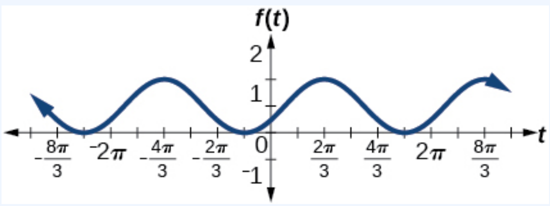
amplitude: 5; period: $\frac{2\pi}{5}$; midline: $y = -2$; maximum: $y = 3$ occurs at $x = 0.08$; minimum: $y = -7$ occurs at $x = 0.71$; phase shift: -4; vertical translation: -2; one full period can be graphed on $x = 0$ to $x = \frac{2\pi}{5}$

For the following exercises, graph one full period of each function, starting at $x = 0$. For each function, state the amplitude, period, and midline. State the maximum and minimum *y*-values and their corresponding *x*-values on one period for $x > 0$. State the phase shift and vertical translation, if applicable. Round answers to two decimal places if necessary.

18. $f\left(t\right) = 2\sin\left(t - \frac{5\pi}{6}\right)$

19. $f\left(t\right) = -\cos\left(t + \frac{\pi}{3}\right) + 1$

Show Solution



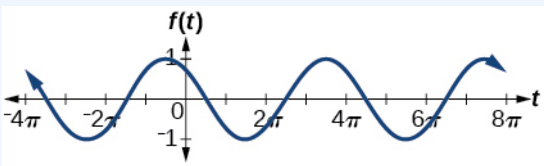
amplitude: 1 ; period: 2π; midline: $y = 1$; maximum: $y = 2$ occurs at $x = 2.09$; maximum:

$y=2$ occurs at $t=2.09;$ minimum: $y=0$ occurs at $t=5.24;$ phase shift: $-\frac{\pi}{3};$ vertical translation: 1; one full period is from $t=0$ to $t=2\pi$

20. $f\left(t\right)=4\text{cos}\left(2\left(t+\frac{\pi}{4}\right)\right)-3$

21. $f\left(t\right)=-\text{sin}\left(\frac{1}{2}t+\frac{5\pi}{3}\right)$

Show Solution



amplitude: 1; period: $4\pi;$ midline: $y=0;$ maximum: $y=1$ occurs at $t=11.52;$ minimum: $y=-1$ occurs at $t=5.24;$ phase shift: $-\frac{10\pi}{3};$ vertical shift: 0

22. $f\left(x\right)=4\text{sin}\left(\frac{\pi}{2}\left(x-3\right)\right)+7$

23. Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in (Figure).

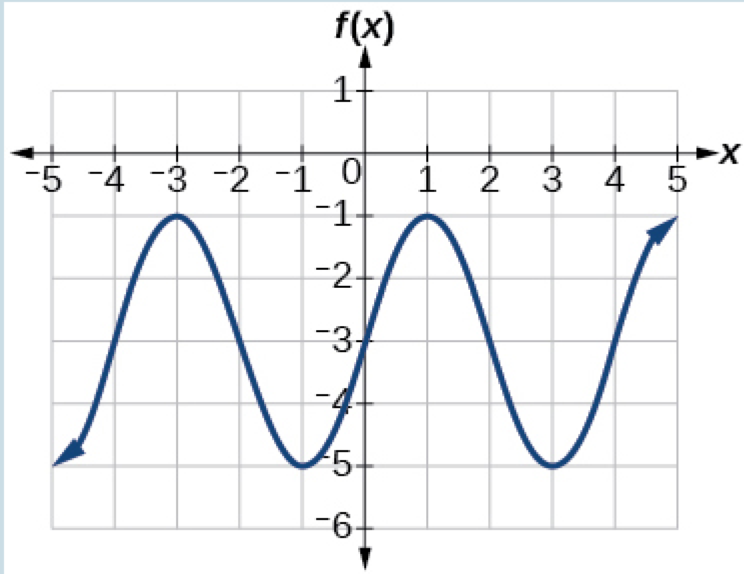


Figure 26.

Show Solution

amplitude: 2; midline: $y = -3;$ period: 4; equation: $f\left(x\right) = 2\sin\left(\frac{\pi}{2}x\right) - 3$

24. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in (Figure).

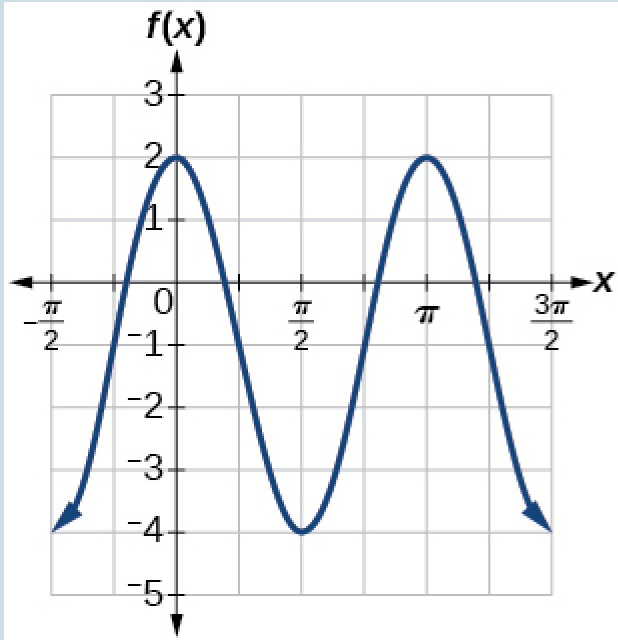


Figure 27.

25. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in (Figure).

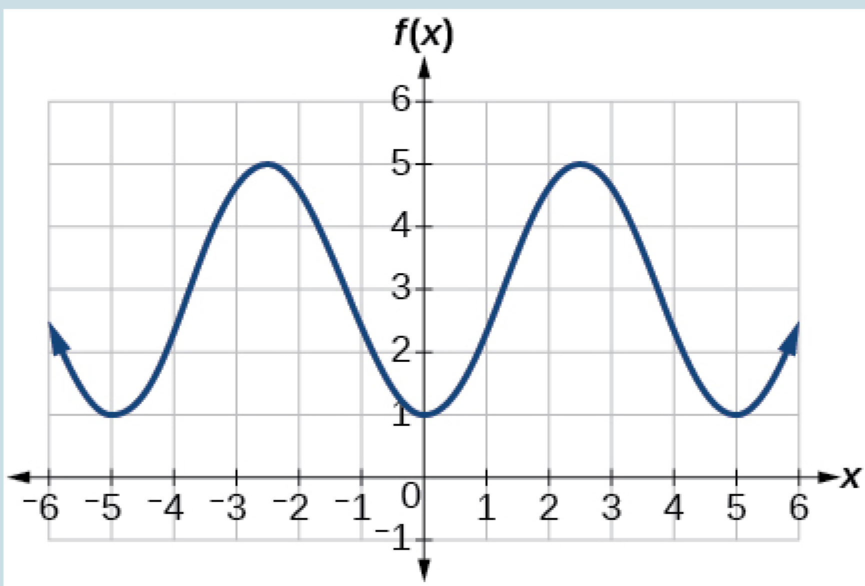


Figure 28.

Show Solution

amplitude: 2; period: 5; midline: $y=3$; equation: $f\left(x\right)=-2\cos\left(\frac{2\pi}{5}x\right)+3$

26. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in (Figure).

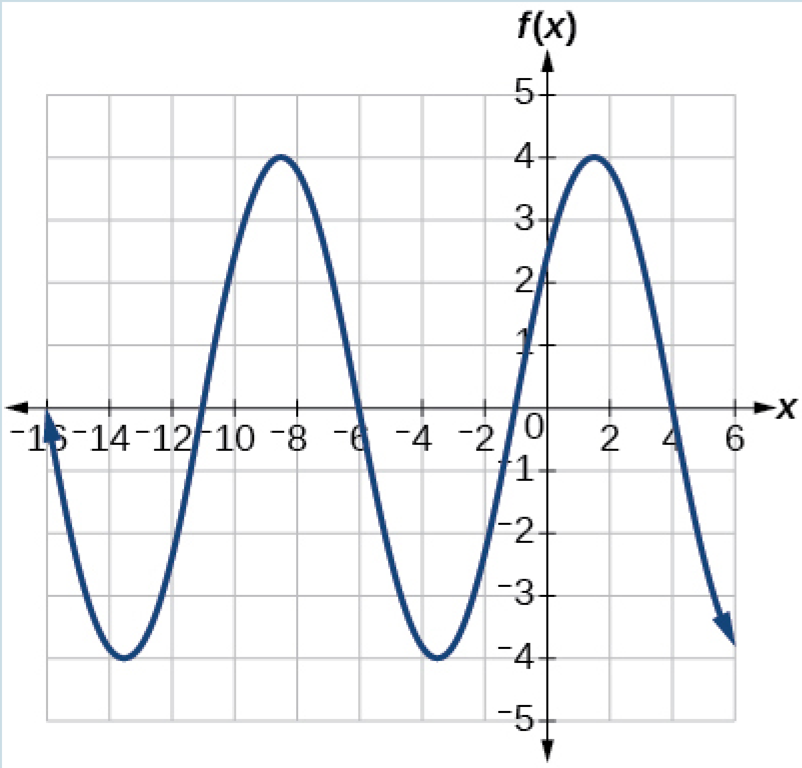


Figure 29.

27. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in (Figure).

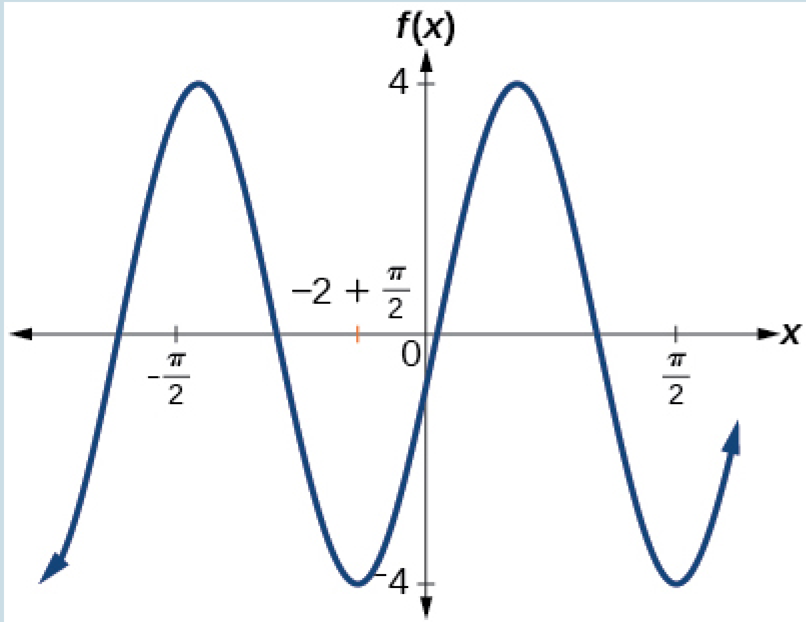


Figure 30.

Show Solution

amplitude: 4; period: 2; midline: $y=0$; equation: $f\left(x\right)=-4\cos\left(\pi\left(x-\frac{\pi}{2}\right)\right)$

28. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in (Figure).

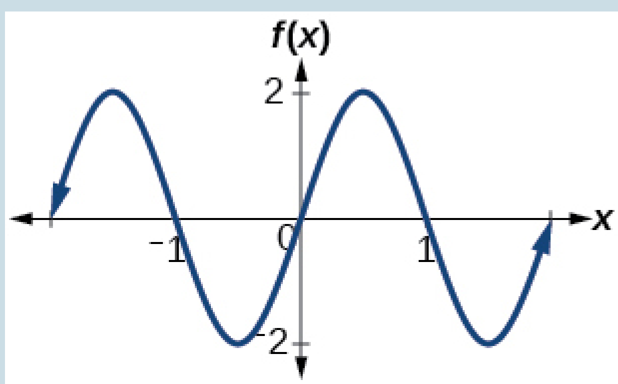


Figure 31.

29. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in (Figure).

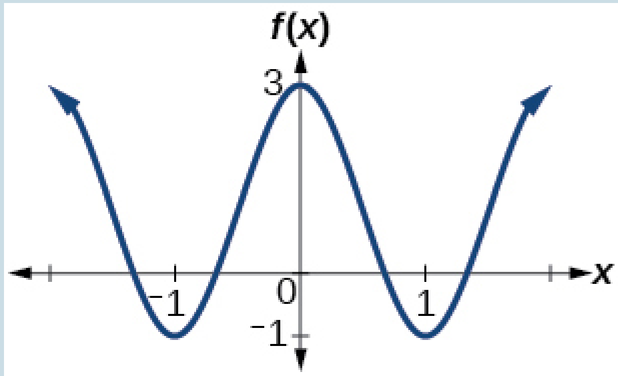


Figure 32.

Show Solution

amplitude: 2; period: 2; midline $y = 1$; equation: $f\left(x\right) = 2\cos\left(\pi x\right) + 1$

30. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in (Figure).

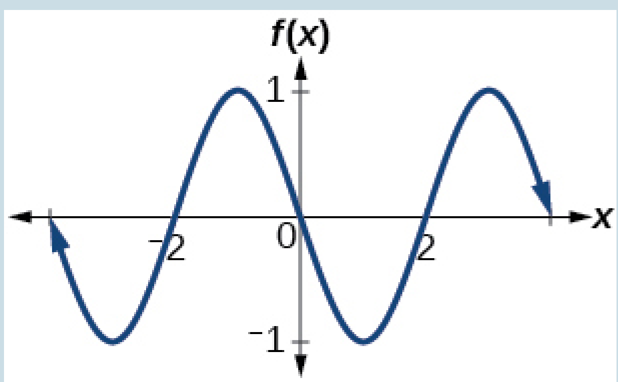


Figure 33.

Algebraic

For the following exercises, let $f\left(x\right) = \sin x$.

31. On $[0, 2\pi)$, solve $f\left(x\right) = 0.$

On $[0, 2\pi)$, solve $f\left(x\right) = \frac{1}{2}.$

Show Solution

$\frac{\pi}{6}, \frac{5\pi}{6}$

32. Evaluate $f\left(\frac{\pi}{2}\right)$.

33. On $[0, 2\pi)$, $f\left(x\right) = \frac{\sqrt{2}}{2}$. Find all values of x.

Show Solution

$\frac{\pi}{4}, \frac{3\pi}{4}$

34. On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what *x*-value(s)?

35. On $[0, 2\pi)$, the minimum value(s) of the function occur(s) at what *x*-value(s)?

Show Solution

$\frac{3\pi}{2}$

36. Show that $f\left(-x\right) = -f\left(x\right)$. This means that $f\left(x\right) = \sin x$ is an odd function and possesses symmetry with respect to ________________.

For the following exercises, let $f\left(x\right) = \cos x.$

37. On $[0, 2\pi)$, solve the equation $f\left(x\right) = \cos x = 0.$

Show Solution

$\frac{\pi}{2}, \frac{3\pi}{2}$

38. On $[0, 2\pi)$, solve $f\left(x\right) = \frac{1}{2}$.

39. On $[0, 2\pi)$, find the *x*-intercepts of $f\left(x\right) = \cos x$.

Show Solution

$\frac{\pi}{2}, \frac{3\pi}{2}$

40. On $[0, 2\pi)$, find the *x*-values at which the function has a maximum or minimum value.

41. On $[0, 2\pi)$, solve the equation $f\left(x\right) = \frac{\sqrt{3}}{2}$.

Show Solution

$\frac{\pi}{6}, \frac{11\pi}{6}$

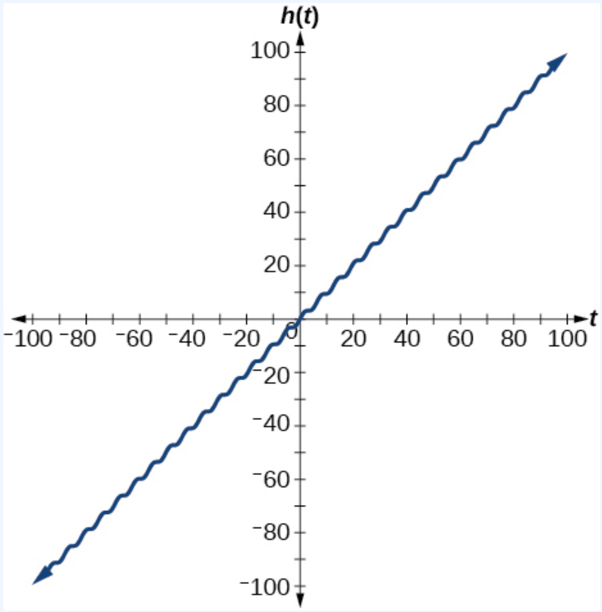
Technology

42. Graph $h\left(x\right) = x + \sin x$ on $[0, 2\pi]$. Explain why the graph appears as it does.

43. Graph $h\left(x\right) = x + \sin x$ on $[-100, 100]$. Did the graph appear as predicted in the previous exercise?

Show Solution

The graph appears linear. The linear functions dominate the shape of the graph for large values of x.

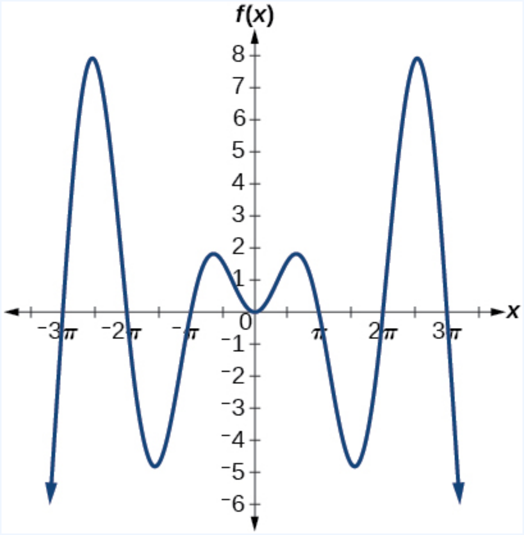


44. Graph $f\left(x\right)=x\sin x$ on $\left[0,2\pi\right]$ and verbalize how the graph varies from the graph of $f\left(x\right)=\sin x.$

45. Graph $f\left(x\right)=x\sin x$ on the window $\left[-10,10\right]$ and explain what the graph shows.

Show Solution

The graph is symmetric with respect to the *y*-axis and there is no amplitude because the function is not periodic.



46. Graph $f\left(x\right)=\frac{\sin x}{x}$ on the window $\left[-5\pi,5\pi\right]$ and explain what the graph shows.

Real-World Applications

47. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h\left(t\right)$ gives a person's height in meters above the ground t minutes after the wheel begins to turn.

a. Find the amplitude, midline, and period of $h\left(t\right)$.
b. Find a formula for the height function $h\left(t\right)$.
c. How high off the ground is a person after 5 minutes?

Show Solution

a. Amplitude: 12.5; period: 10; midline: $y=13.5;$
b. $h\left(t\right)=12.5\sin\left(\frac{\pi}{5}\left(t-2.5\right)\right)+13.5;$
c. 26 ft

Glossary

amplitude
: the vertical height of a function; the constant A appearing in the definition of a sinusoidal function

midline
: the horizontal line $y = D$, where D appears in the general form of a sinusoidal function

periodic function
: a function $f(x)$ that satisfies $f(x + P) = f(x)$ for a specific constant P and any value of x

phase shift
: the horizontal displacement of the basic sine or cosine function; the constant $\frac{C}{B}$

sinusoidal function
: any function that can be expressed in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$

CHAPTER 7.3: GRAPHS OF THE OTHER TRIGONOMETRIC FUNCTIONS

Learning Objectives

In this section, you will:

- Analyze the graph of $y=\tan x$.
- Graph variations of $y=\tan x$.
- Analyze the graphs of $y=\sec x$ and $y=\csc x$.
- Graph variations of $y=\sec x$ and $y=\csc x$.
- Analyze the graph of $y=\cot x$.
- Graph variations of $y=\cot x$.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that
$\tan x=\frac{\sin x}{\cos x}$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{array}{ll} \tan(-x) = \frac{\sin(-x)}{\cos(-x)} & \text{Definition of tangent.} \\ = \frac{-\sin x}{\cos x} & \text{Sine is an odd function, cosine is even.} \\ = -\frac{\sin x}{\cos x} & \text{The quotient of an odd and an even function is odd.} \\ = -\tan x & \text{Definition of tangent.} \end{array}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in (Figure).

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan(x)$	undefined	$-\sqrt{3}$	–1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in (Figure).

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in (Figure).

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. (Figure) represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

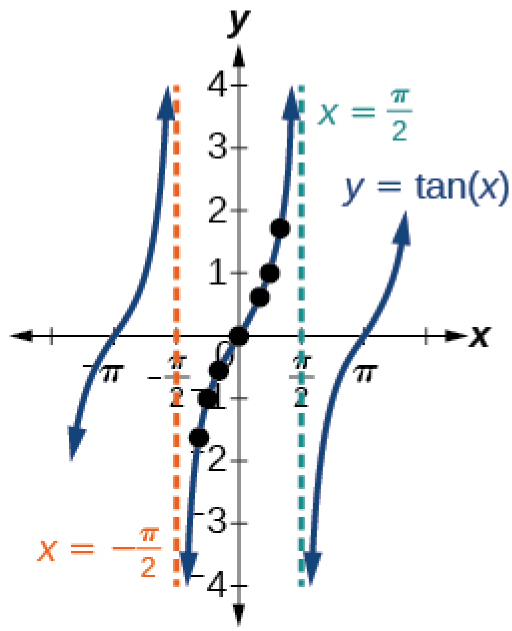


Figure 1. Graph of the tangent function

Graphing Variations of *y* = tan *x*

As with the sine and cosine functions, the tangent function can be described by a general equation.
$y = A\tan\left(Bx\right)$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

Features of the Graph of *y* = *A*tan(*Bx*)

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is all real numbers x, where $x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ such that k is an integer.
- The range is $\left(-\infty, \infty\right)$.
- The asymptotes occur at $x = \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\tan\left(Bx\right)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the tangent function to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f\left(x\right)=A\tan\left(Bx\right)$. We focus on a single period of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is then the interval $\left(-\frac{P}{2},\frac{P}{2}\right)$ and the graph has vertical asymptotes at $\frac{P}{2}$ where $P=\frac{\pi}{B}$. On $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x=-\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x=\frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right)=A\tan\left(B\frac{P}{4}\right)=A\tan\left(B\frac{\pi}{4B}\right)=A$$

because $\tan\left(\frac{\pi}{4}\right)=1$.

Given the function $f\left(x\right)=A\tan\left(Bx\right)$, graph one period.

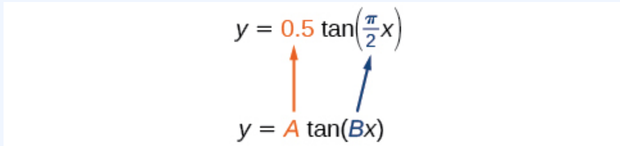
1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P=\frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x=-\frac{P}{2}$ and $x=\frac{P}{2}$.
4. For $A>0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A<0$).
5. Plot reference points at $\left(\frac{P}{4},A\right)$, $\left(0,0\right)$, and $\left(-\frac{P}{4},-A\right)$, and draw the graph through these points.

Sketching a Compressed Tangent

Sketch a graph of one period of the function $y=0.5\tan\left(\frac{\pi}{2}x\right)$.

Show Solution

First, we identify A and B.



Because $A=0.5$ and $B=\frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}}=2$, so the asymptotes are at $x=1$. At a quarter period from the origin, we have

$$\begin{aligned} f\left(0.5\right) &= 0.5\tan\left(\frac{0.5\pi}{2}\right) \\ &= 0.5\tan\left(\frac{\pi}{4}\right) \\ &= 0.5 \end{aligned}$$

This means the curve must pass through the points $\left(0.5, 0.5\right)$, $\left(0, 0\right)$, and $\left(-0.5, -0.5\right)$. The only inflection point is at the origin. (Figure) shows the graph of one period of the function.

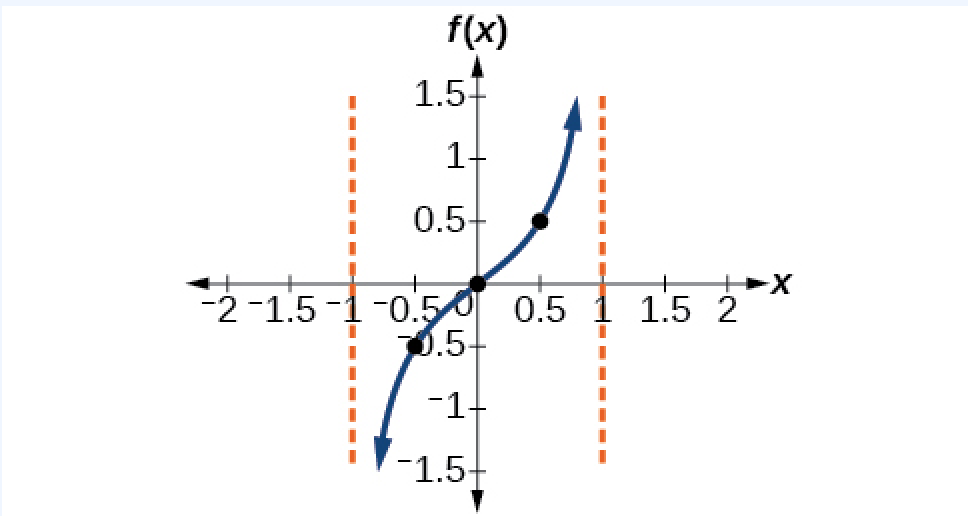
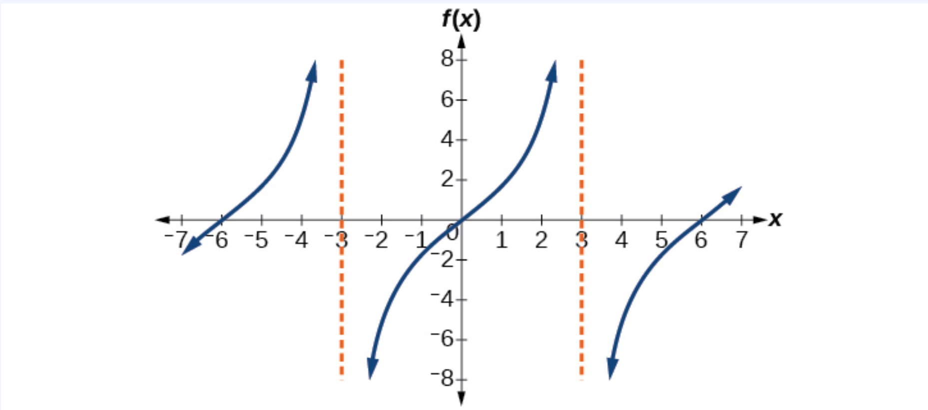


Figure 2.

Try It

Sketch a graph of $f\left(x\right)=3\tan\left(\frac{\pi}{6}x\right).$

Show Solution



Graphing One Period of a Shifted Tangent Function

Now that we can graph a tangent function that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$f\left(x\right)=A\tan\left(Bx-C\right)+D$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

Features of the Graph of *y* = *A*tan(*Bx*−*C*)+*D*

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x\neq\frac{C}{B}+\frac{\pi}{|B|}k$, where k is an integer.
- The range is $\left(-\infty,\infty\right).$

- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx - C) + D$ is an odd function because it is the quotient of odd and even functions (sin and cosine respectively).

Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Show Solution

- *Step 1.* The function is already written in the form $y = A\tan(Bx - C) + D$.
- *Step 2.* $A = -2$, so the stretching factor is $|A| = 2$.
- *Step 3.* $B = \pi$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\pi} = 1$.
- *Step 4.* $C = -\pi$, so the phase shift is $\frac{C}{B} = \frac{-\pi}{\pi} = -1$.
- *Step 5-7.* The asymptotes are at $x = -\frac{3}{2}$ and $x = -\frac{1}{2}$ and the three recommended reference points are $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in (Figure).

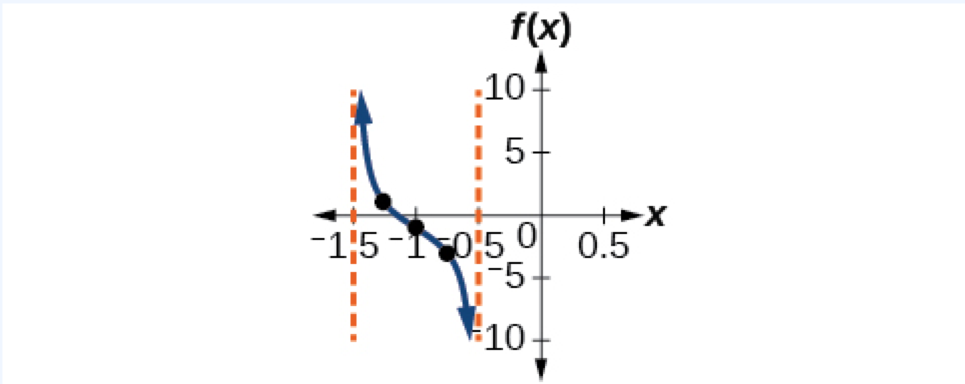


Figure 3.

Analysis

Note that this is a decreasing function because $A < 0$.

Try It

How would the graph in (Figure) look different if we made $A = 2$ instead of -2?

Show Solution
It would be reflected across the line $y = -1$, becoming an increasing function.

Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or x-intercepts.

2. Write $f\left(x\right)=A\tan\left(\frac{\pi}{P}x\right)$.
3. Determine a convenient point $(x,f\left(x\right))$ on the given graph and use it to determine A.

Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in (Figure).

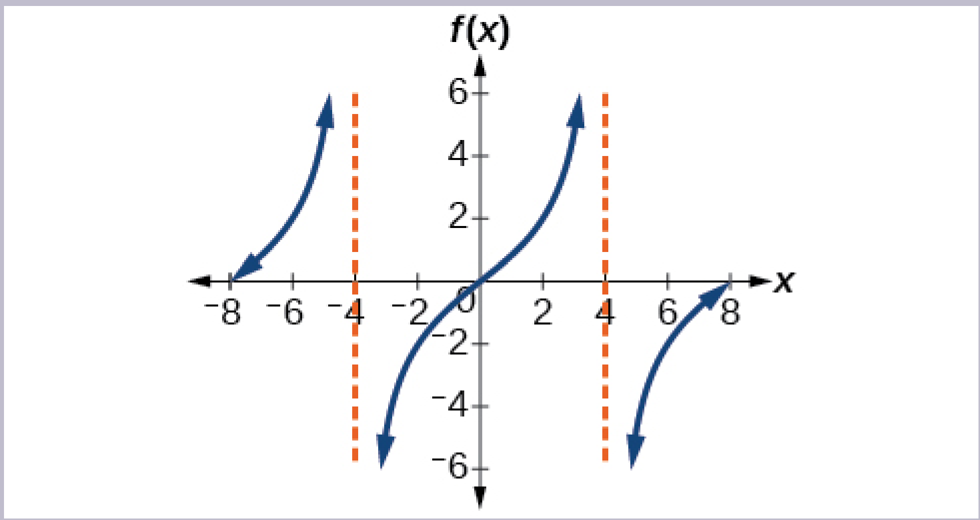


Figure 4. A stretched tangent function

Show Solution

The graph has the shape of a tangent function.

- *Step 1.* One cycle extends from –4 to 4, so the period is $P=8.$ Since $P=\frac{\pi}{|B|}$, we have $B=\frac{\pi}{P}=\frac{\pi}{8}$.
- *Step 2.* The equation must have the form $f\left(x\right)=A\tan\left(\frac{\pi}{8}x\right)$.
- *Step 3.* To find the vertical stretch A, we can use the point $\left(2,2\right)$.
  $2=A\tan\left(\frac{\pi}{8}\cdot 2\right)=A\tan\left(\frac{\pi}{4}\right)$

Because $\tan\left(\frac{\pi}{4}\right)=1, A=2.$

This function would have a formula $f\left(x\right)=2\tan\left(\frac{\pi}{8}x\right)$.

Try It

Find a formula for the function in (Figure).

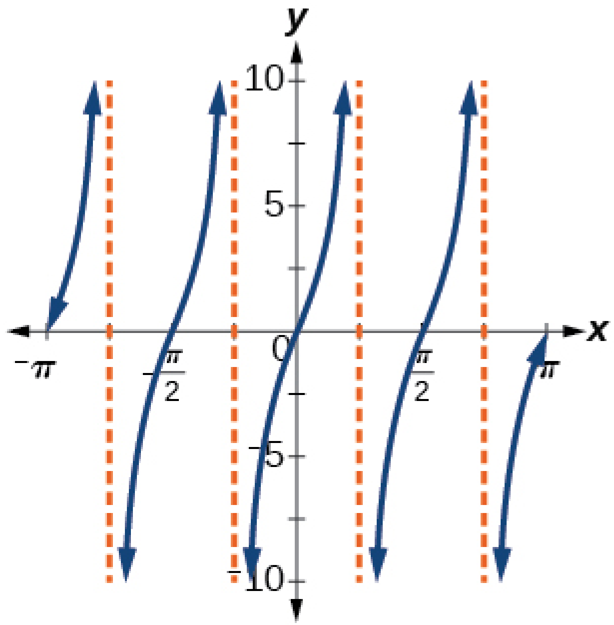


Figure 5.

Show Solution

$g\left(x\right)=4\tan\left(2x\right)$

Analyzing the Graphs of *y* = sec *x* and *y* = csc*x*

The secant was defined by the reciprocal identity $\sec x=\frac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y=\sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See (Figure). The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases.

Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the *x*-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

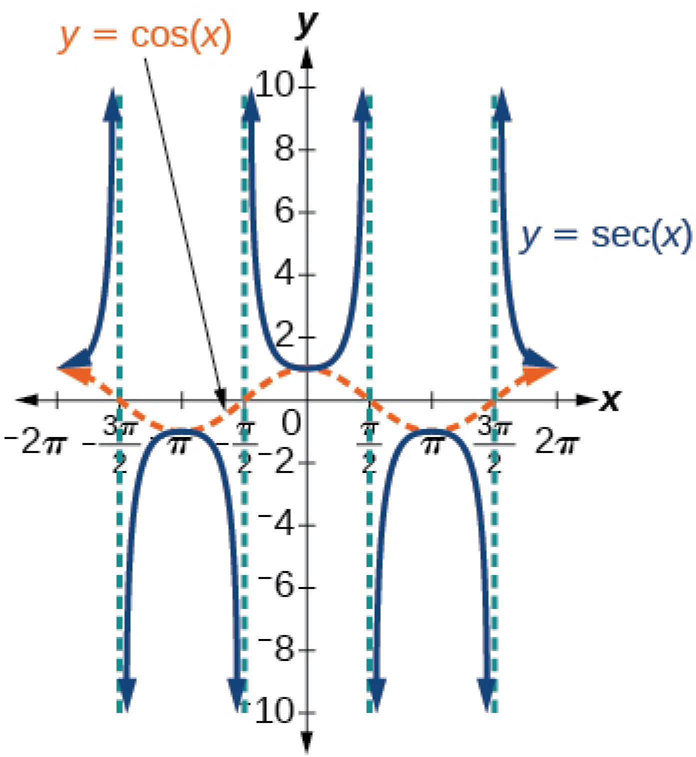


Figure 6. Graph of the secant function, $f(x) = \sec x = \frac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

Features of the Graph of *y* = *A*sec(*Bx*)

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.

- $y = A\sec\left(Bx\right)$ is an even function because cosine is an even function.

Similar to the secant, the cosecant is defined by the reciprocal identity $\csc x = \frac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See (Figure). The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the cosecant function decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the *x*-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc\left(-x\right) = -\csc x$.

The graph of cosecant, which is shown in (Figure), is similar to the graph of secant.

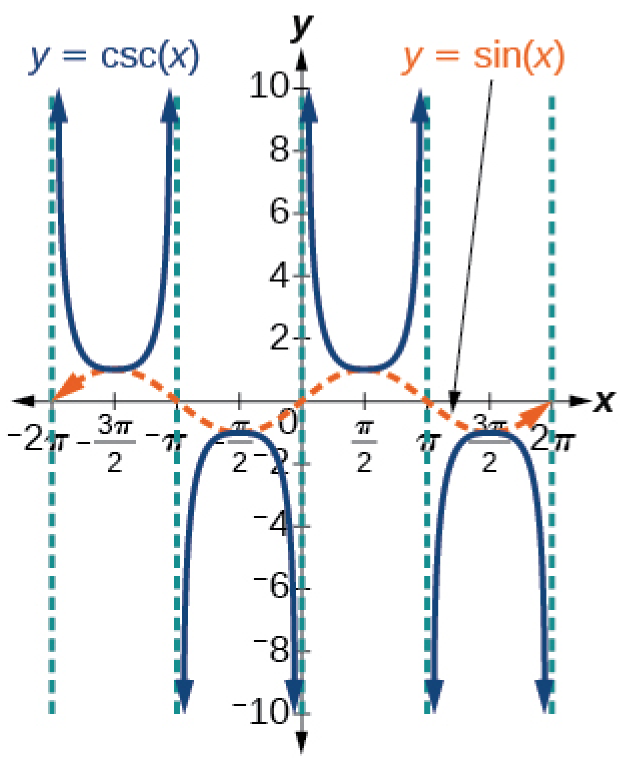


Figure 7. The graph of the cosecant function, $f\left(x\right) = \csc x = \frac{1}{\sin x}$

Features of the Graph of *y* = *A*csc(*Bx*)

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.

- The domain is $x \neq \frac{\pi}{|B|}k$, where k is an integer.
- The range is $\left(-\infty, -|A|\right] \cup \left[|A|, \infty\right)$.
- The asymptotes occur at $x = \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc\left(Bx\right)$ is an odd function because sine is an odd function.

Graphing Variations of *y* = sec *x* and *y*= csc *x*

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$y = A\sec\left(Bx - C\right) + D$
$y = A\csc\left(Bx - C\right) + D$

Features of the Graph of *y* = *A*sec(*Bx*−*C*)+*D*

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $\left(-\infty, -|A| + D\right] \cup \left[|A| + D, \infty\right)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec\left(Bx - C\right) + D$ is an even function because cosine is an even function.

Features of the Graph of *y* = *A*csc(*Bx*−*C*)+*D*

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
- The range is $\left(-\infty, -|A| + D\right] \cup \left[|A| + D, \infty\right)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
- There is no amplitude.
- $y = A\csc\left(Bx - C\right) + D$ is an odd function because sine is an odd function.

Given a function of the form $y = A\sec\left(Bx\right)$, graph one period.

1. Express the function given in the form $y = A\sec\left(Bx\right)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \frac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos\left(Bx\right)$.
5. Use the reciprocal relationship between $y = \cos x$ and $y = \sec x$ to draw the graph of $y = A\sec\left(Bx\right)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Graphing a Variation of the Secant Function

Graph one period of $f\left(x\right) = 2.5\sec\left(0.4x\right)$.

Show Solution

- *Step 1.* The given function is already written in the general form, $y = A\sec\left(Bx\right)$.
- *Step 2.* $A = 2.5$ so the stretching factor is $2.5.$
- *Step 3.* $B = 0.4$ so $P = \frac{2\pi}{0.4} = 5\pi.$ The period is 5π units.
- *Step 4.* Sketch the graph of the function $g\left(x\right) = 2.5\cos\left(0.4x\right)$.
- *Step 5.* Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.
- *Steps 6–7.* Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi.$ We can use two reference points, the local minimum at $\left(0, 2.5\right)$ and the local maximum at $\left(2.5\pi, -2.5\right)$. (Figure) shows the graph.

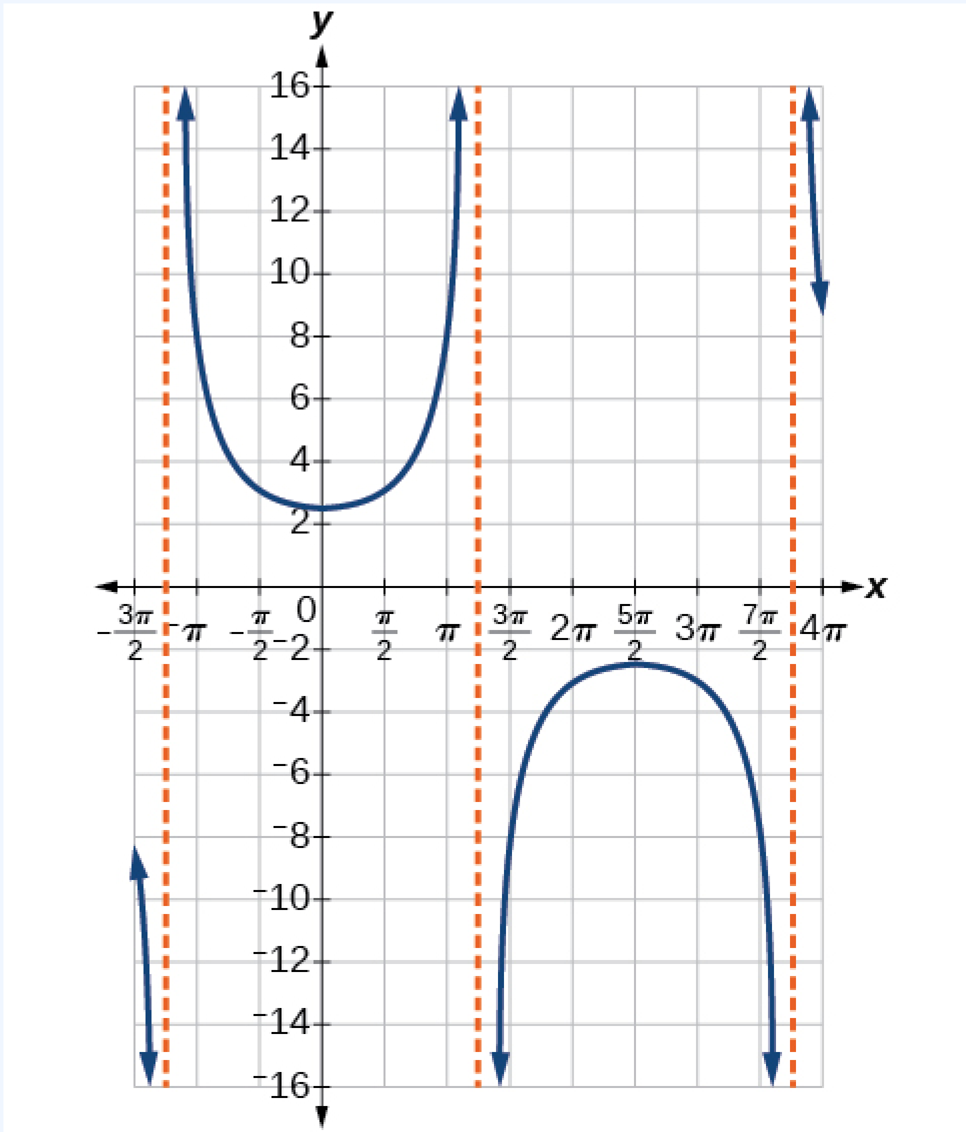

y
x
16
14
12
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
-16
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$
3π
$\frac{7\pi}{2}$
4π

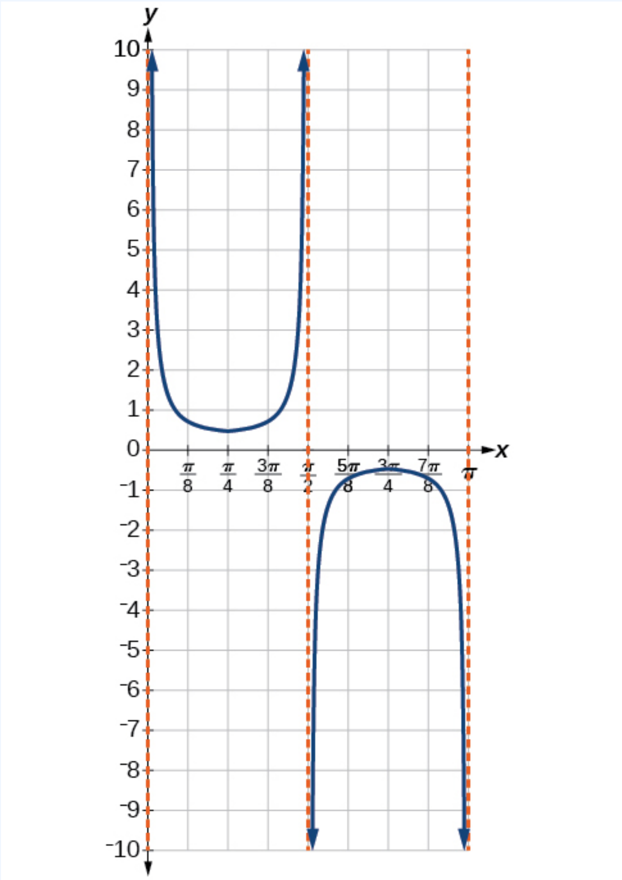
Figure 8.

Try It

Graph one period of $f\left(x\right)=-2.5\text{sec}\left(0.4x\right).$

Show Solution

This is a vertical reflection of the preceding graph because A is negative.



Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f\left(x\right)=A\text{sec}\left(Bx-C\right)+D$ *is* $\left(-\infty,-|A|+D\right]\cup\left[|A|+D,\infty\right).$

Given a function of the form $f\left(x\right)=A\text{sec}\left(Bx-C\right)+D$**, graph one period.**

1. Express the function given in the form $y=A\sec\left(Bx-C\right)+D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y=A\sec\left(Bx\right)$. but shift it to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x=\frac{C}{B}+\frac{\pi}{2|B|}k$, where k is an odd integer.

Graphing a Variation of the Secant Function

Graph one period of $y=4\text{sec}\left(\frac{\pi}{3}x-\frac{\pi}{2}\right)+1.$

Show Solution

- *Step 1.* Express the function given in the form $y=4\text{sec}\left(\frac{\pi}{3}x-\frac{\pi}{2}\right)+1.$
- *Step 2.* The stretching/compressing factor is $|A|=4.$
- *Step 3.* The period is
$$\begin{aligned}\frac{2\pi}{|B|}&=\frac{2\pi}{\frac{\pi}{3}}\\&=\frac{2\pi}{1}\cdot\frac{3}{\pi}\\&=6\end{aligned}$$
- *Step 4.* The phase shift is
$$\begin{aligned}\frac{C}{B}&=\frac{\frac{\pi}{2}}{\frac{\pi}{3}}\\&=\frac{\pi}{2}\cdot\frac{3}{\pi}\\&=1.5\end{aligned}$$
- *Step 5.* Draw the graph of $y=A\text{sec}\left(Bx\right)$, but shift it to the right by $\frac{C}{B}=1.5$ and up by $D=6.$
- *Step 6.* Sketch the vertical asymptotes, which occur at $x=0,x=3,$ and $x=6.$ There

is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. (Figure) shows the graph.

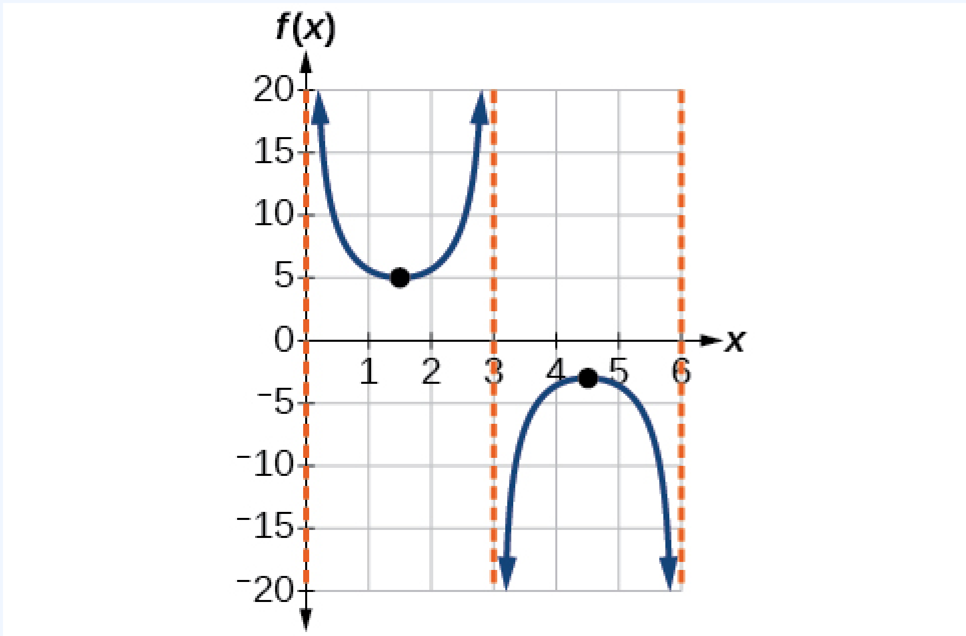
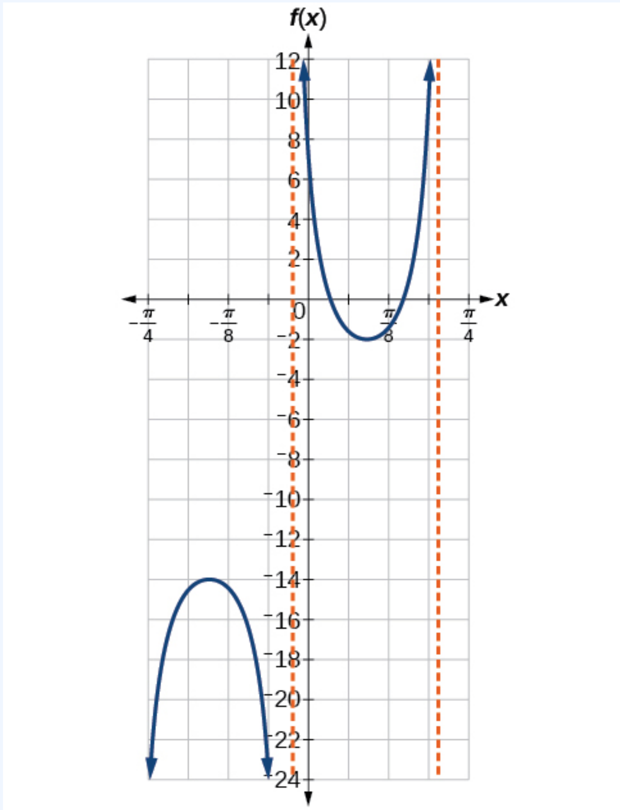


Figure 9.

Try It

Graph one period of $f\left(x\right) = -6\sec\left(4x + 2\right) - 8.$

Show Solution



The domain of $\csc x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc\left(Bx - C\right) + D$ be $x \neq \frac{C+k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

Given a function of the form $y = A\csc\left(Bx\right)$, graph one period.

1. Express the function given in the form $y = A\csc\left(Bx\right)$.

2. $|A|$.
3. Identify B and determine the period, $P = \frac{2\pi}{|B|}$.
4. Draw the graph of $y = A\sin\left(Bx\right)$.
5. Use the reciprocal relationship between $y = \sin x$ and $y = \csc x$ to draw the graph of $y = A\csc\left(Bx\right)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Graphing a Variation of the Cosecant Function

Graph one period of $f\left(x\right) = -3\csc\left(4x\right)$.

Show Solution

- *Step 1.* The given function is already written in the general form, $y = A\csc\left(Bx\right)$.
- *Step 2.* $|A| = |-3| = 3,$ so the stretching factor is 3.
- *Step 3.* $B = 4,$ so $P = \frac{2\pi}{4} = \frac{\pi}{2}.$ The period is $\frac{\pi}{2}$ units.
- *Step 4.* Sketch the graph of the function $g\left(x\right) = -3\sin\left(4x\right)$.
- *Step 5.* Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.
- *Steps 6–7.* Sketch three asymptotes at $x = 0,\ x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. (Figure) shows the graph.

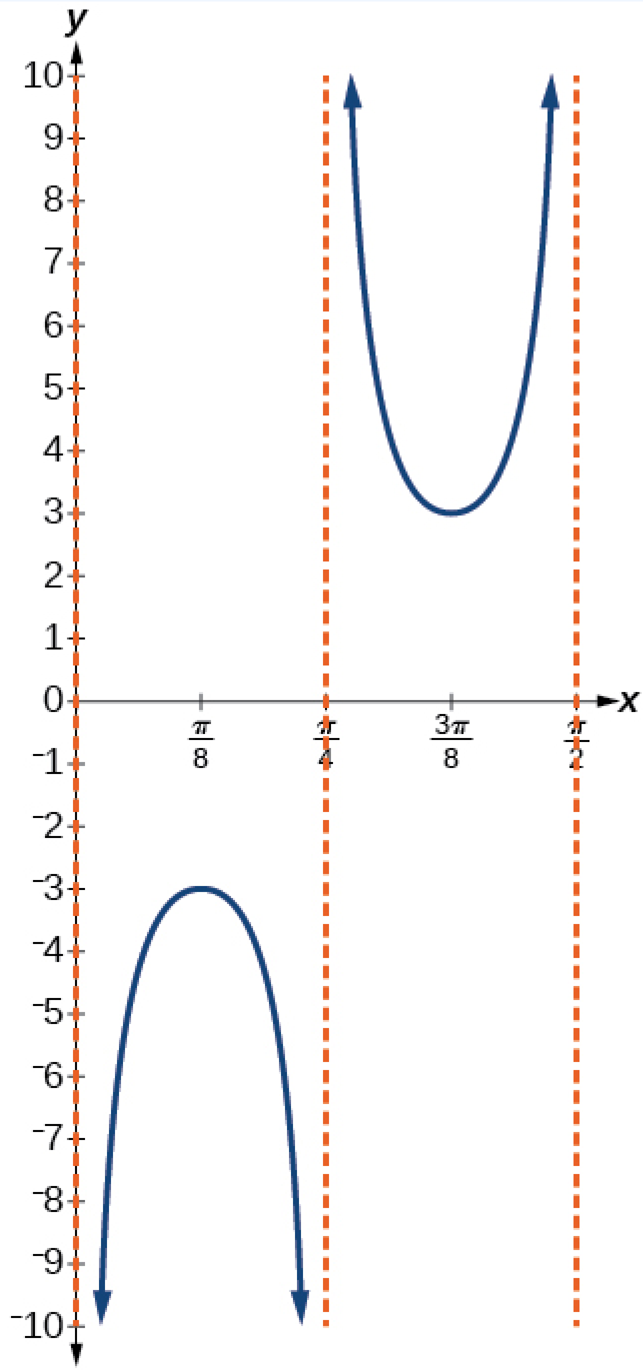
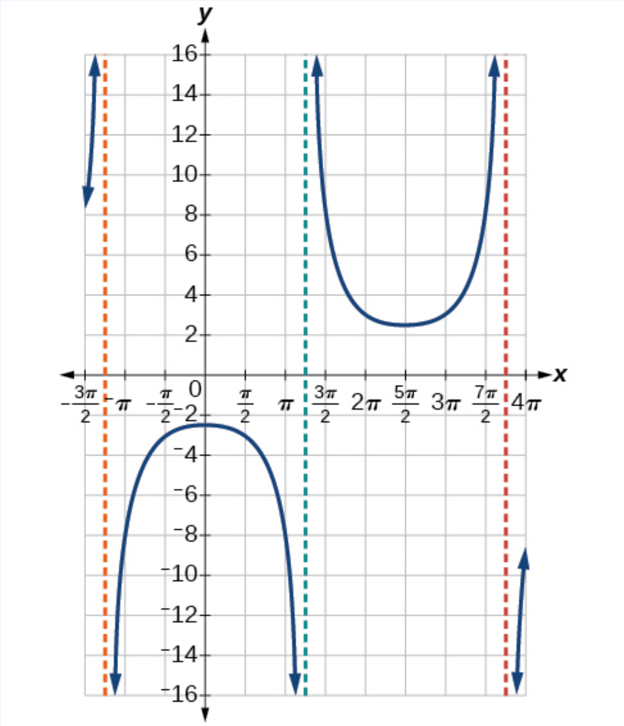


Figure 10.

Try It

Graph one period of $f\left(x\right)=0.5\csc\left(2x\right).$

Show Solution



Given a function of the form $f\left(x\right)=A\csc\left(Bx-C\right)+D$, graph one period.

1. Express the function given in the form $y=A\csc\left(Bx-C\right)+D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y=A\csc\left(Bx\right)$ but shift it to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x=\frac{C}{B}+\frac{\pi}{|B|}k$, where k is an integer.

Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1.$ What are the domain and range of this function?

Show Solution

- *Step 1.* Express the function given in the form $y = 2\csc\left(\frac{\pi}{2}x\right) + 1.$
- *Step 2.* Identify the stretching/compressing factor, $|A| = 2.$
- *Step 3.* The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = \frac{2\pi}{1} \cdot \frac{2}{\pi} = 4.$
- *Step 4.* The phase shift is $\frac{0}{\frac{\pi}{2}} = 0.$
- *Step 5.* Draw the graph of $y = A\csc\left(Bx\right)$ but shift it up $D = 1.$
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0, x = 2, x = 4.$

The graph for this function is shown in (Figure).

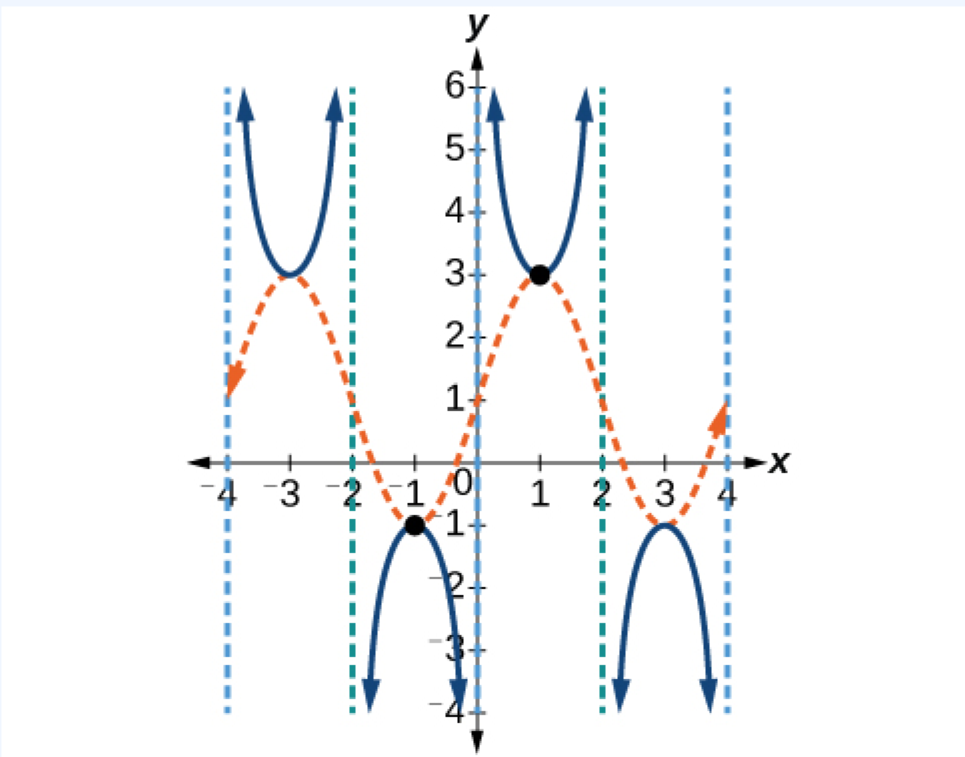


Figure 11. A transformed cosecant function

Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f\left(x\right)=2\sin\left(\frac{\pi}{2}x\right)+1,$ shown as the orange dashed wave.

Try It

Given the graph of $f\left(x\right)=2\cos\left(\frac{\pi}{2}x\right)+1$ shown in (Figure), sketch the graph of $g\left(x\right)=2\sec\left(\frac{\pi}{2}x\right)+1$ on the same axes.

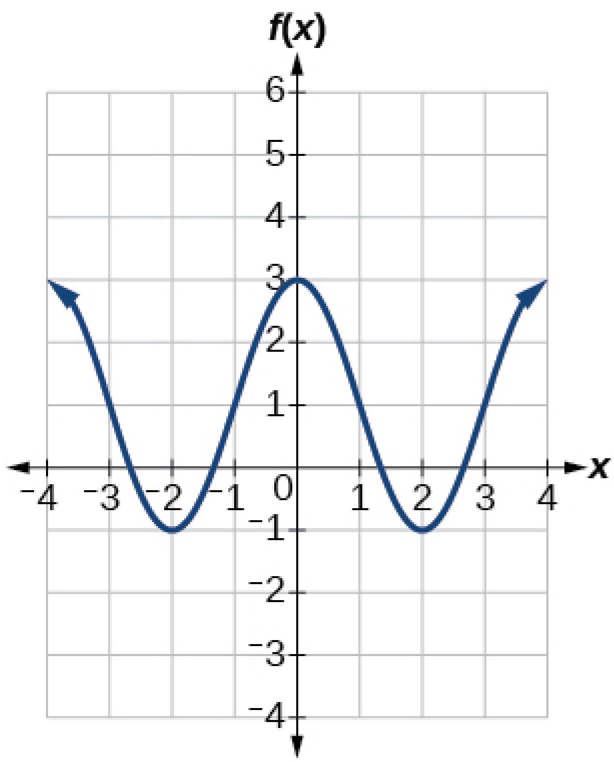
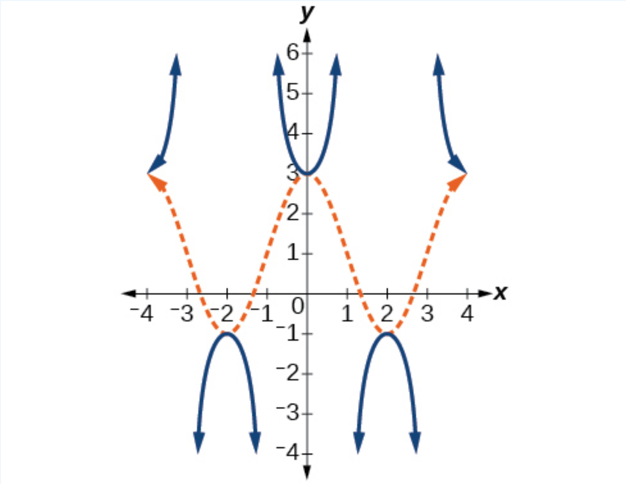


Figure 12.

Show Solution



Analyzing the Graph of *y* = cot *x*

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $\cot x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at $0, \pi$, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See (Figure). Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

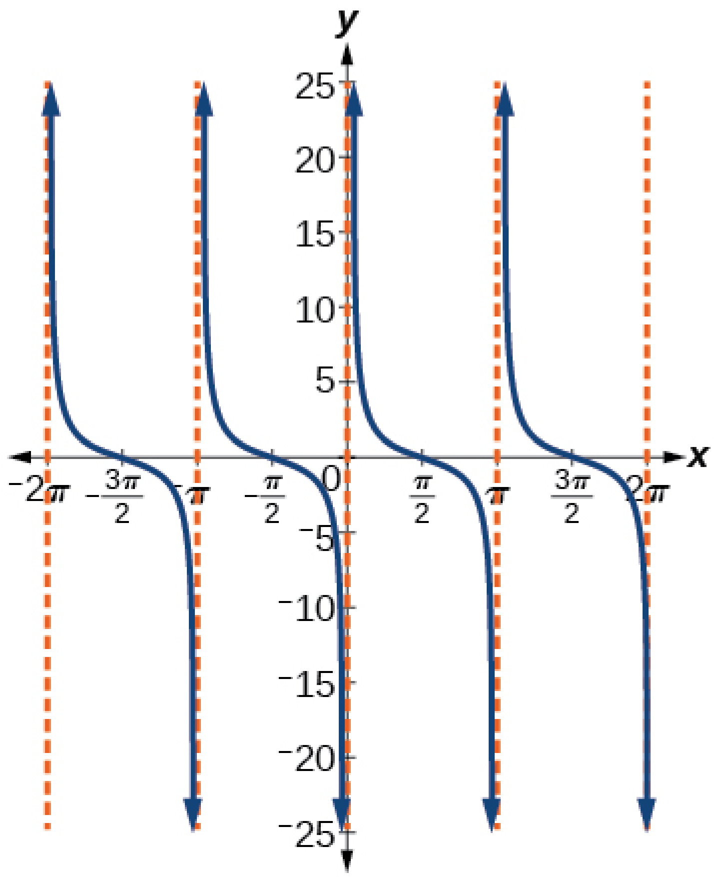


Figure 13. The cotangent function

Features of the Graph of *y* = *A*cot(*Bx*)

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of *y* = cot *x*

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D$$

Properties of the Graph of *y* = *A*cot(*Bx*−C)+*D*

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
- The range is $\left(-\infty, \infty\right)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
- There is no amplitude.
- $y = A\cot\left(Bx\right)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

Given a modified cotangent function of the form $f\left(x\right) = A\cot\left(Bx\right)$, **graph one period.**

1. Express the function in the form $f\left(x\right) = A\cot\left(Bx\right)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Draw the graph of $y = A\tan\left(Bx\right)$.
5. Plot any two reference points.
6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = A\cot\left(Bx\right)$.
7. Sketch the asymptotes.

Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y = 3\cot\left(4x\right)$, and then sketch a graph.

Show Solution

- *Step 1.* Expressing the function in the form $f\left(x\right) = A\cot\left(Bx\right)$ gives $f\left(x\right) = 3\cot\left(4x\right)$.
- *Step 2.* The stretching factor is $|A| = 3$.
- *Step 3.* The period is $P = \frac{\pi}{4}$.

- *Step 4.* Sketch the graph of $y = 3\tan\left(4x\right).$
- *Step 5.* Plot two reference points. Two such points are $\left(\frac{\pi}{16}, 3\right)$ and $\left(\frac{3\pi}{16}, -3\right).$
- *Step 6.* Use the reciprocal relationship to draw $y = 3\cot\left(4x\right).$
- *Step 7.* Sketch the asymptotes, $x = 0,\ \ x = \frac{\pi}{4}.$

The orange graph in (Figure) shows $y = 3\tan\left(4x\right)$ and the blue graph shows $y = 3\cot\left(4x\right).$

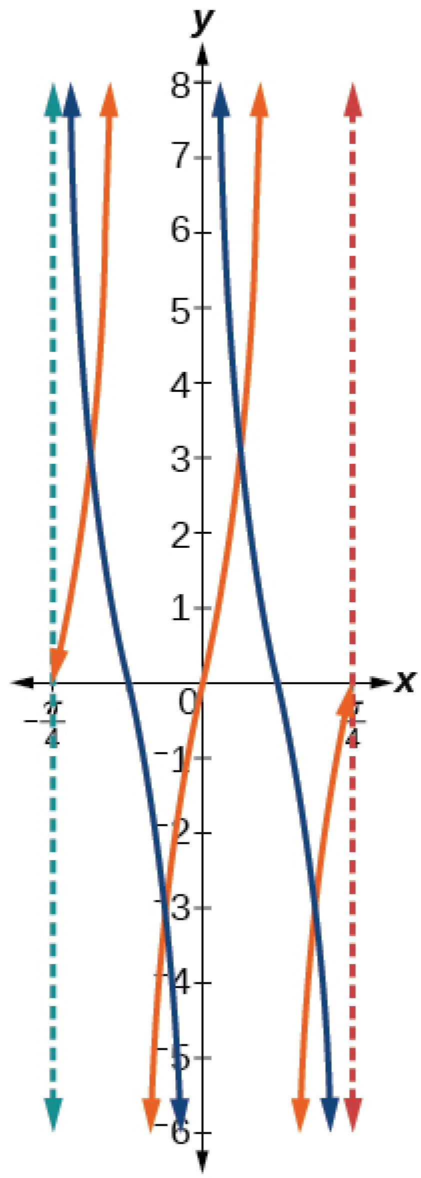


Figure 14.

Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.

2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan\left(Bx\right)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
7. Plot any three reference points and draw the graph through these points.

Graphing a Modified Cotangent

Sketch a graph of one period of the function $f\left(x\right) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2.$

Show Solution

- *Step 1.* The function is already written in the general form $f\left(x\right) = A\cot\left(Bx - C\right) + D.$
- *Step 2.* $A = 4,$ so the stretching factor is 4.
- *Step 3.* $B = \frac{\pi}{8},$ so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8.$
- *Step 4.* $C = \frac{\pi}{2},$ so the phase shift is $\frac{C}{B} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4.$
- *Step 5.* We draw $f\left(x\right) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2.$
- *Step 6-7.* Three points we can use to guide the graph are $\left(6, 2\right)$, $\left(8, -2\right)$, and $\left(10, -6\right)$. We use the reciprocal relationship of tangent and cotangent to draw $f\left(x\right) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2.$
- *Step 8.* The vertical asymptotes are $x = 4$ and $x = 12.$

The graph is shown in (Figure).

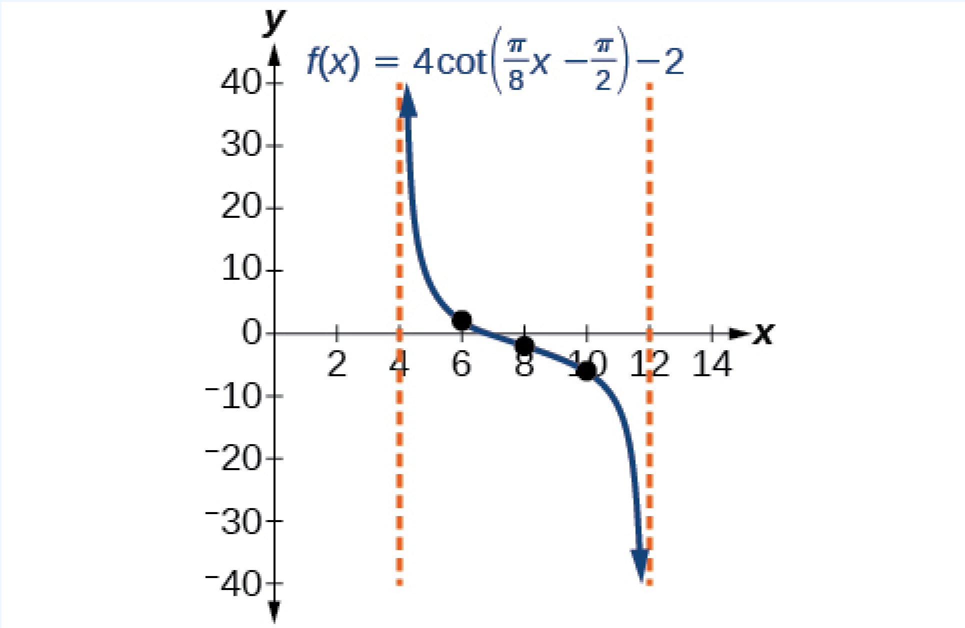


Figure 15. One period of a modified cotangent function

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the tangent function.

Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan\left(\frac{\pi}{4}t\right)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Show Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

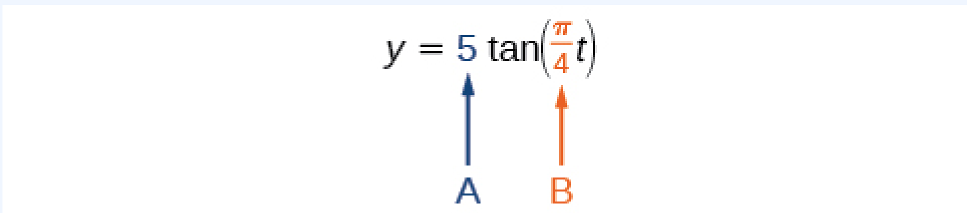


Figure 16.

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4.$ This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See (Figure)

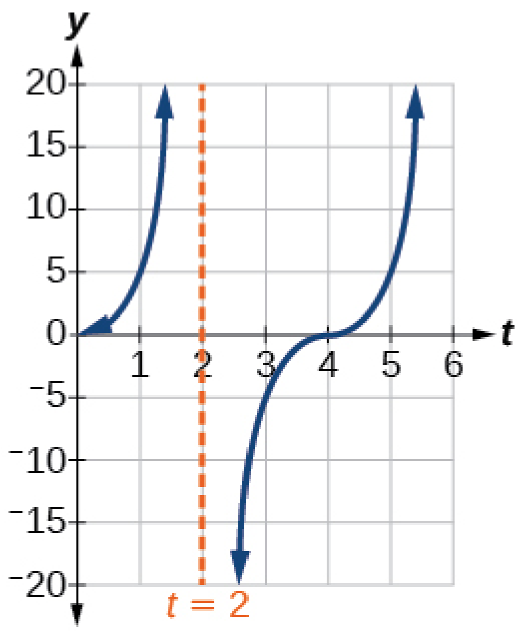


Figure 17.

c. period: $f\left(1\right)=5\tan\left(\frac{\pi}{4}\left(1\right)\right)=5\left(1\right)=5;$ after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent
- Graphing Cosecant and Secant
- Graphing the Cotangent

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan\left(Bx - C\right) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec\left(Bx - C\right) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc\left(Bx - C\right) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot\left(Bx - C\right) + D$

Key Concepts

- The tangent function has period π.
- $f\left(x\right) = A\text{tan}\left(Bx - C\right) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See (Figure), (Figure), and (Figure).
- The secant and cosecant are both periodic functions with a period of 2π. $f\left(x\right) = A\text{sec}\left(Bx - C\right) + D$ gives a shifted, compressed, and/or stretched secant function graph. See (Figure) and (Figure).
- $f\left(x\right) = A\text{csc}\left(Bx - C\right) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See (Figure) and (Figure).
- The cotangent function has period π and vertical asymptotes at $0, \pi, 2\pi, ...$
- The range of cotangent is $\left(-\infty, \infty\right)$, and the function is decreasing at each point in its range.
- The cotangent is zero at $\frac{\pi}{2}, \frac{3\pi}{2}, ...$
- $f\left(x\right) = A\text{cot}\left(Bx - C\right) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See (Figure) and (Figure).
- Real-world scenarios can be solved using graphs of trigonometric functions. See (Figure).

Section Exercises

Verbal

1. Explain how the graph of the sine function can be used to graph $y = \csc x$.

Show Solution

Since $y = \csc x$ is the reciprocal function of $y = \sin x$, you can plot the reciprocal of the coordinates on the graph of $y = \sin x$ to obtain the *y*-coordinates of $y = \csc x$. The *x*-intercepts of the graph $y = \sin x$ are the vertical asymptotes for the graph of $y = \csc x$.

2. How can the graph of $y = \cos x$ be used to construct the graph of $y = \sec x$?

3. Explain why the period of $\tan x$ is equal to π.

Show Solution

Answers will vary. Using the unit circle, one can show that $\tan\left(x + \pi\right) = \tan x$.

4. Why are there no intercepts on the graph of $y = \csc x$?

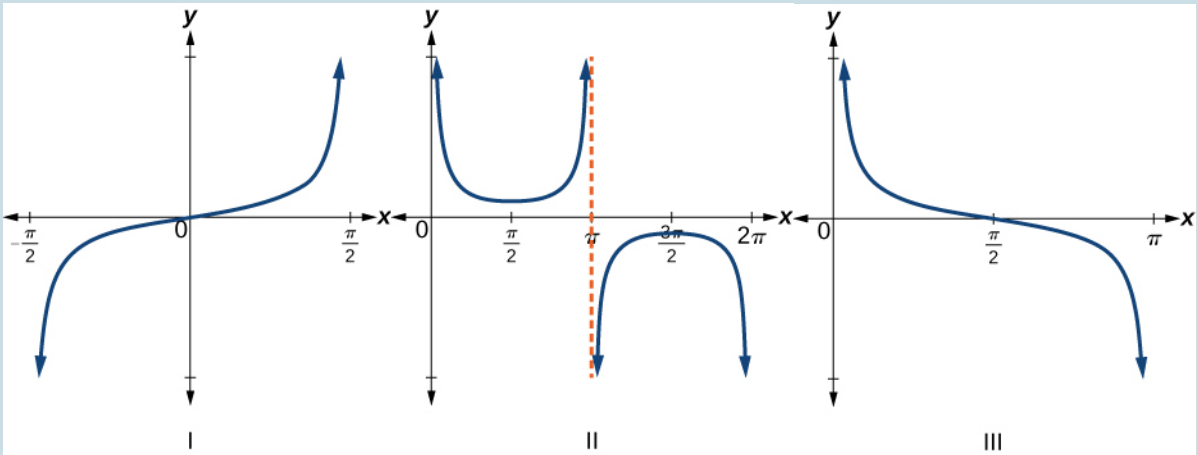
5. How does the period of $y = \csc x$ compare with the period of $y = \sin x$?

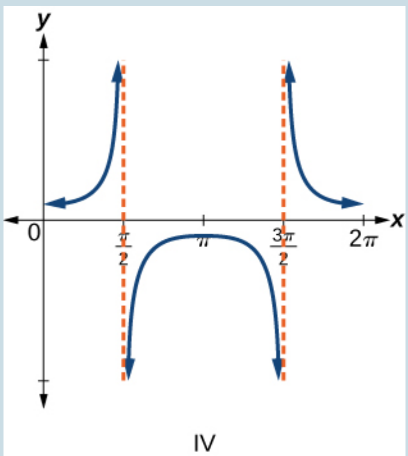
Show Solution

The period is the same: 2π.

Algebraic

For the following exercises, match each trigonometric function with one of the following graphs.





6. $f\left(x\right)=\tan x$

7. $f\left(x\right)=\sec x$

Show Solution

IV

8. $f\left(x\right)=\csc x$

9. $f\left(x\right)=\cot x$

Show Solution
III

For the following exercises, find the period and horizontal shift of each of the functions.

10. $f\left(x\right)=2\tan\left(4x-32\right)$

11. $h\left(x\right)=2\sec\left(\frac{\pi}{4}\left(x+1\right)\right)$

Show Solution
period: 8; horizontal shift: 1 unit to left

12. $m\left(x\right)=6\csc\left(\frac{\pi}{3}x+\pi\right)$

13. If $\tan x=-1.5,$ find $\tan\left(-x\right).$

Show Solution
1.5

14. If $\sec x=2,$ find $\sec\left(-x\right).$

15. If $\csc x=-5,$ find $\csc\left(-x\right).$

Show Solution
5

16. If $x\sin x=2,$ find $\left(-x\right)\sin\left(-x\right).$

For the following exercises, rewrite each expression such that the argument x is positive.

17. $\cot\left(-x\right)\cos\left(-x\right)+\sin\left(-x\right)$

Show Solution

$-\cot x\cos x-\sin x$

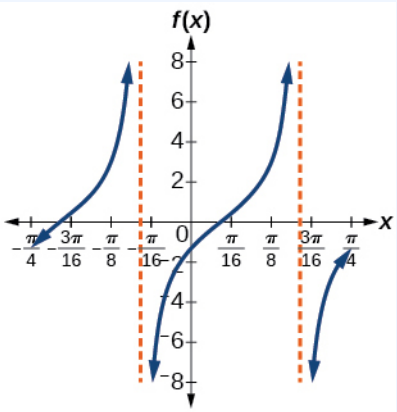
18. $\cos\left(-x\right)+\tan\left(-x\right)\sin\left(-x\right)$

Graphical

For the following exercises, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

19. $f\left(x\right)=2\tan\left(4x-32\right)$

Show Solution



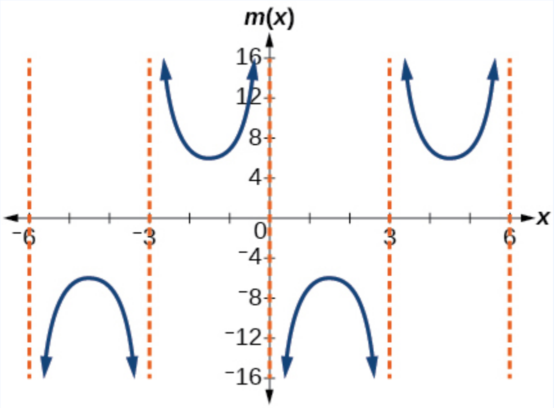
stretching factor: 2; period: $\frac{\pi}{4}$; asymptotes:

$x=\frac{1}{4}\left(\frac{\pi}{2}+\pi k\right)+8,\ \text{where } k \text{ is an integer}$

20. $h\left(x\right)=2\sec\left(\frac{\pi}{4}\left(x+1\right)\right)$

21. $m\left(x\right)=6\csc\left(\frac{\pi}{3}x+\pi\right)$

Show Solution

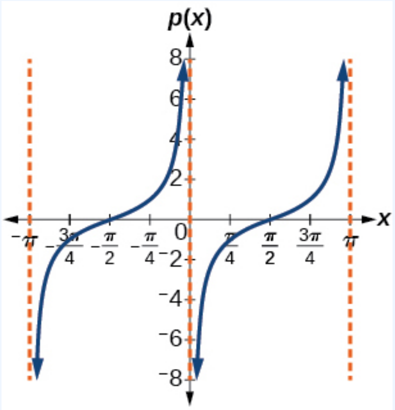


stretching factor: 6; period: 6; asymptotes: $x = 3k,\ \text{where } k \text{ is an integer}$

22. $j\left(x\right) = \tan\left(\frac{\pi}{2}x\right)$

23. $p\left(x\right) = \tan\left(x - \frac{\pi}{2}\right)$

Show Solution

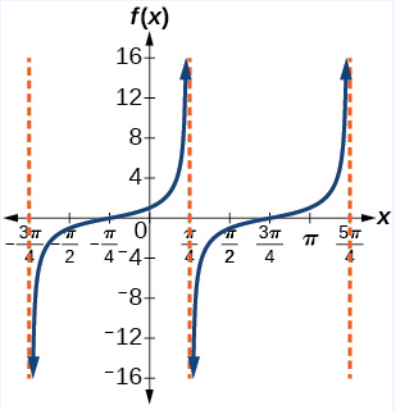


stretching factor: 1; period: π; asymptotes: $x = \pi k,\ \text{where } k \text{ is an integer}$

24. $f\left(x\right) = 4\tan\left(x\right)$

25. $f\left(x\right) = \tan\left(x + \frac{\pi}{4}\right)$

Show Solution

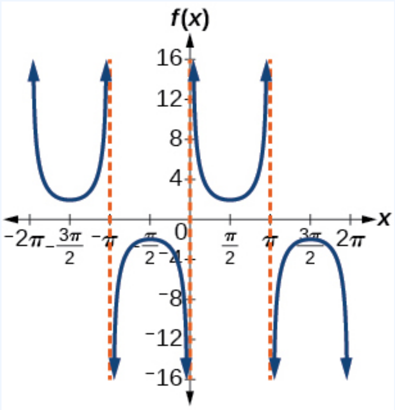


Stretching factor: 1; period: π; asymptotes: $x = \frac{\pi}{4} + \pi k, \text{ where } k \text{ is an integer}$

26. $f\left(x\right) = \pi\tan\left(\pi x - \pi\right) - \pi$

27. $f\left(x\right) = 2\csc\left(x\right)$

Show Solution

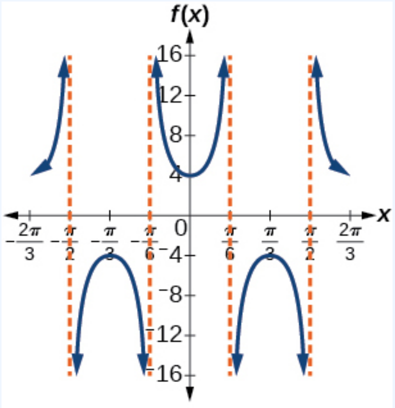


stretching factor: 2; period: 2π; asymptotes: $x = \pi k, \text{ where } k \text{ is an integer}$

28. $f\left(x\right) = -\frac{1}{4}\csc\left(x\right)$

29. $f\left(x\right) = 4\sec\left(3x\right)$

Show Solution

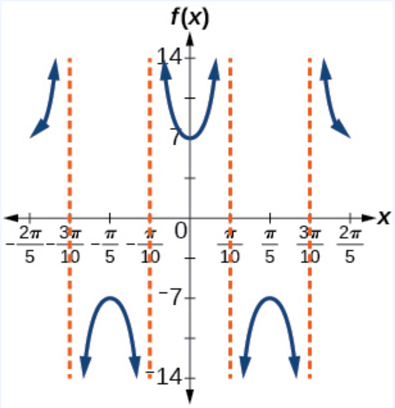


stretching factor: 4; period: $\frac{2\pi}{3}$; asymptotes: $x=\frac{\pi}{6}k,\text{ where } k \text{ is an odd integer}$

30. $f\left(x\right)=-3\cot\left(2x\right)$

31. $f\left(x\right)=7\sec\left(5x\right)$

Show Solution

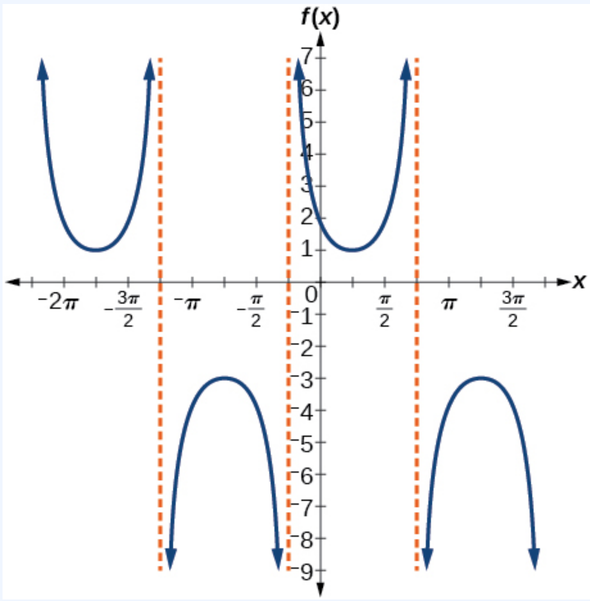


stretching factor: 7; period: $\frac{2\pi}{5}$; asymptotes: $x=\frac{\pi}{10}k,\text{ where } k \text{ is an odd integer}$

32. $f\left(x\right)=\frac{9}{10}\csc\left(\pi x\right)$

33. $f\left(x\right)=2\csc\left(x+\frac{\pi}{4}\right)-1$

Show Solution

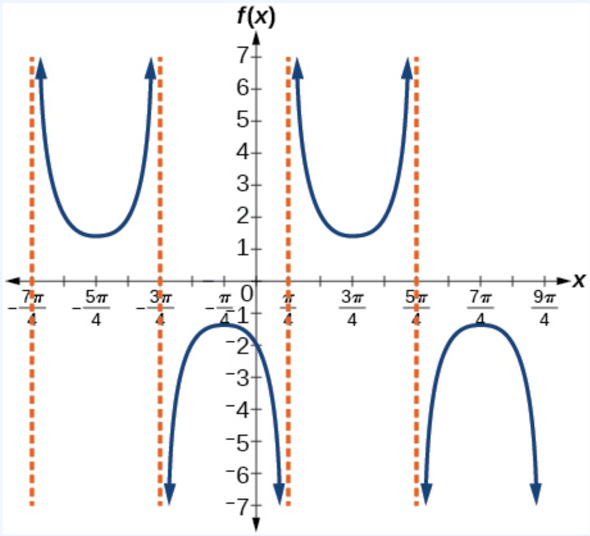


stretching factor: 2; period: $2\pi;$ asymptotes: $x=-\frac{\pi}{4}+\pi k,\text{ where }k\text{ is an integer}$

34. $f\left(x\right)=-\sec\left(x-\frac{\pi}{3}\right)-2$

35. $f\left(x\right)=\frac{7}{5}\csc\left(x-\frac{\pi}{4}\right)$

Show Solution



stretching factor: $\frac{7}{5}$; period: 2π; asymptotes: $x=\frac{\pi}{4}+\pi k,\text{ where }k\text{ is an integer}$

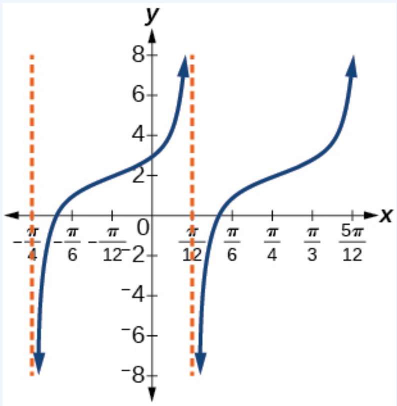
36. $f\left(x\right)=5\left(\cot\left(x+\frac{\pi}{2}\right)-3\right)$

For the following exercises, find and graph two periods of the periodic function with the given stretching factor, $|A|$, period, and phase shift.

37. A tangent curve, $A=1$, period of $\frac{\pi}{3}$; and phase shift $(h,\ k)=\left(\frac{\pi}{4},2\right)$

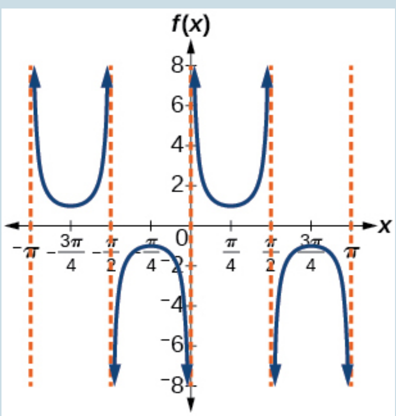
Show Solution

$y=\tan\left(3\left(x-\frac{\pi}{4}\right)\right)+2$



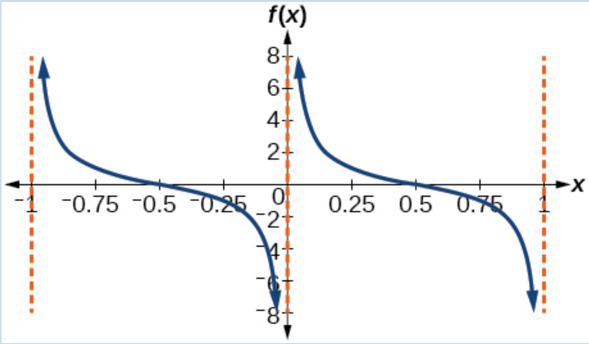
38. A tangent curve, $A = -2$, period of $\frac{\pi}{4}$, and phase shift $(h,\ k) = \left(-\frac{\pi}{4},\ -2\right)$

For the following exercises, find an equation for the graph of each function.

39. 

Show Solution

$f\left(x\right) = \csc\left(2x\right)$

40. 

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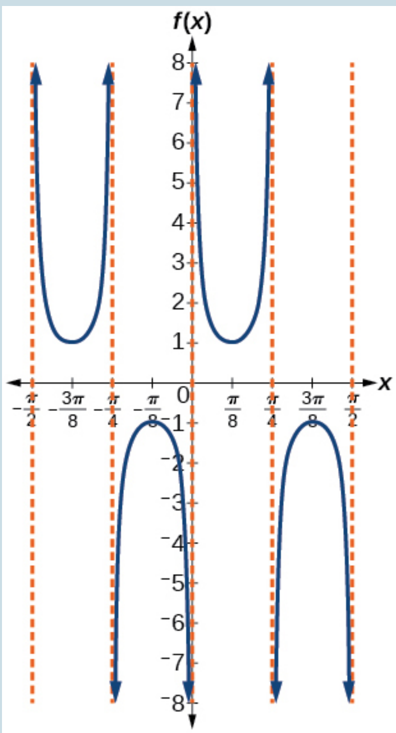
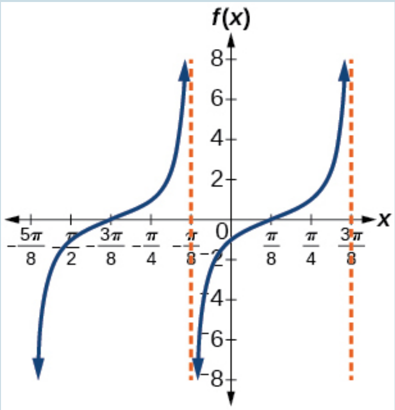
41. 

Show Solution

$f\left(x\right)=\csc\left(4x\right)$

42. 

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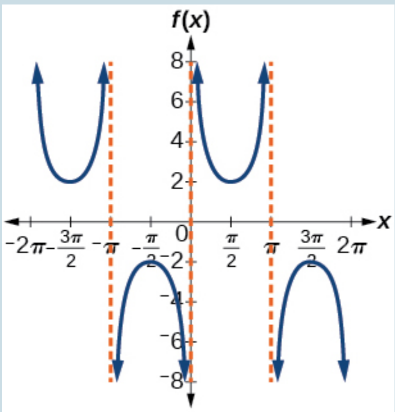
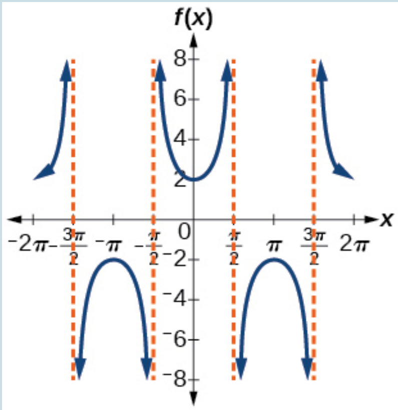
43. 

Show Solution

$f\left(x\right)=2\csc x$

44. 

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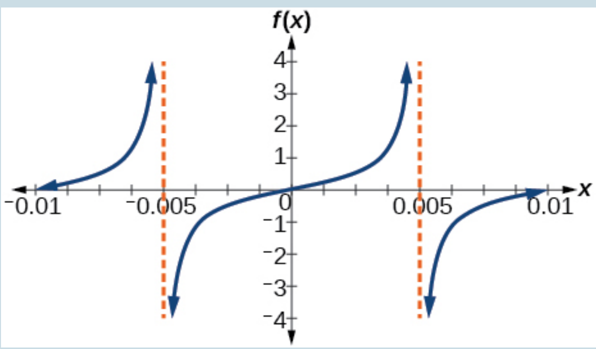
45. 

Show Solution

$f\left(x\right)=\frac{1}{2}\tan\left(100\pi x\right)$

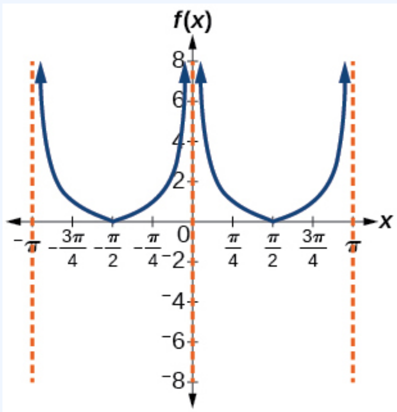
Technology

For the following exercises, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input $\csc x$ as $\frac{1}{\sin x}$.

46. $f\left(x\right)=\left|\csc\left(x\right)\right|$

47. $f\left(x\right)=\left|\cot\left(x\right)\right|$

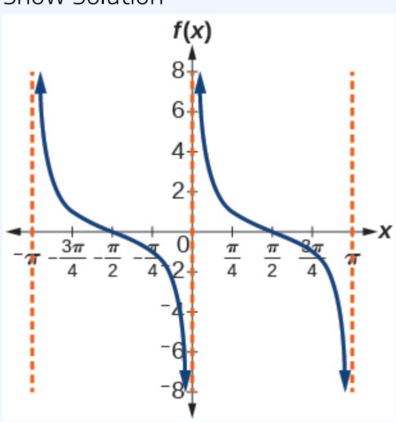
Show Solution



48. $f\left(x\right)=2^{\csc(x)}$

49. $f\left(x\right)=\frac{\csc(x)}{\sec(x)}$

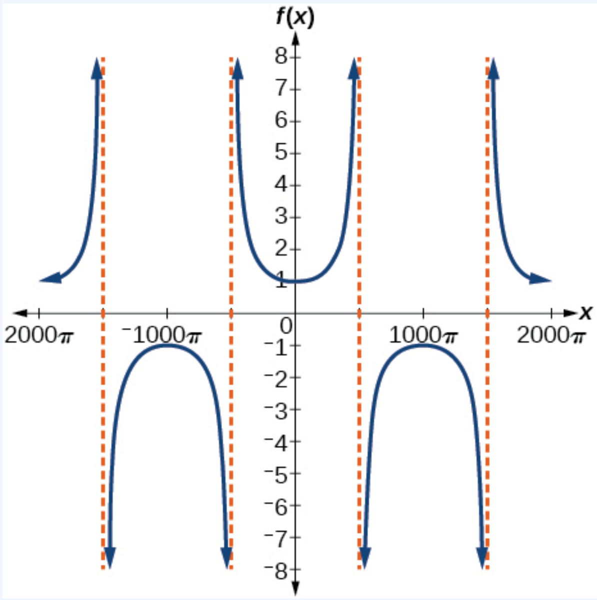
Show Solution



50. Graph $f\left(x\right)=1+\sec^2\left(x\right)-\tan^2\left(x\right)$. What is the function shown in the graph?

51. $f\left(x\right)=\sec\left(0.001x\right)$

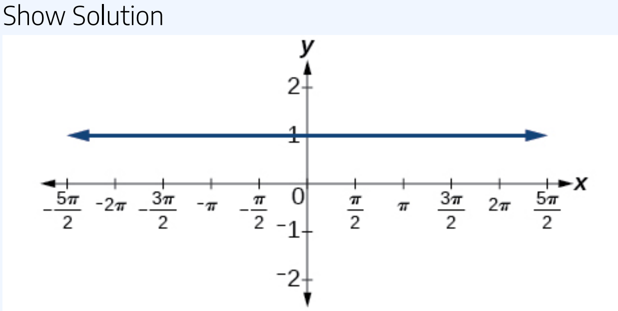
Show Solution



52. $f\left(x\right)=\cot\left(100\pi x\right)$

53. $f\left(x\right)=\sin^2 x+\cos^2 x$

Show Solution



Real-World Applications

54. The function $f\left(x\right)=20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from

a police car across a wall for time x, in seconds, and distance $f\left(x\right)$, in feet.

a. Graph on the interval $\left[0,\ 5\right]$.
b. Find and interpret the stretching factor, period, and asymptote.
c. Evaluate $f\left(1\right)$ and $f\left(2.5\right)$ and discuss the function's values at those inputs.

55. Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. (See (Figure).) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d\left(x\right)$, in kilometers, from the fisherman to the boat is given by the function $d\left(x\right)=1.5\sec\left(x\right)$.

a. What is a reasonable domain for $d\left(x\right)$?
b. Graph $d\left(x\right)$ on this domain.
c. Find and discuss the meaning of any vertical asymptotes on the graph of $d\left(x\right)$.
d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.
e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the second decimal place.
f. What is the minimum distance between the fisherman and the boat? When does this occur?

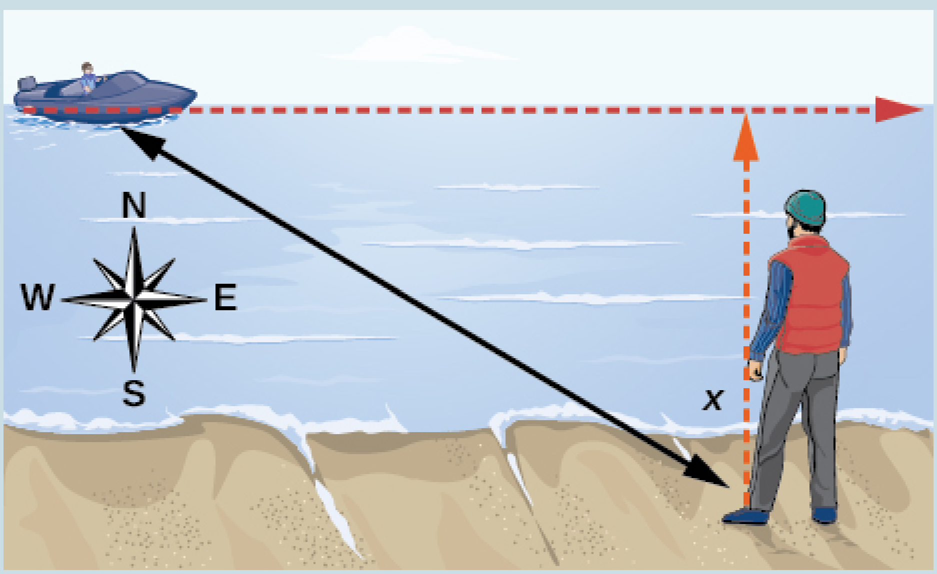
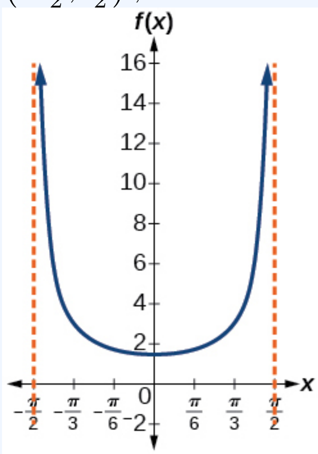


Figure 18.

Show Solution

a. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$;

b. 

c. $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$; the distance grows without bound as $|x|$ approaches $\frac{\pi}{2}$—i.e., at right angles to the line representing due north, the boat would be so far away, the fisherman could not see it;

d. 3; when $x = -\frac{\pi}{3}$, the boat is 3 km away;

e. 1.73; when $x = \frac{\pi}{6}$, the boat is about 1.73 km away;

f. 1.5 km; when $x = 0$

56. A laser rangefinder is locked on a comet approaching Earth. The distance $g\left(x\right)$, in kilometers, of the comet after x days, for x in the interval 0 to 30 days, is given by $g\left(x\right) = 250,000\csc\left(\frac{\pi}{30}x\right)$.

a. Graph $g\left(x\right)$ on the interval $\left[0,\ 35\right]$.
b. Evaluate $g\left(5\right)$ and interpret the information.
c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?
d. Find and discuss the meaning of any vertical asymptotes.

57. A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.

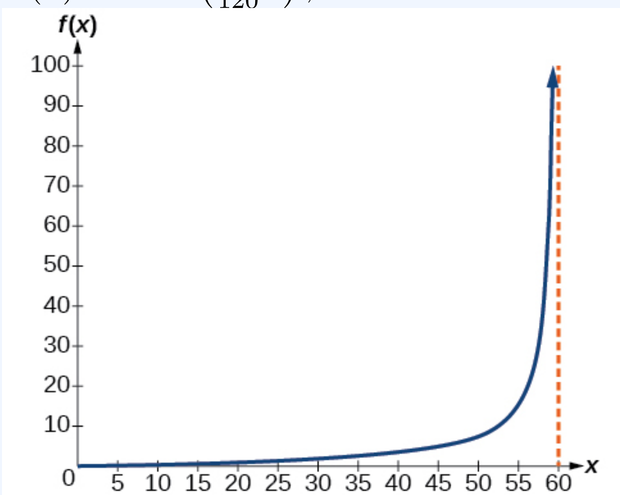
a. Write a function expressing the altitude $h\left(x\right)$, in miles, of the rocket above the ground after

x seconds. Ignore the curvature of the Earth.

b. Graph $h(x)$ on the interval $(0,\ 60)$.
c. Evaluate and interpret the values $h(0)$ and $h(30)$.
d. What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.

Show Solution

a. $h(x) = 2\tan\left(\frac{\pi}{120}x\right)$;

b. 

c. $h(0) = 0$: after 0 seconds, the rocket is 0 mi above the ground; $h(30) = 2$: after 30 seconds, the rockets is 2 mi high;
d. As x approaches 60 seconds, the values of $h(x)$ grow increasingly large. The distance to the rocket is growing so large that the camera can no longer track it.

CHAPTER 7.4: INVERSE TRIGONOMETRIC FUNCTIONS

Learning Objectives

In this section, you will:

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in (Figure).

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure 1.

For example, if $f\left(x\right)=\sin x$, then we would write $f^{-1}\left(x\right)=\sin^{-1}x$. Be aware that $\sin^{-1}x$ does not mean $\frac{1}{\sin x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$, then $\frac{\pi}{6}=\sin^{-1}\left(\frac{1}{2}\right)$.
- Since $\cos\left(\pi\right)=-1$, then $\pi=\cos^{-1}\left(-1\right)$.
- Since $\tan\left(\frac{\pi}{4}\right)=1$, then $\frac{\pi}{4}=\tan^{-1}\left(1\right)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f\left(a\right)=b$, then an inverse function would satisfy $f^{-1}\left(b\right)=a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the domain of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. (Figure) shows the graph of the sine function limited to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and the graph of the cosine function limited to $\left[0,\pi\right]$.

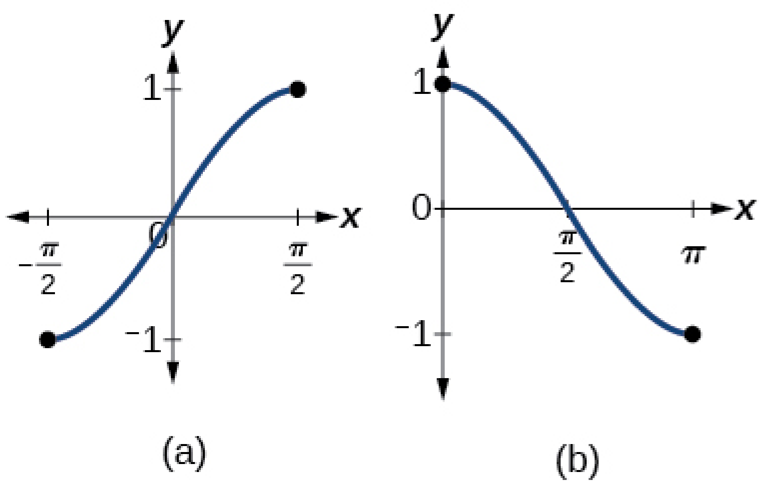


Figure 2. (a) Sine function on a restricted domain of $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $\left[0,\pi\right]$

(Figure) shows the graph of the tangent function limited to $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

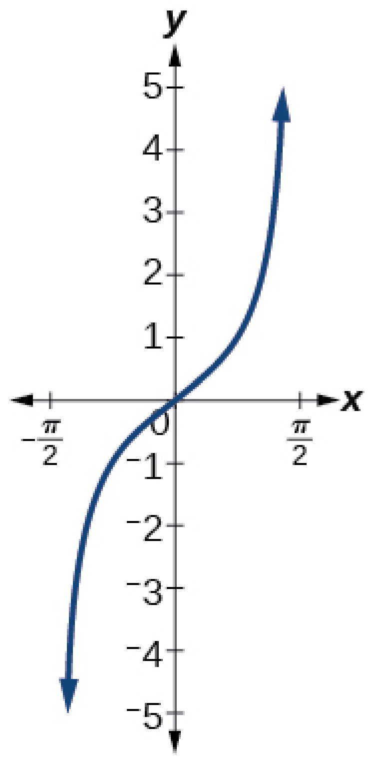


Figure 3. Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one vertical asymptote to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1}x$ means $x = \sin y$. The inverse sine function is sometimes called the arcsine function, and notated $\arcsin x$.
  $y = \sin^{-1}x \text{ has domain } [-1, 1] \text{ and range } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- The inverse cosine function $y = \cos^{-1}x$ means $x = \cos y$. The inverse cosine function is sometimes called the arccosine function, and notated $\arccos x$.
  $y = \cos^{-1}x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$
- The inverse tangent function $y = \tan^{-1}x$ means $x = \tan y$. The inverse tangent function is sometimes called the arctangent function, and notated $\arctan x$.
  $y = \tan^{-1}x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The graphs of the inverse functions are shown in (Figure), (Figure), and (Figure). Notice that the output of each of these inverse functions is a *number*, an angle in radian measure. We see that $\sin^{-1}x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1}x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1}x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

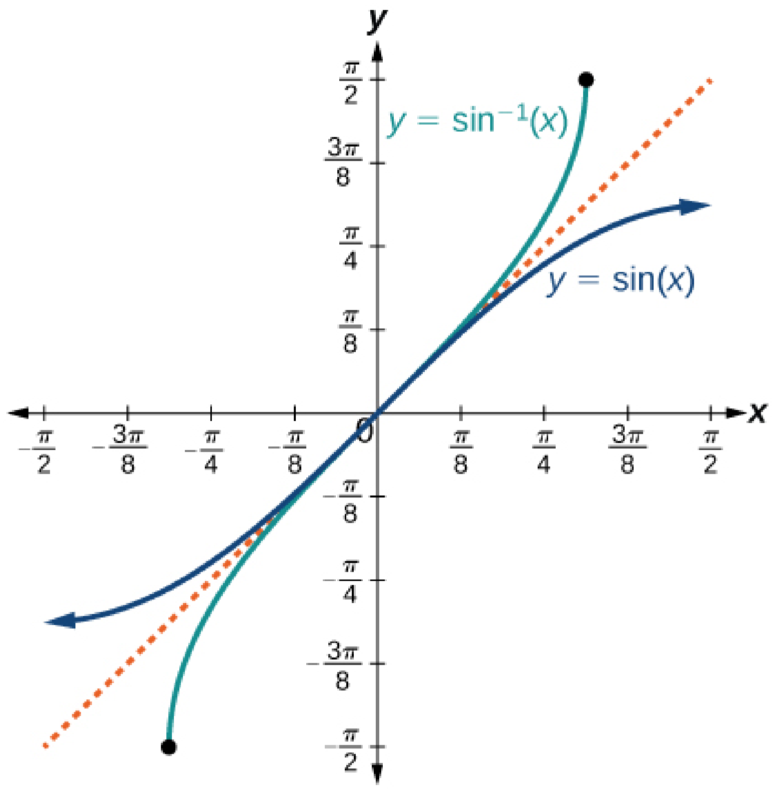


Figure 4. The sine function and inverse sine (or arcsine) function

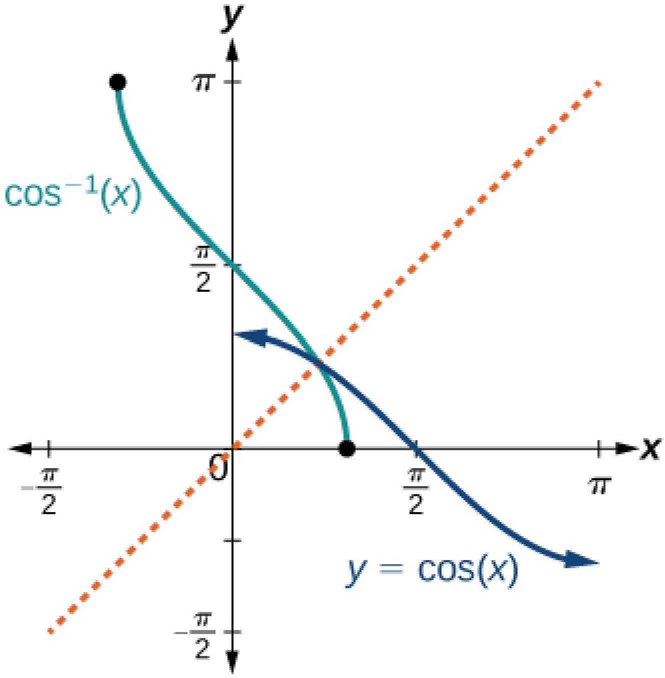


Figure 5. The cosine function and inverse cosine (or arccosine) function

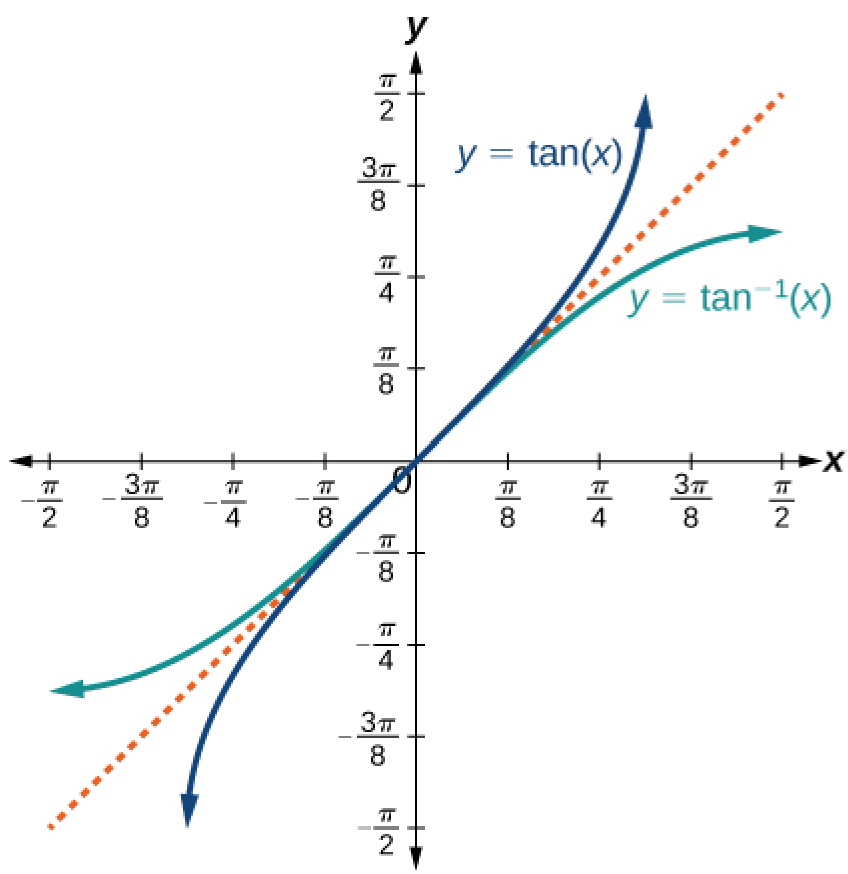


Figure 6. The tangent function and inverse tangent (or arctangent) function

Relations for Inverse Sine, Cosine, and Tangent Functions

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.
For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.
For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$, then $\tan^{-1} x = y$.

Writing a Relation for an Inverse Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Show Solution

Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

Try It

Given $\cos(0.5) \approx 0.8776$, write a relation involving the inverse cosine.

Show Solution

$\cos^{-1}(0.8776) \approx 0.5$

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°), and their reflections into other quadrants.

How To

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x, depending on which corresponds to the given inverse function.

Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$
b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
d. $\tan^{-1}(1)$

Show Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$ $: \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.
c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.
d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1) = \frac{\pi}{4}$.

Try It

Evaluate each of the following.

a. $\sin^{-1}(-1)$
b. $\tan^{-1}(-1)$
c. $\cos^{-1}(-1)$
d. $\cos^{-1}\left(\frac{1}{2}\right)$

Show Solution
a. $-\frac{\pi}{2}$; b. $-\frac{\pi}{4}$; c. π; d. $\frac{\pi}{3}$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate inverse trigonometric functions that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN $^{-1}$, ARCSIN, or ASIN.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

Evaluating the Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Show Solution

Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93°$. Note that in calculus and beyond we will use radians in almost all cases.

Try It

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Show Solution
1.9823 or 113.578°

How To

Given two sides of a right triangle like the one shown in (Figure), find an angle.

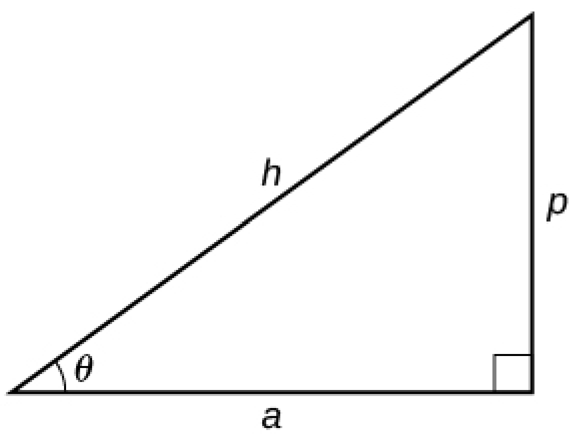


Figure 7.

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Applying the Inverse Cosine to a Right Triangle

Solve the triangle in (Figure) for the angle θ.

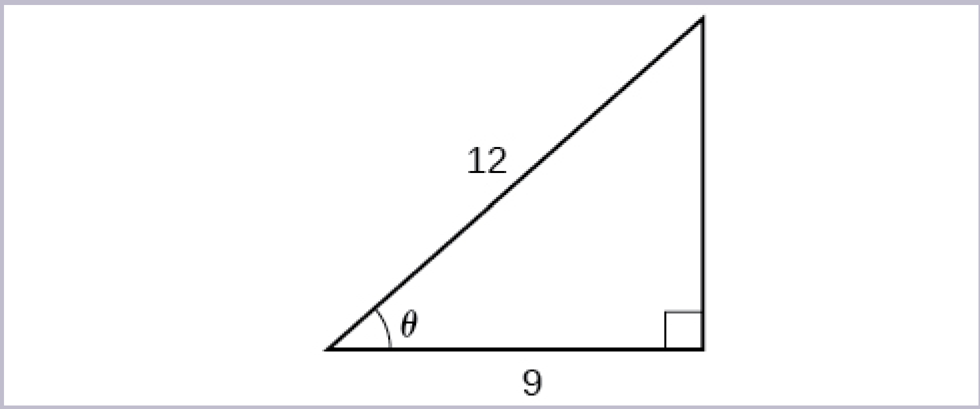


Figure 8.

Show Solution

Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\begin{aligned} \cos\theta &= \tfrac{9}{12} & \\ \theta &= \cos^{-1}\left(\tfrac{9}{12}\right) & \text{Apply definition of the inverse.} \\ \theta &\approx 0.7227 \text{ or about } 41.4096^\circ & \text{Evaluate.} \end{aligned}$$

Try It

Solve the triangle in (Figure) for the angle θ.

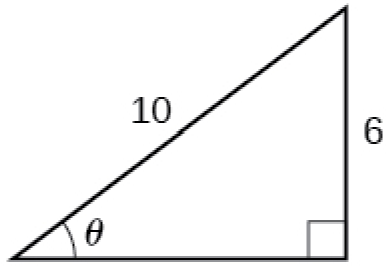


Figure 9.

Show Solution

$\sin^{-1}(0.6) = 36.87° = 0.6435$ radians

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set $\{\sin(x), \cos(x), \tan(x)\}$ and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function, $f\left(f^{-1}(y)\right) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

Compositions of a trigonometric function and its inverse

$$\sin\left(\sin^{-1}x\right) = x \text{ for } -1 \leq x \leq 1$$
$$\cos\left(\cos^{-1}x\right) = x \text{ for } -1 \leq x \leq 1$$
$$\tan\left(\tan^{-1}x\right) = x \text{ for } -\infty < x < \infty$$
$$\sin^{-1}\left(\sin x\right) = x \text{ only for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\cos^{-1}\left(\cos x\right) = x \text{ only for } 0 \leq x \leq \pi$$
$$\tan^{-1}\left(\tan x\right) = x \text{ only for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Is it correct that $\sin^{-1}\left(\sin x\right) = x$?

No. This equation is correct if x *belongs to the restricted domain* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *but sine is defined for all real input values, and for* x *outside the restricted interval, the equation is not correct because its inverse always returns a value in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *The situation is similar for cosine and tangent and their inverses. For example,* $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

Given an expression of the form f^{-1}(f(θ)) where $f\left(\theta\right) = \sin\theta,\ \cos\theta,\ \text{or}\ \tan\theta,$ **evaluate.**

1. If θ is in the restricted domain of $f,\ \text{then}\ f^{-1}\left(f\left(\theta\right)\right) = \theta.$
2. If not, then find an angle φ within the restricted domain of f such that $f\left(\varphi\right) = f\left(\theta\right)$. Then $f^{-1}\left(f\left(\theta\right)\right) = \varphi.$

Using Inverse Trigonometric Functions

Evaluate the following:

1. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
2. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
3. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

4. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Show Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.
b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.
c. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.
d. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function.
e. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Try It

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Show Solution
$\frac{\pi}{8}; \frac{2\pi}{9}$

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}\left(g\left(x\right)\right)$. For special values of x, we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2} - \theta$. Consider the sine and cosine of each angle of the right triangle in (Figure).

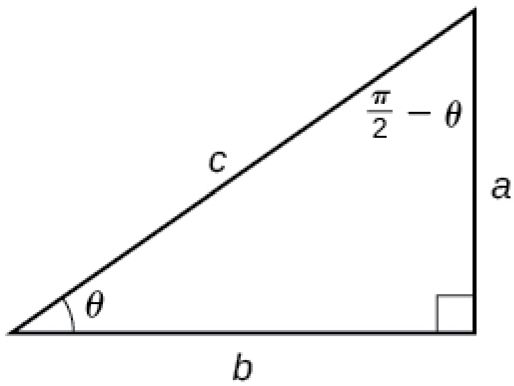


Figure 10. Right triangle illustrating the cofunction relationships

Because $\cos\theta = \frac{b}{c} = \sin\left(\frac{\pi}{2} - \theta\right)$, we have $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$ if $0 \leq \theta \leq \pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$. Similarly, $\sin\theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$, so $\cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$ if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

Given functions of the form $\sin^{-1}(\cos x)$ **and** $\cos^{-1}(\sin x)$, **evaluate them.**

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.
   $\sin^{-1}(\cos x) = \frac{\pi}{2} - y$
3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \sin x$.
   $\cos^{-1}(\sin x) = \frac{\pi}{2} - y$

Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation.
b. by the method described previously.

Show Solution

a. Here, we can directly evaluate the inside of the composition.
$\cos\left(\frac{13\pi}{6}\right) = \cos\left(\frac{\pi}{6} + 2\pi\right)$
$= \cos\left(\frac{\pi}{6}\right)$
$= \frac{\sqrt{3}}{2}$
Now, we can evaluate the inverse function as we did earlier.
$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$
b. We have $x = \frac{13\pi}{6}$, $y = \frac{\pi}{6}$, and
$\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) = \frac{\pi}{2} - \frac{\pi}{6}$
$= \frac{\pi}{3}$

Try It

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Show Solution
$\frac{3\pi}{4}$

Evaluating Compositions of the Form *f*(*g*$^{-1}$(*x*))

To evaluate compositions of the form $f\left(g^{-1}(x)\right)$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin\left(\cos^{-1}x\right) = \sqrt{1-x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Show Solution

Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && \text{Use our known value for cosine.} \\ \sin^2\theta + \left(\frac{4}{5}\right)^2 &= 1 && \text{Solve for sine.} \\ \sin^2\theta &= 1 - \frac{16}{25} \\ \sin\theta &= \sqrt{\frac{9}{25}} = \frac{3}{5} \end{aligned}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is $\frac{3}{5}$. See (Figure).

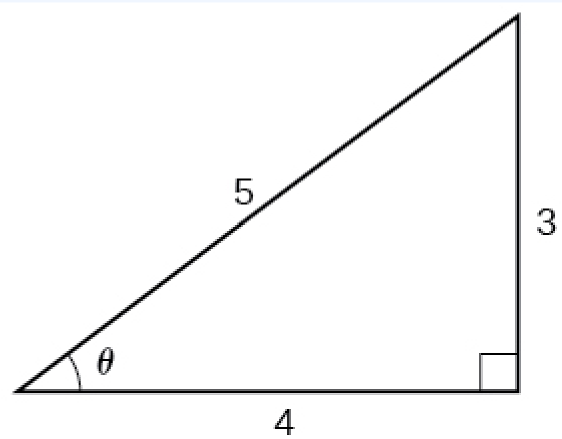


Figure 11. Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive; therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \frac{3}{5}$.

Try It

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Show Solution
$\frac{12}{13}$

Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Show Solution
While we could use a similar technique as in (Figure), we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in (Figure).

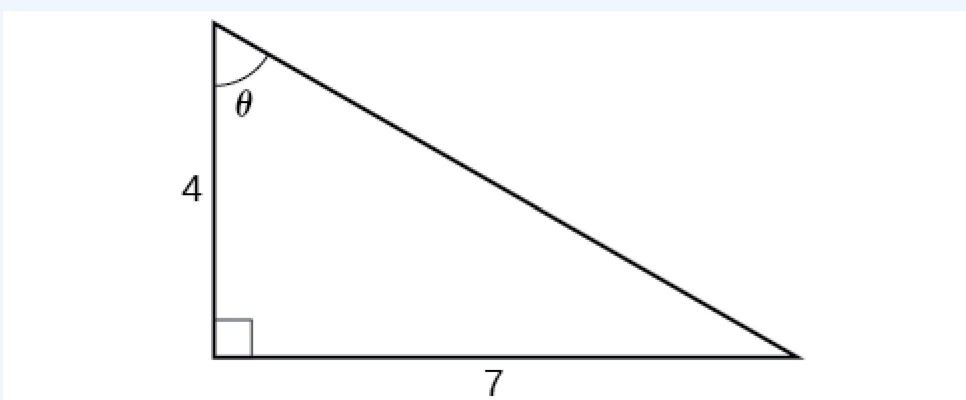


Figure 12. A right triangle with two sides known

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$4^2 + 7^2 = \text{hypotenuse}^2$$
$$\text{hypotenuse} = \sqrt{65}$$

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned}\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65}\end{aligned}$$

Try It

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Show Solution

$\frac{4\sqrt{2}}{9}$

Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Show Solution

We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$\sin^2\theta + \cos^2\theta = 1$ Use the Pythagorean Theorem.

$\left(\frac{x}{3}\right)^2 + \cos^2\theta = 1$ Solve for cosine.

$\cos^2\theta = 1 - \frac{x^2}{9}$

$\cos\theta = \sqrt{\frac{9-x^2}{9}} = \frac{\sqrt{9-x^2}}{3}$

Because we know that the inverse sine must give an angle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can deduce that the cosine of that angle must be positive.

$\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \frac{\sqrt{9-x^2}}{3}$

Try It

Find a simplified expression for $\sin\left(\tan^{-1}(4x)\right)$ for $-\frac{1}{4} \le x \le \frac{1}{4}$.

Show Solution

$\frac{4x}{\sqrt{16x^2+1}}$

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions

Visit this website for additional practice questions from Learningpod.

Key Concepts

- An inverse function is one that “undoes” another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f\left(x\right)$, if $x=f^{-1}\left(y\right)$, then $f\left(x\right)=y$. However, $f\left(x\right)=y$ only implies $x=f^{-1}\left(y\right)$ if x is in the restricted domain of f. See (Figure).
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4}=\tan^{-1}\left(1\right)\text{ and }\frac{\pi}{6}=\sin^{-1}\left(\frac{1}{2}\right)$. See (Figure).
- A calculator will return an angle within the restricted domain of the original trigonometric function. See (Figure).
- Inverse functions allow us to find an angle when given two sides of a right triangle. See (Figure).
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example, $\sin\left(\cos^{-1}\left(x\right)\right)=\sqrt{1-x^2}$. See (Figure).
- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}\left(\cos x\right)=\frac{\pi}{2}-x$ if $0\le x\le\pi$ and $\cos^{-1}\left(\sin x\right)=\frac{\pi}{2}-x$ if $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$. See (Figure) and (Figure).
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See (Figure).
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See (Figure).

Section Exercises

Verbal

1. Why do the functions $f\left(x\right)=\sin^{-1}x$ and $g\left(x\right)=\cos^{-1}x$ have different ranges?

Show Solution

The function $y = \sin x$ is one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; thus, this interval is the range of the inverse function of $y = \sin x, f(x) = \sin^{-1} x$. The function $y = \cos x$ is one-to-one on $[0, \pi]$; thus, this interval is the range of the inverse function of $y = \cos x, f(x) = \cos^{-1} x$.

2. Since the functions $y = \cos x$ and $y = \cos^{-1} x$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?

3. Explain the meaning of $\frac{\pi}{6} = \arcsin(0.5)$.

Show Solution

$\frac{\pi}{6}$ is the radian measure of an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is 0.5.

4. Most calculators do not have a key to evaluate $\sec^{-1}(2)$. Explain how this can be done using the cosine function or the inverse cosine function.

5. Why must the domain of the sine function, $\sin x$, be restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the inverse sine function to exist?

Show Solution

In order for any function to have an inverse, the function must be one-to-one and must pass the horizontal line test. The regular sine function is not one-to-one unless its domain is restricted in some way. Mathematicians have agreed to restrict the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that it is one-to-one and possesses an inverse.

6. Discuss why this statement is incorrect: $\arccos(\cos x) = x$ for all x.

7. Determine whether the following statement is true or false and explain your answer: $\arccos(-x) = \pi - \arccos x$.

Show Solution

True . The angle, θ_1 that equals $\arccos\left(-x\right)$, $x>0$, will be a second quadrant angle with reference angle, θ_2 , where θ_2 equals $\arccos x$, $x>0$. Since θ_2 is the reference angle for θ_1 , $\theta_2=\pi-\theta_1$ and $\arccos\left(-x\right)$ = $\pi-\arccos x$ -

Algebraic

For the following exercises, evaluate the expressions.

8. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

9. $\sin^{-1}\left(-\frac{1}{2}\right)$

Show Solution

$-\frac{\pi}{6}$

10. $\cos^{-1}\left(\frac{1}{2}\right)$

11. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Show Solution

$\frac{3\pi}{4}$

12. $\tan^{-1}\left(1\right)$

13. $\tan^{-1}\left(-\sqrt{3}\right)$

Show Solution
$-\frac{\pi}{3}$

14. $\tan^{-1}\left(-1\right)$

15. $\tan^{-1}\left(\sqrt{3}\right)$

Show Solution
$\frac{\pi}{3}$

16. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

For the following exercises, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

17. $\cos^{-1}\left(-0.4\right)$

Show Solution
1.98

18. $\arcsin\left(0.23\right)$

19. $\arccos\left(\frac{3}{5}\right)$

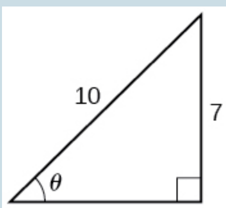
Show Solution
0.93

20. $\cos^{-1}(0.8)$

21. $\tan^{-1}(6)$

Show Solution
1.41

For the following exercises, find the angle θ in the given right triangle. Round answers to the nearest hundredth.

22. 

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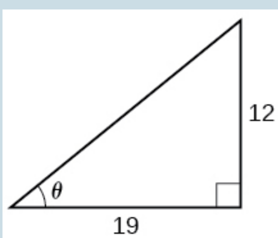
23. 

Show Solution
0.56 radians

For the following exercises, find the exact value, if possible, without a calculator. If it is not possible, explain why.

24. $\sin^{-1}(\cos(\pi))$

25. $\tan^{-1}\left(\sin\left(\pi\right)\right)$

Show Solution
0

26. $\cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

27. $\tan^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

Show Solution
0.71

28. $\sin^{-1}\left(\cos\left(\frac{-\pi}{2}\right)\right)$

29. $\tan^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

Show Solution
-0.71

30. $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

31. $\tan^{-1}\left(\sin\left(\frac{-5\pi}{2}\right)\right)$

Show Solution
$-\frac{\pi}{4}$

32. $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

33. $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

Show Solution
0.8

34. $\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

35. $\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)$

Show Solution
$\frac{5}{13}$

36. $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

For the following exercises, find the exact value of the expression in terms of x with the help of a reference triangle.

37. $\tan\left(\sin^{-1}\left(x-1\right)\right)$

Show Solution
$\frac{x-1}{\sqrt{-x^2+2x}}$

38. $\sin\left(\cos^{-1}\left(1-x\right)\right)$

39. $\cos\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$

Show Solution
$\frac{\sqrt{x^2-1}}{x}$

40. $\cos\left(\tan^{-1}(3x-1)\right)$

41. $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

Show Solution
$\frac{x+0.5}{\sqrt{-x^2-x+\frac{3}{4}}}$

Extensions

For the following exercises, evaluate the expression without using a calculator. Give the exact value.

42. $\frac{\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\cos^{-1}(1)}{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0)}$

For the following exercises, find the function if $\sin t=\frac{x}{x+1}$.

43. $\cos t$

Show Solution
$\frac{\sqrt{2x+1}}{x+1}$

44. $\sec t$

45. $\cot t$

Show Solution

$\frac{\sqrt{2x+1}}{x}$

46. $\cos\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right)$

47. $\tan^{-1}\left(\frac{x}{\sqrt{2x+1}}\right)$

Show Solution

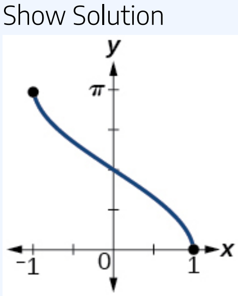
t

Graphical

48. Graph $y = \sin^{-1} x$ and state the domain and range of the function.

49. Graph $y = \arccos x$ and state the domain and range of the function.

Show Solution



domain $[-1, 1]$; range $[0, \pi]$

50. Graph one cycle of $y = \tan^{-1} x$ and state the domain and range of the function.

51. For what value of x does $\sin x = \sin^{-1} x$? Use a graphing calculator to approximate the answer.

Show Solution
approximately $x = 0.00$

52. For what value of x does $\cos x = \cos^{-1} x$? Use a graphing calculator to approximate the answer.

Real-World Applications

53. Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?

Show Solution
0.395 radians

54. Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road?

55. An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch side.

Show Solution
1.11 radians

56. Without using a calculator, approximate the value of $\arctan(10,000)$. Explain why your answer is reasonable.

57. A truss for the roof of a house is constructed from two identical right triangles. Each has a base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

Show Solution

1.25 radians

58. The line $y = \frac{3}{5}x$ passes through the origin in the *x*,*y*-plane. What is the measure of the angle that the line makes with the positive *x*-axis?

59. The line $y = \frac{-3}{7}x$ passes through the origin in the *x*,*y*-plane. What is the measure of the angle that the line makes with the negative *x*-axis?

Show Solution

0.405 radians

60. What percentage grade should a road have if the angle of elevation of the road is 4 degrees? (The percentage grade is defined as the change in the altitude of the road over a 100-foot horizontal distance. For example a 5% grade means that the road rises 5 feet for every 100 feet of horizontal distance.)

61. A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

Show Solution

No. The angle the ladder makes with the horizontal is 60 degrees.

62. Suppose a 15-foot ladder leans against the side of a house so that the angle of elevation of the ladder is 42 degrees. How far is the foot of the ladder from the side of the house?

Chapter Review Exercises

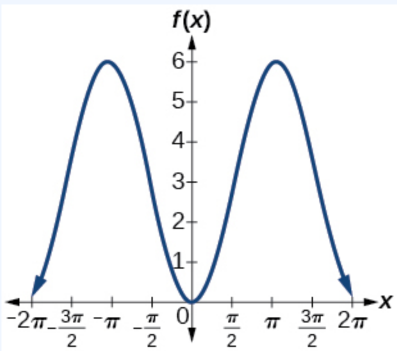
Graphs of the Sine and Cosine Functions

For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

1. $f(x) = -3\cos x + 3$

Show Solution

amplitude: 3; period: 2π; midline: $y = 3$; no asymptotes

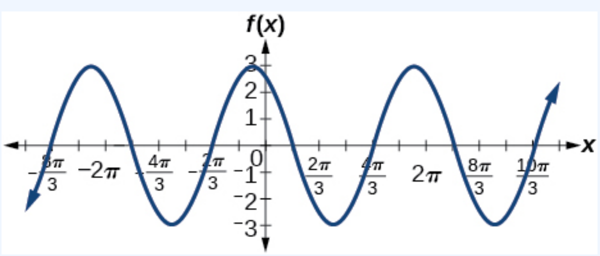


2. $f(x) = \frac{1}{4}\sin x$

3. $f(x) = 3\cos\left(x + \frac{\pi}{6}\right)$

Show Solution

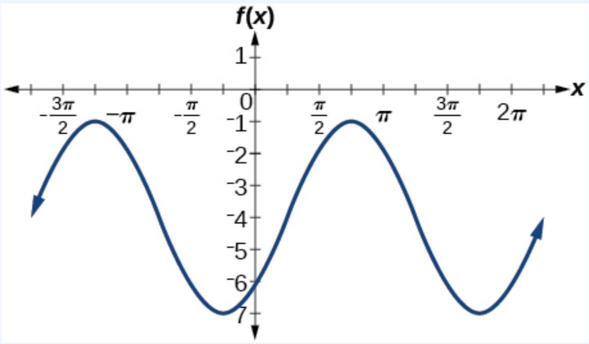
amplitude: 3; period: 2π; midline: $y = 0$; no asymptotes



4. $f\left(x\right)=-2\sin\left(x-\frac{2\pi}{3}\right)$

5. $f\left(x\right)=3\sin\left(x-\frac{\pi}{4}\right)-4$

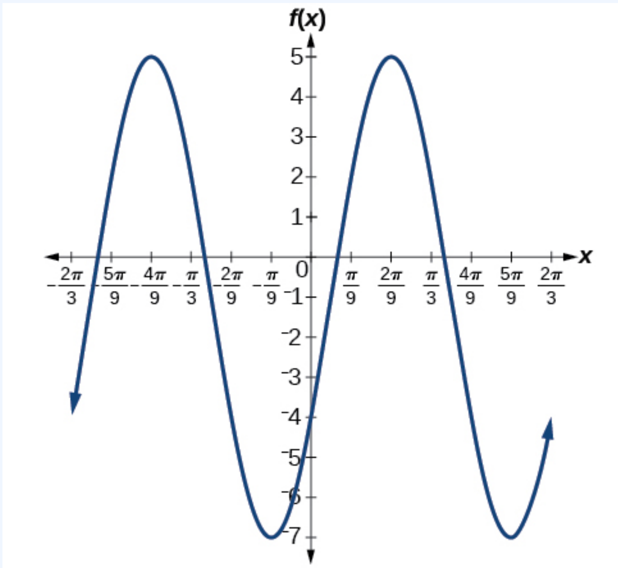
Show Solution
amplitude: 3; period: 2π; midline: $y=-4$; no asymptotes



6. $f\left(x\right)=2\left(\cos\left(x-\frac{4\pi}{3}\right)+1\right)$

7. $f\left(x\right)=6\sin\left(3x-\frac{\pi}{6}\right)-1$

Show Solution
amplitude: 6; period: $\frac{2\pi}{3}$; midline: $y=-1$; no asymptotes



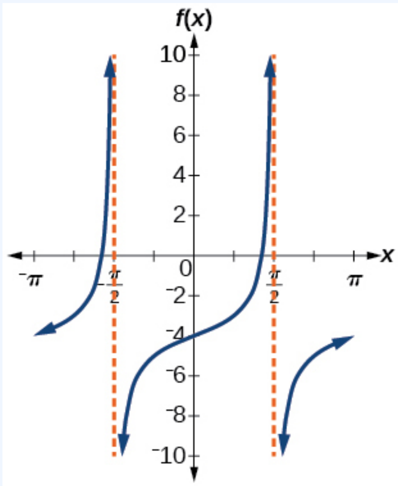
8. $f\left(x\right)=-100\sin\left(50x-20\right)$

Graphs of the Other Trigonometric Functions

For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period,
midline equation, and asymptotes.

9. $f\left(x\right)=\tan x-4$

Show Solution
stretching factor: none; period: π; midline: $y=-4$; asymptotes: $x=\frac{\pi}{2}+\pi k$, where k is an integer

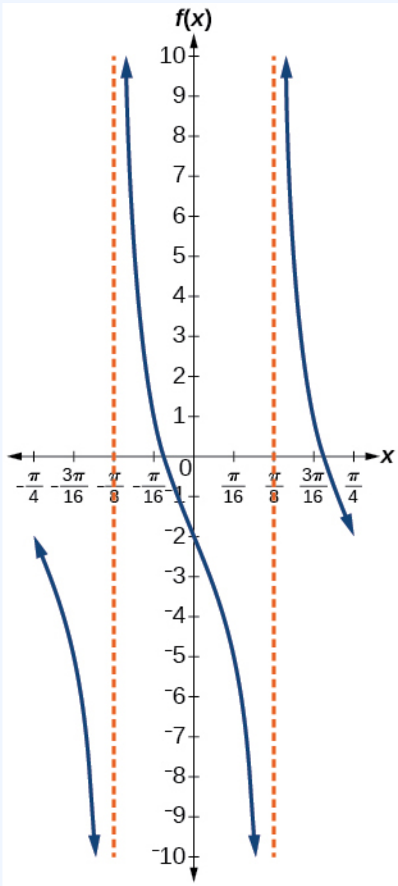


10. $f\left(x\right)=2\tan\left(x-\frac{\pi}{6}\right)$

11. $f\left(x\right)=-3\tan\left(4x\right)-2$

Show Solution

stretching factor: 3; period: $\frac{\pi}{4}$; midline: $y=-2$; asymptotes: $x=\frac{\pi}{8}+\frac{\pi}{4}k,$ where k is an integer



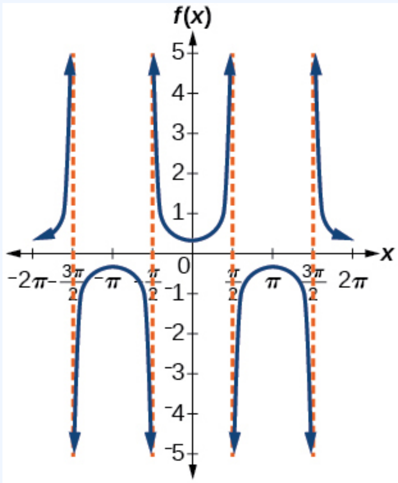
12. $f\left(x\right)=0.2\cos\left(0.1x\right)+0.3$

For the following exercises, graph two full periods. Identify the period, the phase shift, the amplitude, and asymptotes.

13. $f\left(x\right)=\frac{1}{3}\sec x$

Show Solution

amplitude: none; period: $2\pi;$ no phase shift; asymptotes: $x=\frac{\pi}{2}k,$ where k is an odd integer

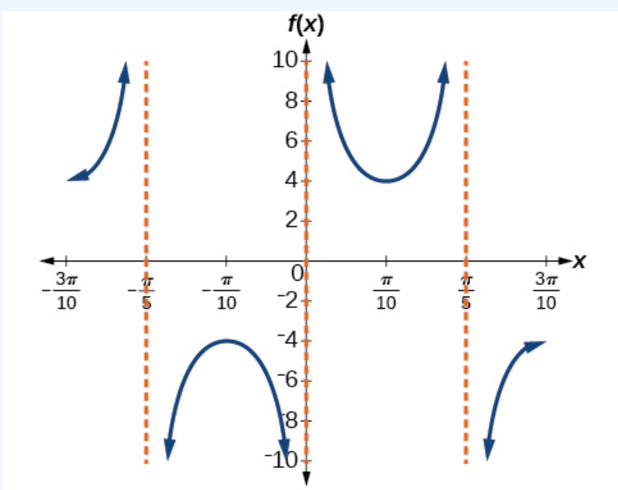


14. $f\left(x\right)=3\cot x$

15. $f\left(x\right)=4\csc\left(5x\right)$

Show Solution

amplitude: none; period: $\frac{2\pi}{5}$; no phase shift; asymptotes: $x=\frac{\pi}{5}k,$ where k is an integer

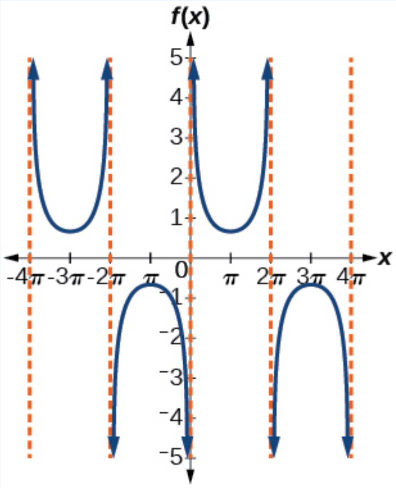


16. $f\left(x\right)=8\sec\left(\frac{1}{4}x\right)$

17. $f\left(x\right)=\frac{2}{3}\text{csc}\left(\frac{1}{2}x\right)$

Show Solution

amplitude: none; period: $4\pi;$ no phase shift; asymptotes: $x=2\pi k,$ where k is an integer



18. $f\left(x\right)=-\text{csc}\left(2x+\pi\right)$

For the following exercises, use this scenario: The population of a city has risen and fallen over a 20-year interval. Its population may be modeled by the following function:

19. $y=12,000+8,000\text{sin}\left(0.628x\right)$, where the domain is the years since 1980 and the range is the population of the city.

What is the largest and smallest population the city may have?

Show Solution

largest: 20,000; smallest: 4,000

20. Graph the function on the domain of $\left[0,40\right]$.

21. What are the amplitude, period, and phase shift for the function?

Show Solution

amplitude: 8,000; period: 10; phase shift: 0

22. Over this domain, when does the population reach 18,000? 13,000?

23. What is the predicted population in 2007? 2010?

Show Solution

In 2007, the predicted population is 4,413. In 2010, the population will be 11,924.

For the following exercises, suppose a weight is attached to a spring and bobs up and down, exhibiting symmetry.

24. Suppose the graph of the displacement function is shown in (Figure), where the values on the x-axis represent the time in seconds and the y-axis represents the displacement in inches. Give the equation that models the vertical displacement of the weight on the spring.

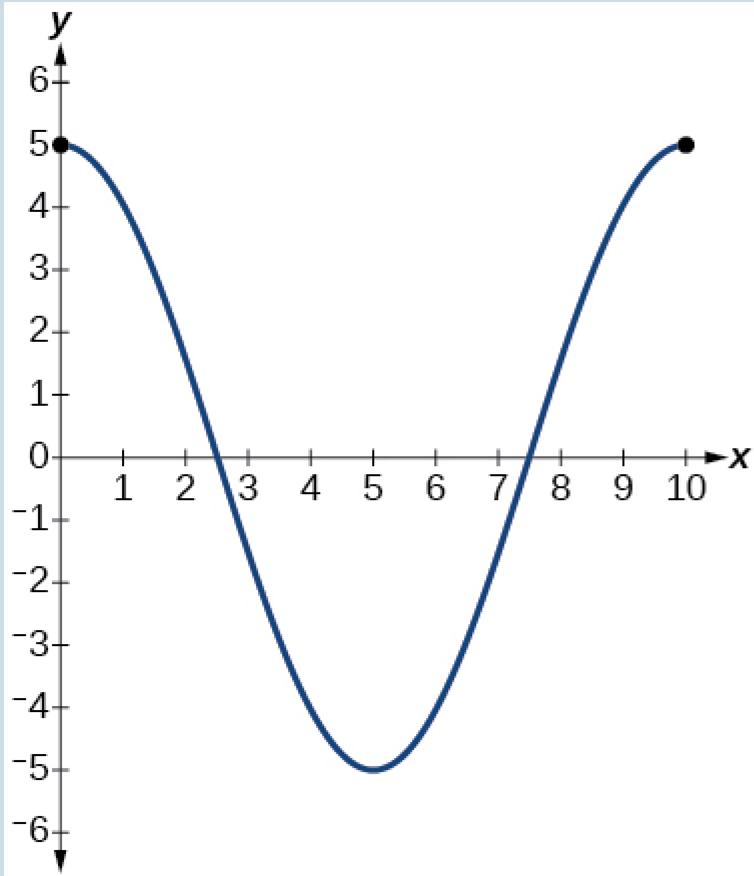


Figure 13.

25. At time = 0, what is the displacement of the weight?

Show Solution

5 in.

26. At what time does the displacement from the equilibrium point equal zero?

27. What is the time required for the weight to return to its initial height of 5 inches? In other words, what is the period for the displacement function?

Show Solution
10 seconds

Inverse Trigonometric Functions

For the following exercises, find the exact value without the aid of a calculator.

28. $\sin^{-1}(1)$

29. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Show Solution
$\frac{\pi}{6}$

30. $\tan^{-1}(-1)$

31. $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Show Solution
$\frac{\pi}{4}$

32. $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

33. $\sin^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$

Show Solution

$\frac{\pi}{3}$

34. $\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

35. $\sin\left(\sec^{-1}\left(\frac{3}{5}\right)\right)$

Show Solution
No solution

36. $\cot\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

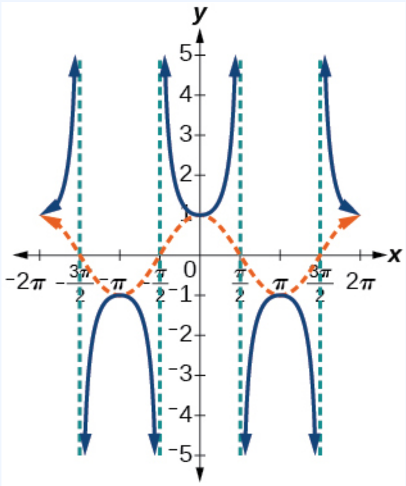
37. $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$

Show Solution
$\frac{12}{5}$

38. $\sin\left(\cos^{-1}\left(\frac{x}{x+1}\right)\right)$

39. Graph $f\left(x\right) = \cos x$ and $f\left(x\right) = \sec x$ on the interval $[0, 2\pi)$ and explain any observations.

Show Solution
The graphs are not symmetrical with respect to the line $y = x.$ They are symmetrical with respect to the y -axis.

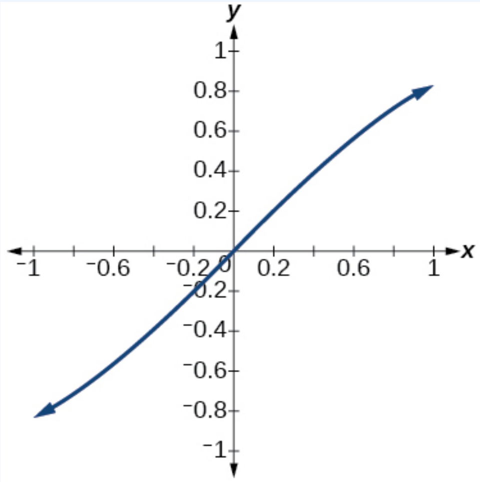


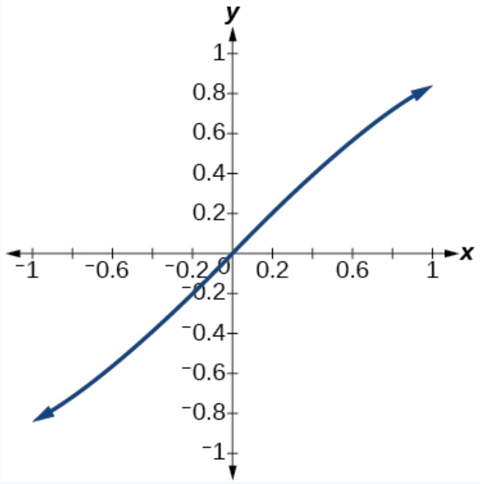
40. Graph $f\left(x\right)=\sin x$ and $f\left(x\right)=\csc x$ and explain any observations.

41. Graph the function $f\ \left(x\right)=\frac{x}{1}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$ on the interval $\left[-1,1\right]$ and compare the graph to the graph of $f\left(x\right)=\sin x$ on the same interval. Describe any observations.

Show Solution

The graphs appear to be identical.





Glossary

arccosine

another name for the inverse cosine; $\arccos x = \cos^{-1} x$

arcsine
: another name for the inverse sine; $\arcsin x = \sin^{-1} x$

arctangent
: another name for the inverse tangent; $\arctan x = \tan^{-1} x$

inverse cosine function
: the function $\cos^{-1} x$, which is the inverse of the cosine function and the angle that has a cosine equal to a given number

inverse sine function
: the function $\sin^{-1} x$, which is the inverse of the sine function and the angle that has a sine equal to a given number

inverse tangent function
: the function $\tan^{-1} x$, which is the inverse of the tangent function and the angle that has a tangent equal to a given number

CHAPTER 8: RADICAL EXPRESSIONS AND EQUATIONS

CHAPTER 8.1: REDUCING SQUARE ROOTS

Square roots are the most common type of radical. A square will take some number and multiply it by itself. A square root of a number gives the number that, when multiplied by itself, gives the number shown beneath the radical. For example, because $5^2 = 25$, the square root of 25 is 5.

The square root of 25 is written as $\sqrt{25}$ or as $25^{½}$.

Example 1

Solve the following square roots:

$\sqrt{1} = 1 \qquad \sqrt{121} = 11 \qquad \sqrt{4} = 2$

$\sqrt{625} = 25 \qquad \sqrt{9} = 3 \qquad \sqrt{-81} = \text{Undefined}$

The final example, $\sqrt{81}$, is classified as being undefined in the real number system since negatives have no square root. This is because if you square a positive or a negative, the answer will be positive. This means that when using the real number system, take only square roots of positive numbers. There are solutions to negative square roots, but they require a new number system to be created that is termed the imaginary number system. For now, simply say they are undefined in the real number system or that they have no real solution

Not all numbers have a nice even square root. For example, if you look up $\sqrt{8}$ on your calculator, the answer would be 2.828427124746190097603377448419..., with this number being a rounded approximation of the square root. The standard for radicals that have large, rounded solutions is that the calculator is not used to find decimal approximations of square roots. Instead, express roots in simplest radical form.

There are a number of properties that can be used when working with radicals. One is known as the product rule:

$$\text{Product Rule of Square Roots: } \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

Use the product rule to simplify an expression by finding perfect squares that divide evenly into the radicand (the number under the radical). Commonly used perfect squares are:

$$\begin{array}{llllll} 4 = 2^2 & 9 = 3^2 & 16 = 4^2 & 25 = 5^2 & 36 = 6^2 & 49 = 7^2 \\ 64 = 8^2 & 81 = 9^2 & 100 = 10^2 & 121 = 11^2 & 144 = 12^2 & 169 = 13^2 \\ 196 = 14^2 & 225 = 15^2 & 400 = 20^2 & 625 = 25^2 & 900 = 30^2 & 1600 = 40^2 \end{array}$$

The challenge in reducing radicals is often simplified to finding the perfect square to divide into the radicand.

Example 2

Find the perfect squares that divide evenly into the radicand.

1. $18 = 2 \cdot 9$
2. $75 = 3 \cdot 25$
3. $125 = 5 \cdot 25$
4. $72 = 2 \cdot 36$
5. $98 = 2 \cdot 47$
6. $45 = 5 \cdot 9$

Combining the strategies used in the above two examples makes the simplest strategy to reduce radicals.

Example 3

Reduce $\sqrt{75}$.

$\sqrt{75} = \sqrt{25} \cdot \sqrt{3}$

$\sqrt{25} \cdot \sqrt{3}$ reduces to $5 \cdot \sqrt{3}$ or $5\sqrt{3}$

$\sqrt{75} = 5\sqrt{3}$

If there is a coefficient in front of the radical to begin with, the problem merely becomes a big multiplication problem.

Example 4

Reduce $5\sqrt{63}$.

$$\begin{array}{rl} 5\sqrt{63} & \text{63 equals } 9 \times 7, \text{ and 9 is a perfect square} \\ 5\sqrt{9 \cdot 7} & \text{Take the square root of 9} \\ 5 \cdot 3\sqrt{7} & \text{Multiply 5 and 3} \\ 15\sqrt{7} & \end{array}$$

Variables often are part of the radicand as well. When taking the square roots of variables, divide the exponent by 2.

For example, $\sqrt{x^8} = x^4$, because you divide the exponent 8 by 2. This follows from the power of a power rule of exponents, $(x^4)^2 = x^8$. When squaring, multiply the exponent by two, so when taking a square root, divide the exponent by 2. This is shown in the following example.

Example 5

Reduce $-5\sqrt{18x^4y^6z^{10}}$.

$$\begin{array}{rl} -5\sqrt{18x^4y^6z^{10}} & \text{18 is divisible by 9, a perfect square} \\ -5\sqrt{9 \cdot 2x^4y^6z^{10}} & \text{Split into factors} \\ -5\sqrt{9} \cdot \sqrt{2} \cdot \sqrt{x^4} \cdot \sqrt{y^6} \cdot \sqrt{z^{10}} & \text{Divide exponents by 2} \\ -5 \cdot 3x^2y^3z^5\sqrt{2} & \text{Multiply coefficients} \\ -15x^2y^3z^5\sqrt{2} & \end{array}$$

Sometimes, you cannot evenly divide the exponent on a variable by 2. Sometimes, there is a remainder. If there is a remainder, this means the remainder is left inside the radical, and the whole number part goes outside the radical. This is shown in the following example.

Example 6

Reduce $\sqrt{20x^5y^9z^6}$.

$$\sqrt{20x^5y^9z^6}$$

$$\sqrt{4\cdot 5x^4xy^8yz^6} \quad \text{Break into square root factors}$$

$$\sqrt{4}\cdot\sqrt{5}\cdot\sqrt{x^4}\cdot\sqrt{x}\cdot\sqrt{y^8}\cdot\sqrt{y}\cdot\sqrt{z^6}$$

$$2x^2y^4z^3\sqrt{5xy}$$

Example 7

Reduce $\sqrt{42x^{11}y^{10}z^9}$.

$$\sqrt{42x^{11}y^{10}z^9}$$

$$\sqrt{42x^{10}xy^{10}z^8z} \quad \text{Break into square root factors}$$

$$\sqrt{42}\cdot\sqrt{x^{10}}\cdot\sqrt{x}\cdot\sqrt{y^{10}}\cdot\sqrt{z^8}\cdot\sqrt{z}$$

$$x^5y^5z^4\sqrt{42xz}$$

Questions

Simplify the following radicals.

1. $\sqrt{245}$
2. $\sqrt{125}$
3. $2\sqrt{36}$
4. $5\sqrt{196}$
5. $\sqrt{12}$
6. $\sqrt{72}$
7. $3\sqrt{12}$
8. $5\sqrt{32}$
9. $6\sqrt{128}$
10. $7\sqrt{128}$
11. $-7\sqrt{64x^4}$
12. $-2\sqrt{128n}$
13. $-5\sqrt{36m}$
14. $8\sqrt{112p^2}$
15. $\sqrt{45x^2y^2}$
16. $\sqrt{72a^3b^4}$
17. $\sqrt{16x^3y^3}$
18. $\sqrt{512a^4b^2}$
19. $\sqrt{320x^4y^4}$
20. $\sqrt{512m^4n^3}$

Answers to odd questions

1. $\sqrt{5 \cdot 49}$

$\pm 7\sqrt{5}$

3. $2 \cdot (\pm 6)$

± 12

5. $\sqrt{4 \cdot 3}$

$\pm 2\sqrt{3}$

7. $3\sqrt{4 \cdot 3}$

$3 \cdot 2\sqrt{3}$

$\pm 6\sqrt{3}$

9. $6\sqrt{64 \cdot 2}$

$6 \cdot 8\sqrt{2}$

$\pm 48\sqrt{2}$

11. $-7 \cdot 8x^2$

$\pm 56x^2$

13. $-5 \cdot 6\sqrt{m}$

$\pm 30\sqrt{m}$

15. $\sqrt{5 \cdot 9 \cdot x^2 \cdot y^2}$

$\pm 3xy\sqrt{5}$

17. $\sqrt{16 \cdot x^2 \cdot x \cdot y^2 \cdot y}$

$\pm 4xy\sqrt{xy}$

19. $\sqrt{5 \cdot 64 \cdot x^4 \cdot y^4}$

$\pm 8x^2y^2\sqrt{5}$

CHAPTER 8.2: REDUCING HIGHER POWER ROOTS

While square roots are the most common type of radical, there are higher roots of numbers as well: cube roots, fourth roots, fifth roots, and so on. The following is a definition of radicals:

$$\sqrt[m]{a} = b \text{ if } b^m = a$$

The small letter m inside the radical is called the index. It dictates which root you are taking. For square roots, the index is 2, which, since it is the most common root, is not usually written.

Example 1

Here are several higher powers of positive numbers and their roots:

$2^2 = 4$	$2^3 = 8$	$2^4 = 16$	$2^5 = 32$	$2^6 = 64$	$2^7 = 128$
$3^2 = 9$	$3^3 = 27$	$3^4 = 81$	$3^5 = 243$	$3^6 = 729$	$3^7 = 2187$
$4^2 = 16$	$4^3 = 64$	$4^4 = 256$	$4^5 = 1024$	$4^6 = 4096$	$4^7 = 16384$
$5^2 = 25$	$5^3 = 125$	$5^4 = 625$	$5^5 = 3125$	$5^6 = 15625$	$5^7 = 78125$
$6^2 = 36$	$6^3 = 216$	$6^4 = 1296$	$6^5 = 7776$	$6^6 = 46656$	$6^7 = 279936$
$7^2 = 49$	$7^3 = 343$	$7^4 = 2401$	$7^5 = 16807$	$7^6 = 117649$	$7^7 = 823543$
$8^2 = 64$	$8^3 = 512$	$8^4 = 4096$	$8^5 = 32768$	$8^6 = 262144$	$8^7 = 2097152$
$9^2 = 81$	$9^3 = 729$	$9^4 = 6561$	$9^5 = 59049$	$9^6 = 531441$	$9^7 = 4782969$
$10^2 = 100$	$10^3 = 1000$	$10^4 = 10000$	$10^5 = 100000$	$10^6 = 1000000$	

$2 = \sqrt{4}$	$2 = \sqrt[3]{8}$	$2 = \sqrt[4]{16}$	$2 = \sqrt[5]{32}$	$2 = \sqrt[6]{64}$	$2 = \sqrt[7]{128}$
$3 = \sqrt{9}$	$3 = \sqrt[3]{27}$	$3 = \sqrt[4]{81}$	$3 = \sqrt[5]{243}$	$3 = \sqrt[6]{729}$	$3 = \sqrt[7]{2187}$
$4 = \sqrt{16}$	$4 = \sqrt[3]{64}$	$4 = \sqrt[4]{256}$	$4 = \sqrt[5]{1024}$	$4 = \sqrt[6]{4096}$	$4 = \sqrt[7]{16384}$
$5 = \sqrt{25}$	$5 = \sqrt[3]{125}$	$5 = \sqrt[4]{625}$	$5 = \sqrt[5]{3125}$	$5 = \sqrt[6]{15625}$	$5 = \sqrt[7]{78125}$
$6 = \sqrt{36}$	$6 = \sqrt[3]{216}$	$6 = \sqrt[4]{1296}$	$6 = \sqrt[5]{7776}$	$6 = \sqrt[6]{46656}$	$6 = \sqrt[7]{279936}$
$7 = \sqrt{49}$	$7 = \sqrt[3]{343}$	$7 = \sqrt[4]{2401}$	$7 = \sqrt[5]{16807}$	$7 = \sqrt[6]{117649}$	$7 = \sqrt[7]{823543}$
$8 = \sqrt{64}$	$8 = \sqrt[3]{512}$	$8 = \sqrt[4]{4096}$	$8 = \sqrt[5]{32768}$	$8 = \sqrt[6]{262144}$	$8 = \sqrt[7]{2097152}$
$9 = \sqrt{81}$	$9 = \sqrt[3]{729}$	$9 = \sqrt[4]{6561}$	$9 = \sqrt[5]{59049}$	$9 = \sqrt[6]{531441}$	$9 = \sqrt[7]{4782969}$
$10 = \sqrt{100}$	$10 = \sqrt[3]{1000}$	$10 = \sqrt[4]{10000}$	$10 = \sqrt[5]{100000}$	$10 = \sqrt[6]{1000000}$	

Note there is a notable distinction between solutions of even roots and of odd roots. For even-powered roots, the solution is always +/− or ±. The reason for this can shown in the following examples.

Example 2

Find the solutions to √4.

There are two ways to multiple identical numbers to equal 4:

$$(2)(2) = 4 \text{ and } (-2)(-2) = 4$$

This means that the √4 is either +2 or −2, which is often written as ±2.

The ± solution occurs for all even roots and can be seen in:

$$\sqrt[4]{16} = \pm 2 \text{ and } \sqrt[6]{64} = \pm 2 \text{ and } \sqrt[8]{256} = \pm 2$$

All roots that have an even index will always have ± solutions.

Odd-powered roots do not share this feature and will only maintain the sign of the number that you are taking the root of.

Example 3

Find the solutions to $\sqrt[3]{8}$ and $\sqrt[3]{-8}$.

The solution of $\sqrt[3]{8}$ is 2 and $\sqrt[3]{-8}$ is −2.

The reason for this is $(2)^3 = 8$ and $(-2)^3 = -8$.

All negative-indexed roots will keep the sign of the number being rooted.

Higher roots can be simplified in much the same way one simplifies square roots: through using the product property of radicals.

$$\text{Product Property of Radicals: } m\sqrt{ab} = m(\sqrt{a} \cdot m\sqrt{b})$$

Examples 4

Use the product property of radicals to simplify the following.

1. $\sqrt[3]{32}$32 can be broken down into 2^5. Since you are taking the cube root of this number, you can only take out numbers that have a cube root. This means that 32 is broken into 8 × 4, with the number 8 being the only number that you can take the cube root of.
   $$\sqrt[3]{32} = \sqrt[3]{8} \cdot \sqrt[3]{4}$$
   This simplifies to:
   $$\sqrt[3]{32} = 2\sqrt[3]{4}$$
2. $\sqrt[5]{96}$
   $$\sqrt[5]{96} = \sqrt[5]{32} \cdot \sqrt[5]{3}$$
   96 can be broken down into 2^5 × 3. Since you are taking the fifth root of this number, you can only take out numbers that have a fifth root. This means that 96 is broken into 32 × 3, with the number 32 being the only number that you can take the fifth root of.This simplifies to:
   $$\sqrt[5]{96} = 2\sqrt[5]{3}$$

This strategy is used to take the higher roots of variables. In this case, only take out whole number multiples of the root index. This is shown in the following examples.

Example 5

Reduce $\sqrt[4]{x^{25}y^{16}z^4}$.

For this root, you will break the exponent into multiples of the index 4.

This means that $x^{25}y^{16}z^4$ will be broken up into $x^{24}xy^{16}z^4$.

The fourth roots of $x^{24}y^{16}z^4$ are x^6y^4z and the solitary x remains under the fourth root radical. This looks like:

$$\sqrt[4]{x^{25}y^{16}z^4} = \sqrt[4]{x^{24}} \cdot \sqrt[4]{x} \cdot \sqrt[4]{y^{16}} \cdot \sqrt[4]{z^4}$$

Which simplifies to:

$$x^6y^4z\sqrt[4]{x}$$

Example 6

Reduce $\sqrt[5]{64x^{25}y^{16}z^4}$.

For this root, you will break the exponent into multiples of the index 5.

This means that $x^{25}y^{16}z^4$ will be broken up into $x^{25}y^{15}yz^4$ and 64 broken up into 32 × 2.

The fifth roots of $32x^{25}y^{15}$ are $2x^5y^3$ and the remainder $2yz^4$ remains under the fifth root radical.

This looks like:

$$\sqrt[5]{64x^{25}y^{16}z^4} = \sqrt[5]{32} \cdot \sqrt[5]{2} \cdot \sqrt[5]{x^{25}} \cdot \sqrt[5]{y^{15}} \cdot \sqrt[5]{y} \cdot \sqrt[5]{z^4}$$

Which simplifies to:

$$2x^5y^3\sqrt[5]{2yz^4}$$

Questions

Simplify the following radicals.

1. $\sqrt[3]{64}$
2. $\sqrt[3]{-125}$
3. $\sqrt[3]{625}$
4. $\sqrt[3]{250}$
5. $\sqrt[3]{192}$

6. $\sqrt[3]{-24}$
7. $-4\sqrt[4]{96}$
8. $-8\sqrt[4]{48}$
9. $6\sqrt[4]{112}$
10. $5\sqrt[4]{243}$
11. $6\sqrt[4]{648x^5y^7z^2}$
12. $-6\sqrt[4]{405a^5b^8c}$
13. $\sqrt[5]{224n^3p^7q^5}$
14. $\sqrt[5]{-96x^3y^6z^5}$
15. $\sqrt[5]{224p^5q^{10}r^{15}}$
16. $\sqrt[6]{256x^6y^6z^7}$
17. $-3\sqrt[7]{896rs^7t^{14}}$
18. $-8\sqrt[7]{384b^8c^7d^6}$

Answers to odd questions

1. 4
3. $\sqrt[3]{5 \cdot 125}$

$5\sqrt[3]{5}$

5. $\sqrt[3]{3 \cdot 64}$

$4\sqrt[3]{3}$

7. $-4\sqrt[4]{6 \cdot 16}$

$-4 \cdot 2\sqrt[4]{6}$

$\pm 8\sqrt[4]{6}$

9. $6 \cdot \sqrt[4]{7 \cdot 16}$

$6 \cdot 2\sqrt[4]{7}$

$\pm 12\sqrt[4]{7}$

11. $6\sqrt[4]{8 \cdot 81 \cdot x^4 \cdot x \cdot y^4 \cdot y^3 \cdot z^2}$

$6 \cdot 3 \cdot x \cdot y\sqrt[4]{8xy^3z^2}$

$\pm 18xy\sqrt[4]{8xy^3z^2}$

13. $\sqrt[5]{7 \cdot 32 \cdot n^3 \cdot p^2 \cdot p^5 \cdot q^5}$

$2pq\sqrt[5]{7n^3p^2}$

15. $\sqrt[5]{7 \cdot 32 \cdot p^5 \cdot q^{10} \cdot r^{15}}$

$2pq^2r^3\sqrt[5]{7}$

17. $-3\sqrt[7]{7 \cdot 128 \cdot r \cdot s^7 \cdot t^{14}}$

$-3 \cdot 2 \cdot s \cdot t^2\sqrt[7]{7r}$

$-6st^2\sqrt[7]{7r}$

CHAPTER 8.3: ADDING AND SUBTRACTING RADICALS

Adding and subtracting radicals is similar to adding and subtracting variables. The condition is that the variables, like the radicals, must be identical before they can be added or subtracted. Recall the addition and subtraction of like variables:

Example 1

Simplify $4x^2 + 5x - 6x^2 + 3x - 2x$.

First, we sort out like variables and reorder them to be combined.

$$\begin{array}{rl} & 4x^2 + 5x - 6x^2 + 3x - 2x \\ \text{becomes} & 4x^2 - 6x^2 \text{ and } 5x + 3x - 2x \end{array}$$

Combining like variables yields:

$$-2x^2 + 6x$$

When adding and subtracting radicals, follow the same logic. Radicals must be the same before they can be combined.

Example 2

Simplify $5\sqrt{11} + 5\sqrt{13} - 2\sqrt{13} + 6\sqrt{11} - 2\sqrt{11}$.

First, we sort out like variables and reorder them to be combined.

$$\begin{array}{rl} & 5\sqrt{11}+5\sqrt{13}-2\sqrt{13}+6\sqrt{11}-2\sqrt{11} \\ \text{becomes} & 5\sqrt{13}-2\sqrt{13} \text{ and } 5\sqrt{11}+6\sqrt{11}-2\sqrt{11} \end{array}$$

Combining like radicals yields:

$$3\sqrt{13}+9\sqrt{11}$$

Generally, it is required to simplify radicals before combining them. For example:

Example 3

Simplify $4\sqrt{45}+3\sqrt{18}-\sqrt{98}+2\sqrt{20}$.

All of these radicals need to be simplified before they can be combined.

$$\begin{array}{ll} & 4\sqrt{45}+3\sqrt{18}-\sqrt{98}+2\sqrt{20} \\ \text{becomes} & 4\sqrt{9\cdot 5}+3\sqrt{9\cdot 2}-\sqrt{49\cdot 2}+2\sqrt{5\cdot 4} \\ \text{simplifying to} & 4\cdot 3\sqrt{5}+3\cdot 3\sqrt{2}-7\sqrt{2}+2\cdot 2\sqrt{5} \\ \text{and reduces to} & 12\sqrt{5}+9\sqrt{2}-7\sqrt{2}+4\sqrt{5} \end{array}$$

Recombining these so they can be added and subtracted yields:

$$12\sqrt{5}+4\sqrt{5} \text{ and } 9\sqrt{2}-7\sqrt{2}$$

Combining like radicals yields:

$$16\sqrt{5}+2\sqrt{2}$$

Higher order radicals are treated in the same fashion as square roots. For example:

Example 4

Simplify $4\sqrt[3]{54} - 9\sqrt[3]{16} + 5\sqrt[3]{9}$.

Like example 9.3.3, these radicals need to be simplified before they can be combined.

$$\begin{array}{ll} & 4\sqrt[3]{54} - 9\sqrt[3]{16} + 5\sqrt[3]{9} \\ \text{becomes} & 4\sqrt[3]{27\cdot 2} - 9\sqrt[3]{8\cdot 2} + 5\sqrt[3]{9} \\ \text{simplifying to} & 4\cdot 3\sqrt[3]{2} - 9\cdot 2\sqrt[3]{2} + 5\sqrt[3]{9} \\ \text{and reduces to} & 12\sqrt[3]{2} - 18\sqrt[3]{2} + 5\sqrt[3]{9} \end{array}$$

Combining like radicals yields:

$$5\sqrt[3]{9} - 6\sqrt[3]{2}$$

Questions

Simplify.

1. $2\sqrt{5} + 2\sqrt{5} + 2\sqrt{5}$
2. $-3\sqrt{6} - 3\sqrt{3} - 2\sqrt{3}$
3. $-3\sqrt{2} + 3\sqrt{5} + 3\sqrt{5}$
4. $-2\sqrt{6} - \sqrt{3} - 3\sqrt{6}$
5. $2\sqrt{2} - 3\sqrt{18} - \sqrt{2}$
6. $-\sqrt{54} - 3\sqrt{6} + 3\sqrt{27}$
7. $-3\sqrt{6} - \sqrt{12} + 3\sqrt{3}$
8. $-\sqrt{5} - \sqrt{5} - 2\sqrt{54}$
9. $3\sqrt{2} + 2\sqrt{8} - 3\sqrt{18}$
10. $2\sqrt{20} + 2\sqrt{20} - \sqrt{3}$
11. $3\sqrt{18} - \sqrt{2} - 3\sqrt{2}$
12. $-3\sqrt{27} + 2\sqrt{3} - \sqrt{12}$
13. $-3\sqrt{6} - 3\sqrt{6} - \sqrt{3} + 3\sqrt{6}$
14. $-2\sqrt{2} - \sqrt{2} + 3\sqrt{8} + 3\sqrt{6}$
15. $-2\sqrt{18} - 3\sqrt{8} - \sqrt{20} + 2\sqrt{20}$
16. $-3\sqrt{18} - \sqrt{8} + 2\sqrt{8} + 2\sqrt{8}$
17. $-2\sqrt{24} - 2\sqrt{6} + 2\sqrt{6} + 2\sqrt{20}$
18. $-3\sqrt{8} - \sqrt{5} - 3\sqrt{6} + 2\sqrt{18}$
19. $3\sqrt{24} - 3\sqrt{27} + 2\sqrt{6} + 2\sqrt{8}$
20. $2\sqrt{6} - \sqrt{54} - 3\sqrt{27} - \sqrt{3}$

Answers to odd questions

1. $(2+2+2)\sqrt{5}$
$6\sqrt{5}$
3. $-3\sqrt{2}+6\sqrt{5}$
5. $\sqrt{2}-3\sqrt{9\cdot 2}$
$\sqrt{2}-3\cdot 3\sqrt{2}$
$-8\sqrt{2}$
7. $-3\sqrt{6}-\sqrt{4\cdot 3}+3\sqrt{3}$
$-3\sqrt{6}-2\sqrt{3}+3\sqrt{3}$
$-3\sqrt{6}+\sqrt{3}$
9. $3\sqrt{2}+2\sqrt{4\cdot 2}-3\sqrt{2\cdot 9}$
$3\sqrt{2}+2\cdot 2\sqrt{2}-3\cdot 3\sqrt{2}$
$3\sqrt{2}+4\sqrt{2}-9\sqrt{2}$
$-2\sqrt{2}$
11. $3\sqrt{9\cdot 2}-4\sqrt{2}$
$3\cdot 3\sqrt{2}-4\sqrt{2}$
$9\sqrt{2}-4\sqrt{2}\Rightarrow 5\sqrt{2}$
13. $-3\sqrt{6}-\sqrt{3}$
15. $-2\sqrt{9\cdot 2}-3\sqrt{4\cdot 2}-\sqrt{4\cdot 5}+2\sqrt{4\cdot 5}$
$-6\sqrt{2}-6\sqrt{2}-2\sqrt{5}+4\sqrt{5}$
$-12\sqrt{2}+2\sqrt{5}$
17. $-2\sqrt{4\cdot 6}+2\sqrt{5\cdot 4}$
$-4\sqrt{6}+4\sqrt{5}$
19. $3\sqrt{6\cdot 4}-3\sqrt{3\cdot 9}+2\sqrt{6}+2\sqrt{2\cdot 4}$
$6\sqrt{6}-9\sqrt{3}+2\sqrt{6}+4\sqrt{2}$
$4\sqrt{2}-9\sqrt{3}+8\sqrt{6}$

CHAPTER 8.4: MULTIPLICATION AND DIVISION OF RADICALS

Multiplying radicals is very simple if the index on all the radicals match. The product rule of radicals, which is already been used, can be generalized as follows:

$$\text{Product Rule of Radicals: } a\sqrt[m]{b} \cdot c\sqrt[m]{d} = ac\sqrt[m]{bd}$$

This means that, if the index on the radicals match, then simply multiply the factors outside the radical and also multiply the factors inside the radicals. An example showing this is as follows.

Example 1

Multiply $-5\sqrt{14} \cdot 4\sqrt{6}$.

$$\begin{array}{ll} \text{This results in} & -5 \cdot 4\sqrt{14 \cdot 6} \\ \text{Which simplifies to} & -20\sqrt{84} \\ \text{Reducing inside the radical leaves} & -20\sqrt{4 \cdot 21} \\ \text{Yielding} & -20 \cdot 2\sqrt{21} \\ \text{Or} & -40\sqrt{21} \end{array}$$

This same process works with any higher root radicals having matching indices.

Example 2

Multiply $2\sqrt[3]{18} \cdot 6\sqrt[3]{15}$.

$$\begin{array}{ll} \text{This results in} & 2 \cdot 6\sqrt[3]{18 \cdot 15} \\ \text{Which simplifies to} & 12\sqrt[3]{270} \\ \text{Reducing inside the radical leaves} & 12\sqrt[3]{27 \cdot 10} \\ \text{Yielding} & 12 \cdot 3\sqrt[3]{10} \\ \text{Or} & 36\sqrt[3]{10} \end{array}$$

This process of multiplying radicals is the same when multiplying monomial radicals by binomial radicals, binomial radicals by binomial radicals, trinomial radicals (although these are not shown here), and so on.

Example 3

Multiply $7\sqrt{6}(3\sqrt{10} - 5\sqrt{15})$.

$$\begin{array}{ll} \text{Foiling the radicals will leave you with} & 21\sqrt{60} - 35\sqrt{90} \\ \text{Reducing inside the radical leaves} & 21\sqrt{4 \cdot 15} - 35\sqrt{9 \cdot 10} \\ \text{Yielding} & 21 \cdot 2\sqrt{15} - 35 \cdot 3\sqrt{10} \\ \text{Or} & 42\sqrt{15} - 105\sqrt{10} \end{array}$$

Example 4

Multiply $(\sqrt{5} - 2\sqrt{3})(4\sqrt{10} + 6\sqrt{6})$.

Multiplying the factors inside and outside the radicals yields:

$$4\sqrt{50} + 6\sqrt{30} - 8\sqrt{30} - 12\sqrt{18}$$

Reducing inside these radicals leaves	$4\sqrt{25\cdot 2}+6\sqrt{30}-8\sqrt{30}-12\sqrt{9\cdot 2}$
Yielding	$4\cdot 5\sqrt{2}+6\sqrt{30}-8\sqrt{30}-12\cdot 3\sqrt{2}$
Or	$20\sqrt{2}+6\sqrt{30}-8\sqrt{30}-36\sqrt{2}$
Which simplifies to	$-16\sqrt{2}-2\sqrt{30}$

Division with radicals is very similar to multiplication. If you think about division as reducing fractions, you can reduce the coefficients outside the radicals and reduce the values inside the radicals to get our final solution. There is one catch to dividing with radicals: it is considered bad practice to have a radical in the denominator of a final answer, so if there is a radical in the denominator, it should be rationalized by cancelling or multiplying the radicals.

$$\text{Quotient Rule of Radicals: } \frac{a\sqrt[m]{b}}{c\sqrt[m]{d}}=\left(\frac{a}{c}\right)\sqrt[m]{\frac{b}{d}}$$

The quotient rule means that factors outside the radical are divided by each other and the factors inside the radical are also divided by each other. To see this illustrated, consider the following:

Example 5

Reduce $\dfrac{15\sqrt[3]{108}}{20\sqrt[3]{2}}$.

Using the quotient rule of radicals, this problem is separated into factors inside and outside the radicals. This results in the following:

$$\left(\frac{15}{20}\right)\sqrt[3]{\frac{108}{2}}$$

Simplifying the two resulting divisions leaves us with $\left(\frac{3}{4}\right)\sqrt[3]{54}$

Which we can further reduce to $\left(\frac{3}{4}\right)\sqrt[3]{27\cdot 2}$

Taking the cube root of 27 leaves us with $\left(\frac{3}{4}\right)3\sqrt[3]{2}$

Or $\left(\frac{9}{4}\right)\sqrt[3]{2}$

Which can also be written as $\frac{9\sqrt[3]{2}}{4}$

Removing radicals from the denominator that cannot be divided out by using the numerator is often simply done by multiplying the numerator and denominator by a common radical. This is easily done and is shown by the following examples.

Example 6

Rationalize the denominator of $\frac{\sqrt{6}}{\sqrt{5}}$.

For this pair of radicals, the denominator $\sqrt{5}$ cannot be cancelled by the $\sqrt{6}$, so the solution requires that $\sqrt{5}$ be rationalized through multiplication. This is done as follows:

$$\frac{\sqrt{6\cdot 5}}{\sqrt{5\cdot 5}}$$

This now simplifies to:

$$\frac{\sqrt{30}}{5}$$

This process is similar for radicals in which the index is greater than 2.

Example 7

Rationalize the denominator of $\dfrac{4\sqrt[3]{6}}{5\sqrt[3]{25}}$.

To rationalize the denominator, we need to get a cube root of 125, which will leave us with a denominator of 5 × 5. This requires that both the numerator and the denominator to be multiplied by the cube root of 5. This looks like:

$$\frac{4\sqrt[3]{6\cdot 5}}{5\sqrt[3]{25\cdot 5}}=\frac{4\sqrt[3]{30}}{5\sqrt[3]{125}}$$

This simplifies to:

$$\frac{4\sqrt[3]{30}}{5\cdot 5}$$

Or:

$$\frac{4\sqrt[3]{30}}{25}$$

The last example to be considered involves rationalizing denominators that have variables. Remeber to always reduce any fractions (inside and outside of the radical) before rationalizing.

Example 8

Rationalize the denominator of $\dfrac{18\sqrt[4]{6x^3y^4z}}{8\sqrt[4]{10xy^6z^3}}$.

The first thing to do is cancel all common factors both inside and outside the radicals. This leaves:

$$\frac{9\sqrt[4]{3x^2}}{4\sqrt[4]{5y^2z^2}}$$

The next step is to multiply both the numerator and denominator to rationalize the denominator:

$$\frac{9\sqrt[4]{3x^2}}{4\sqrt[4]{5y^2z^2}}\cdot\frac{\sqrt[4]{125y^2z^2}}{\sqrt[4]{125y^2z^2}}$$

Multiplying these yields:

$$\frac{9\sqrt[4]{375x^2y^2z^2}}{4\sqrt[4]{625x^4y^4z^4}}$$

Taking the fourth root of the denominator leaves:

$$\frac{9\sqrt[4]{375x^2y^2z^2}}{4\cdot 5xyz}$$

Or:

$$\frac{9\sqrt[4]{375x^2y^2z^2}}{20xyz}$$

Questions

Simplify.

1. $3\sqrt{5}\cdot 4\sqrt{16}$
2. $-5\sqrt{10}\cdot\sqrt{15}$
3. $\sqrt{12m}\cdot\sqrt{15m}$
4. $\sqrt{5r^3}-5\sqrt{10r^2}$
5. $\sqrt[3]{4x^3}\cdot\sqrt[3]{2x^4}$
6. $3\sqrt[3]{4a^4}\cdot\sqrt[3]{10a^3}$
7. $\sqrt{6}(\sqrt{2}+2)$
8. $\sqrt{10}(\sqrt{5}+\sqrt{2})$
9. $-5\sqrt{15}(3\sqrt{3}+2)$

10. $5\sqrt{15}(3\sqrt{3}+2)$
11. $5\sqrt{10}(5n+\sqrt{2})$
12. $\sqrt{15}(\sqrt{5}-3\sqrt{3v})$
13. $(2+2\sqrt{2})(-3+\sqrt{2})$
14. $(-2+\sqrt{3})(-5+2\sqrt{3})$
15. $(\sqrt{5}-5)(2\sqrt{5}-1)$
16. $(2\sqrt{3}+\sqrt{5})(5\sqrt{3}+2\sqrt{4})$
17. $(\sqrt{2a}+2\sqrt{3a})(3\sqrt{2a}+\sqrt{5a})$
18. $(-2\sqrt{2p}+5\sqrt{5})(\sqrt{5p}+\sqrt{5p})$
19. $(-5-4\sqrt{3})(-3-4\sqrt{3})$
20. $(5\sqrt{2}-1)(-\sqrt{2m}+5)$
21. $\dfrac{\sqrt{12}}{5\sqrt{100}}$
22. $\dfrac{\sqrt{15}}{2\sqrt{4}}$
23. $\dfrac{\sqrt{5}}{4\sqrt{125}}$
24. $\dfrac{\sqrt{12}}{\sqrt{3}}$
25. $\dfrac{\sqrt{10}}{\sqrt{6}}$
26. $\dfrac{\sqrt{2}}{3\sqrt{5}}$
27. $\dfrac{5x^2}{4\sqrt{3x^3y^3}}$
28. $\dfrac{4}{5\sqrt{3xy^4}}$
29. $\dfrac{\sqrt{2p^2}}{\sqrt{3p}}$
30. $\dfrac{\sqrt{8n^2}}{\sqrt{10n}}$

Answers to odd questions

1. $12\sqrt{5\cdot 16}$

$12\cdot 4\sqrt{5}$

$48\sqrt{5}$

3. $\sqrt{15\cdot 12\cdot m^2}$

$\sqrt{3\cdot 5\cdot 3\cdot 4\cdot m^2}$

$3 \cdot 2m\sqrt{5}$
$6m\sqrt{5}$

5. $\sqrt[3]{8x^7}$
$\sqrt[3]{8 \cdot x^6 \cdot x}$
$2x^2\sqrt[3]{x}$

7. $\sqrt{12} + 2\sqrt{6}$
$\sqrt{4 \cdot 3} + 2\sqrt{6}$
$2\sqrt{3} + 2\sqrt{6}$

9. $-15\sqrt{45} - 10\sqrt{15}$
$-15\sqrt{9 \cdot 5} - 10\sqrt{15}$
$-15 \cdot 3\sqrt{5} - 10\sqrt{15}$
$-45\sqrt{5} - 10\sqrt{15}$

11. $25n\sqrt{10} + 5\sqrt{20}$
$25n\sqrt{10} + 5\sqrt{4 \cdot 5}$
$25n\sqrt{10} + 10\sqrt{5}$

13. $-6 + 2\sqrt{2} - 6\sqrt{2} + 2(\sqrt{2})(\sqrt{2})$
$-6 + 2\sqrt{2} - 6\sqrt{2} + 2(2)$
$-6 + 4 + 2\sqrt{2} - 6\sqrt{2}$
$-2 - 4\sqrt{2}$

15. $(2\sqrt{5})(\sqrt{5}) - \sqrt{5} - 10\sqrt{5} + 5$
$2(5) - \sqrt{5} - 10\sqrt{5} + 5$
$10 + 5 - \sqrt{5} - 10\sqrt{5}$
$15 - 11\sqrt{5}$

17. $3(2a) + 6\sqrt{6a^2} + \sqrt{10a^2} + 2\sqrt{15a^2}$
$6a + 6a\sqrt{6} + a\sqrt{10} + 2a\sqrt{15}$

19. $15 + 12\sqrt{3} + 20\sqrt{3} + 16(3)$
$63 + 32\sqrt{3}$

21.
$$\frac{\sqrt{12}}{5\sqrt{100}} \div \sqrt{4}$$
$$\frac{\sqrt{3}}{5\sqrt{25}} \Rightarrow \frac{\sqrt{3}}{5 \cdot 5} \Rightarrow \frac{\sqrt{3}}{25}$$

23.
$$\frac{\sqrt{5}}{4\sqrt{125}} \div \sqrt{5}$$
$$\frac{\sqrt{1}}{4\sqrt{25}} \Rightarrow \frac{1}{4 \cdot 5} \Rightarrow \frac{1}{20}$$

25.
$$\frac{\sqrt{10}}{\sqrt{6}} \div \sqrt{2}$$

$$\frac{\sqrt{5}}{\sqrt{3}}$$

27. $$\frac{5x^2}{4\sqrt{3 \cdot x^2 \cdot x \cdot y^2 \cdot y}} \Rightarrow \frac{5x^2}{4xy\sqrt{3xy}} \Rightarrow \frac{5x}{4y\sqrt{3xy}}$$

29.
$$\frac{\sqrt{2p^2}}{\sqrt{3p}} \div \sqrt{p}$$

$$\frac{\sqrt{2p}}{\sqrt{3}}$$

CHAPTER 8.5: RATIONALIZING DENOMINATORS

It is considered non-conventional to have a radical in the denominator. When this happens, generally the numerator and denominator are multiplied by the same factors to remove the radical denominator. The problems in the previous section dealt with removing a monomial radical. In this section, the previous strategy is expanded to include binomial radicals.

Example 1

Rationalize $\dfrac{\sqrt{3}-9}{2\sqrt{6}}$.

To rationalize the denominator, multiply out the $\sqrt{6}$.

This will look like:

$$\frac{(\sqrt{3}-9)(\sqrt{6})}{2\sqrt{6}(\sqrt{6})}$$

Multiplying the $\sqrt{6}$ throughout yields:

$$\frac{(\sqrt{3})(\sqrt{6})-(9)(\sqrt{6})}{2\sqrt{36}}$$

Which reduces to:

$$\frac{3\sqrt{2}-9\sqrt{6}}{2\cdot 6}$$

And simplifies to:

$$\frac{\sqrt{2}-3\sqrt{6}}{4}$$

Please note that, in reducing the numerator and denominator by the factor 3, reduce each term in the numerator by 3.

Quite often, there will be a denominator binomial that contains radicals. For these problems, it is easiest to use a feature from the sum and difference of squares: $a^2 - b^2 = (a+b)(a-b)$.

$(a+b)(a-b)$ are termed conjugates of each other. They are identical binomials, except that their signs are opposite. When encountering radical binomials, simply multiply by the conjugates to square out the radical.

Example 2

Square out the radical of the binomial $(\sqrt{3} - \sqrt{5})$ using its conjugate.

The conjugate of $(\sqrt{3} - \sqrt{5})$ is $(\sqrt{3} + \sqrt{5})$.

When multiplied, these conjugates yield $(\sqrt{3} - \sqrt{5})(\sqrt{3} + \sqrt{5})$ or $(\sqrt{3})^2 - (\sqrt{5})^2$.

This yields 3 − 5 = −2.

When encountering a radicalized binomial denominator, the best solution is to multiply both the numerator and denominator by the conjugate of the denominator.

Example 3

Rationalize the denominator of $\dfrac{\sqrt{6}}{\sqrt{6}+\sqrt{13}}$.

Multiplying the numerator and denominator by the denominator's conjugate yields:

$$\frac{\sqrt{6}}{\sqrt{6}+\sqrt{13}} \cdot \frac{(\sqrt{6}-\sqrt{13})}{(\sqrt{6}-\sqrt{13})}$$

When multiplied out, this yields:

$$\frac{(\sqrt{6})^2 - \sqrt{6}\sqrt{13}}{(\sqrt{6})^2 - (\sqrt{13})^2}$$

Which reduces to:

$$\frac{6-\sqrt{78}}{6-13}$$

Or:

$$\frac{6-\sqrt{78}}{-7}$$

Questions

Rationalize the following radical fractions.

1. $\dfrac{4+2\sqrt{3}}{\sqrt{3}}$
2. $\dfrac{-4+\sqrt{3}}{4\sqrt{3}}$
3. $\dfrac{4+2\sqrt{3}}{5\sqrt{6}}$
4. $\dfrac{2\sqrt{3}-2}{2\sqrt{3}}$
5. $\dfrac{2-5\sqrt{5}}{4\sqrt{3}}$
6. $\dfrac{\sqrt{5}+4}{4\sqrt{5}}$
7. $\dfrac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}$
8. $\dfrac{\sqrt{5}-\sqrt{2}}{3\sqrt{6}}$
9. $\dfrac{5}{3\sqrt{5}+\sqrt{2}}$
10. $\dfrac{5}{\sqrt{3}+4\sqrt{5}}$
11. $\dfrac{2}{5+\sqrt{2}}$
12. $\dfrac{5}{2\sqrt{3}-\sqrt{2}}$
13. $\dfrac{3}{4-\sqrt{3}}$

14. $\dfrac{4}{\sqrt{2}-2}$
15. $\dfrac{4}{3+\sqrt{5}}$
16. $\dfrac{2}{\sqrt{5}+2\sqrt{3}}$
17. $\dfrac{-3+2\sqrt{3}}{\sqrt{3}+2}$
18. $\dfrac{4+\sqrt{5}}{2+2\sqrt{5}}$
19. $\dfrac{2-\sqrt{3}}{1+\sqrt{2}}$
20. $\dfrac{-1+\sqrt{3}}{\sqrt{3}-1}$

Answers to odd Questions

1. $\dfrac{4+2\sqrt{3}}{\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}\Rightarrow\dfrac{4\sqrt{3}+2(3)}{3}\Rightarrow\dfrac{4\sqrt{3}+6}{3}$
3. $\dfrac{4+2\sqrt{3}}{5\sqrt{6}}\cdot\dfrac{\sqrt{6}}{\sqrt{6}}\Rightarrow\dfrac{4\sqrt{6}+2\sqrt{18}}{5(6)}\Rightarrow\dfrac{4\sqrt{6}+6\sqrt{2}}{30}\Rightarrow\dfrac{2\sqrt{6}+3\sqrt{2}}{15}$
5. $\dfrac{2-5\sqrt{5}}{4\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}\Rightarrow\dfrac{2\sqrt{3}-5\sqrt{15}}{4(3)}\Rightarrow\dfrac{2\sqrt{3}-5\sqrt{15}}{12}$
7. $\dfrac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}\Rightarrow\dfrac{\sqrt{6}-3(3)}{3}\Rightarrow\dfrac{\sqrt{6}-9}{3}$
9. $\dfrac{5}{3\sqrt{5}+\sqrt{2}}\cdot\dfrac{3\sqrt{5}-\sqrt{2}}{3\sqrt{5}-\sqrt{2}}\Rightarrow\dfrac{15\sqrt{5}-5\sqrt{2}}{9(5)-2}\Rightarrow\dfrac{15\sqrt{5}-5\sqrt{2}}{43}$
11. $\dfrac{2}{5+\sqrt{2}}\cdot\dfrac{5-\sqrt{2}}{5-\sqrt{2}}\Rightarrow\dfrac{10-2\sqrt{2}}{25-2}\Rightarrow\dfrac{10-2\sqrt{2}}{23}$
13. $\dfrac{3}{4-\sqrt{3}}\cdot\dfrac{4+\sqrt{3}}{4+\sqrt{3}}\Rightarrow\dfrac{12+3\sqrt{3}}{16-3}\Rightarrow\dfrac{12+3\sqrt{3}}{13}$
15. $\dfrac{4}{3+\sqrt{5}}\cdot\dfrac{3-\sqrt{5}}{3-\sqrt{5}}\Rightarrow\dfrac{12-4\sqrt{5}}{9-5}\Rightarrow\dfrac{12-4\sqrt{5}}{4}\Rightarrow 3-\sqrt{5}$
17. $\dfrac{-3+2\sqrt{3}}{\sqrt{3}+2}\cdot\dfrac{\sqrt{3}-2}{\sqrt{3}-2}\Rightarrow\dfrac{-3\sqrt{3}+6+2(3)-4\sqrt{3}}{3-4}\Rightarrow\dfrac{12-7\sqrt{3}}{-1}\Rightarrow$
$-12+7\sqrt{3}$
19. $\dfrac{2-\sqrt{3}}{1+\sqrt{2}}\cdot\dfrac{1-\sqrt{2}}{1-\sqrt{2}}\Rightarrow\dfrac{2-2\sqrt{2}-\sqrt{3}+\sqrt{6}}{1-2}\Rightarrow\dfrac{2-2\sqrt{2}-\sqrt{3}+\sqrt{6}}{-1}\Rightarrow$
$2\sqrt{2}+\sqrt{3}-\sqrt{6}-2$

CHAPTER 8.6: RADICALS AND RATIONAL EXPONENTS

When simplifying radicals that use fractional exponents, the numerator on the exponent is divided by the denominator. All radicals can be shown as having an equivalent fractional exponent. For example:

$$\sqrt{x} = x^{\frac{1}{2}} \qquad \sqrt[3]{x} = x^{\frac{1}{3}} \qquad \sqrt[4]{x} = x^{\frac{1}{4}} \qquad \sqrt[5]{x} = x^{\frac{1}{5}}$$

Radicals having some exponent value inside the radical can also be written as a fractional exponent. For example:

$$\sqrt{x^3} = x^{\frac{3}{2}} \qquad \sqrt[3]{x^2} = x^{\frac{2}{3}} \qquad \sqrt[4]{x^5} = x^{\frac{5}{4}} \qquad \sqrt[5]{x^9} = x^{\frac{9}{5}}$$

The general form that radicals having exponents take is:

$$x^{\frac{b}{a}} = \sqrt[a]{x^b} \text{ or } (\sqrt[a]{x})^b$$

Should the reciprocal of a radical having an exponent, it would look as follows:

$$x^{-\frac{b}{a}} = \frac{1}{\sqrt[a]{x^b}} \text{ or } \frac{1}{(\sqrt[a]{x})^b}$$

In both cases shown above, the power of the radical is b and the root of the radical is a. These are the two forms that a radical having an exponent is commonly written in. It is convenient to work with a radical containing an exponent in one of these two forms.

Example 1

Evaluate $27^{-\frac{4}{3}}$.

Converting to a radical form:

$$\frac{1}{\sqrt[3]{27^4}} \text{ or } \frac{1}{(\sqrt[3]{27})^4}$$

First, the cube root of 27 will reduce to 3, which leaves:

$$\frac{1}{3^4} \text{ or } \frac{1}{81}$$

Once the radical having an exponent is converted into a pure fractional exponent, then the following rules can be used.

Properties of Exponents

$$a^m a^n = a^{m+n} \qquad (ab)^m = a^m b^m \qquad a^{-m} = \frac{1}{a^m}$$

$$\frac{a^m}{a^n} = a^{m-n} \qquad \left(\frac{a}{b}\right) = \frac{a^m}{b^m} \qquad \frac{1}{a^{-m}} = a^m$$

$$(a^m)^n = a^{mn} \qquad a^0 = 1 \qquad \left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}$$

Example 2

Simplify $(x^2y^{\frac{4}{3}})(x^{-1}y^{\frac{2}{3}})$.

First, you need to separate the different variables:

$$(x^2y^{\frac{4}{3}})(x^{-1}y^{\frac{2}{3}}) \text{ becomes } x^2 \cdot x^{-1} \cdot y^{\frac{4}{3}} \cdot y^{\frac{2}{3}}$$

Combining the exponents yields:

$$x^{2-1} \cdot y^{\frac{4}{3}+\frac{2}{3}}$$

Which results in:

$$x^1 \cdot y^{\frac{6}{3}}$$

Which simplifies to:

$$xy^2$$

Example 3

Simplify $\dfrac{ab^{\frac{2}{3}}3b^{-\frac{5}{3}}}{5a^{-\frac{3}{2}}b^{-\frac{4}{3}}}$.

First, separate the different variables:

$$\frac{ab^{\frac{2}{3}}3b^{-\frac{5}{3}}}{5a^{-\frac{3}{2}}b^{-\frac{4}{3}}} \text{ becomes } 3\cdot 5^{-1}\cdot a\cdot a^{\frac{3}{2}}\cdot b^{\frac{2}{3}}\cdot b^{-\frac{5}{3}}\cdot b^{\frac{4}{3}}$$ [1]

Combining the exponents yields:

$$3\cdot 5^{-1}\cdot a^{1+\frac{3}{2}}\cdot b^{\frac{2}{3}-\frac{5}{3}+\frac{4}{3}}$$

Which gives:

$$3\cdot 5^{-1}\cdot a^{\frac{5}{2}}\cdot b^{\frac{1}{3}}$$

Which simplifies to:

$$\frac{3\cdot a^{\frac{5}{2}}\cdot b^{\frac{1}{3}}}{5}$$

Questions

Write each of the following fractional exponents in radical form.

1. $m^{\frac{3}{5}}$
2. $(10r)^{-\frac{3}{4}}$
3. $(7x)^{\frac{3}{2}}$
4. $(6b)^{-\frac{4}{3}}$
5. $(2x+3)^{-\frac{3}{2}}$
6. $(x-3y)^{\frac{3}{4}}$

Write each of the following radicals in exponential form.

7. $\sqrt[3]{5}$
8. $\sqrt[5]{2^3}$

1. When we divide by an exponent, we subtract powers.

9. $\sqrt[3]{ab^5}$
10. $\sqrt[5]{x^3}$
11. $\sqrt[3]{(a+5)^2}$
12. $\sqrt[5]{(a-2)^3}$

Evaluate the following.

13. $8^{\frac{2}{3}}$
14. $16^{\frac{1}{4}}$
15. $\sqrt[3]{4^6}$
16. $\sqrt[5]{32^2}$

Simplify. Your answer should only contain positive exponents.

17. $(xy^{\frac{1}{3}})(xy^{\frac{2}{3}})$
18. $(4v^{\frac{2}{3}})(v^{-1})$
19. $(a^{\frac{1}{2}}b^{\frac{1}{2}})^{-1}$
20. $(x^{\frac{5}{3}}y^{-2})^0$
21. $\dfrac{a^2b^0}{3a^4}$
22. $\dfrac{2x^{\frac{1}{2}}y^{\frac{1}{3}}}{2x^{\frac{4}{3}}y^{\frac{7}{4}}}$
23. $\dfrac{a^{\frac{3}{4}}b^{-1}b^{\frac{7}{4}}}{3b^{-1}}$
24. $\dfrac{2x^{-2}y^{\frac{5}{3}}}{x^{-\frac{5}{4}}y^{-\frac{5}{3}}xy^{\frac{1}{2}}}$
25.

$\dfrac{3y^{-\frac{5}{4}}}{y^{-1}2y^{-\frac{1}{3}}}$

26. $\dfrac{ab^{\frac{1}{3}}2b^{-\frac{5}{4}}}{4a^{-\frac{1}{2}}b^{-\frac{2}{3}}}$

Answers to odd questions

1. $\sqrt[5]{m^3}$

3. $\sqrt{(7x)^3}$
5. $\dfrac{1}{\sqrt{(2x+3)^3}}$
7. $5^{\frac{1}{3}}$
9. $(ab^5)^{\frac{1}{3}}$ or $a^{\frac{1}{3}}b^{\frac{5}{3}}$
11. $(a+5)^{\frac{2}{3}}$
13. $8^{\frac{2}{3}} \Rightarrow (2^3)^{\frac{2}{3}} \Rightarrow 2^2 \text{ or } 4$
15. $\sqrt[3]{4^6} \Rightarrow (2^2)^{\frac{6}{3}} \Rightarrow 2^4 \text{ or } 16$
17. $x^2y^{\frac{1}{3}+\frac{2}{3}} \Rightarrow x^2y$
19. $a^{-\frac{1}{2}}b^{-\frac{1}{2}} \Rightarrow \dfrac{1}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$
21. $\dfrac{a^2b^01}{3a^4a^2} \Rightarrow \dfrac{1}{3a^2}$
23. $\dfrac{a^{\frac{3}{4}}b^{-1}b^{\frac{7}{4}}}{3b^{-1}} \Rightarrow \dfrac{a^{\frac{3}{4}}b^{\frac{7}{4}}}{3}$
25. $\dfrac{3}{2}y^{-\frac{5}{4}--1--\frac{1}{3}} \Rightarrow \dfrac{3}{2}y^{\frac{1}{12}}$

CHAPTER 8.7: RATIONAL EXPONENTS (INCREASED DIFFICULTY)

Simplifying rational exponents equations that are more difficult generally involves two steps. First, reduce inside the brackets. Second, multiplu the power outside the brackets for all terms inside.

Example 1

Simplify the following rational exponent expression:

$$\left(\frac{x^{-2}y^{-3}}{x^{-2}y^{4}}\right)^{2}$$

First, simplifying inside the brackets gives:

$$x^{-2--2}y^{-3-4}$$

Or:

$$x^{0}y^{-7}$$

Which simplifies to:

$$y^{-7}$$

Second, taking the exponent 2 outside the brackets and applying it to the reduced expression gives:

$$y^{-7\cdot 2} \text{ or } y^{-14}$$

Therefore:

$$\left(\frac{x^{-2}y^{-3}}{x^{-2}y^{4}}\right)^{2} = y^{-14}$$

Example 2

Simplify the following rational exponent expression:

$$\left(\frac{x^{-4}y^{-6}}{x^{-5}y^{10}}\right)^{-3}$$

First, simplifying inside the brackets gives:

$$x^{-4--5}y^{-6-10}$$

Or:

$$x^{1}y^{-16}$$

Which simplifies to:

$$xy^{-16}$$

Second, taking the exponent −3 outside the brackets and applying it to the reduced expression gives:

$$(xy^{-16})^{-3} \text{ or } x^{-3}y^{48}$$

Therefore:

$$\left(\frac{x^{-4}y^{-6}}{x^{-5}y^{10}}\right)^{-3} = x^{-3}y^{48} = \frac{y^{48}}{x^{3}}$$

Example 3

Simplify the following rational exponent expression:

$$\left(\frac{a^{0}b^{3}}{c^{6}d^{-12}}\right)^{\frac{1}{3}}$$

First, simplifying inside the brackets gives:

$$\frac{b^3}{c^6d^{-12}}$$

Second, taking the exponent $\frac{1}{3}$ outside the brackets and applying it to the reduced expression gives:

$$\frac{b^{3\cdot\frac{1}{3}}}{c^{6\cdot\frac{1}{3}}d^{-12\cdot\frac{1}{3}}}$$

Or:

$$\frac{b}{c^2d^{-4}}$$

Which simplifies to:

$$\frac{bd^4}{c^2}$$

Questions

Simplify the following rational exponents.

1. $\left(\dfrac{x^{-2}y^{-6}}{x^{-2}y^{4}}\right)^2$
2. $\left(\dfrac{x^{-3}y^{-3}}{x^{-1}y^{6}}\right)^3$
3. $\left(\dfrac{x^{-2}y^{-4}}{x^{2}y^{-4}}\right)^2$
4. $\left(\dfrac{x^{-5}y^{-3}}{x^{-4}y^{2}}\right)^4$
5. $\left(\dfrac{x^{-2}y^{-2}}{x^{-3}y^{3}}\right)^8$
6. $\left(\dfrac{x^{-4}y^{-3}}{x^{-3}y^{2}}\right)^5$
7. $\left(\dfrac{x^{-2}y^{-4}}{x^{-2}y^{4}}\right)^{-2}$
8. $\left(\dfrac{x^{-2}y^{-3}}{x^{-5}y^{3}}\right)^{-3}$
9. $\left(\dfrac{x^{-2}y^{-3}}{x^{-2}y^{-3}}\right)^{-1}$

10. $\left(\dfrac{x^{-2}y^{-3}}{x^{-2}y^{4}}\right)^{-2}$
11. $\left(\dfrac{x^{0}y^{-3}}{x^{-2}y^{0}}\right)^{-5}$
12. $\left(\dfrac{x^{-22}y^{-36}}{x^{-24}y^{12}}\right)^{0}$
13. $\left(\dfrac{a^{0}b^{3}}{a^{6}b^{-12}}\right)^{-\frac{1}{3}}$
14. $\left(\dfrac{a^{12}b^{4}}{a^{8}c^{-12}}\right)^{\frac{1}{4}}$
15. $\left(\dfrac{a^{5}c^{10}}{b^{5}d^{-15}}\right)^{\frac{2}{5}}$
16. $\left(\dfrac{a^{2}b^{8}}{a^{6}b^{-12}}\right)^{-\frac{3}{4}}$
17. $\left(\dfrac{a^{0}b^{3}}{c^{6}d^{-12}}\right)^{\frac{0}{3}}$
18. $\left(\dfrac{a^{0}b^{3}}{c^{6}d^{-12}}\right)^{\frac{1}{10}}$

Answers to odd questions

1. $(x^{-2--2}y^{-6-4})^2$
$(1x^0y^{-10})^2$
y^{-20} or $\dfrac{1}{y^{20}}$

3. $(x^{-2-2}y^{-4--4})^2$
$(x^{-4}y^0 1)^2$
x^{-8} or $\dfrac{1}{x^8}$

5. $(x^{-2--3}y^{-2-3})^8$
$(x^1y^{-5})^8$
x^8y^{-40} or $\dfrac{x^8}{y^{40}}$

7. $(x^{-2--2}y^{-4-4})^{-2}$
$(1x^0y^{-8})^{-2}$
y^{16}

9. $(x^{-2--2}y^{-3--3})^{-1}$
$(x^0y^0 1)^{-1}$
1

11. $\left(\dfrac{1x^0y^{-3}}{x^{-2}y^01}\right)^{-5}$

$\left(\dfrac{x^2}{y^3}\right)^{-5}$

$\dfrac{x^{-10}}{y^{-15}}$

$\dfrac{y^{15}}{x^{10}}$

13. $\left(\dfrac{1a^0b^3}{a^6b^{-12}}\right)^{-\frac{1}{3}}$

$\left(\dfrac{b^15}{a^6}\right)^{-\frac{1}{3}}$

$\dfrac{b^-5}{a^-2}$ or $\dfrac{a^2}{b^5}$

15. $\left(\dfrac{a^5c^{10}d^{15}}{b^5}\right)^{\frac{2}{5}}$

$\dfrac{a^2c^4d^6}{b^2}$

17. 1

CHAPTER 8.8: RADICALS OF MIXED INDEX

Knowing that a radical has the same properties as exponents allows conversion of radicals to exponential form and then reduce according to the various rules of exponents is possible. This is shown in the following examples.

Example 1

Simplify $\sqrt[8]{x^6y^2}$.

$$\begin{array}{ll} \text{First rewrite the radical as a fractional exponent} & (x^6y^2)^{\frac{1}{8}} \\ \text{Multiply all exponents} & x^{6\cdot\frac{1}{8}}y^{2\cdot\frac{1}{8}} \\ \text{This yields} & x^{\frac{6}{8}}y^{\frac{2}{8}} \\ \text{Reducing this gives} & x^{\frac{3}{4}}y^{\frac{1}{4}} \\ \text{Rewrite as} & \sqrt[4]{x^3y} \end{array}$$

Note: In Example 9.8.1, all exponents are reduced by the common factor 2. If there is a common factor in all exponents, reduce by dividing that common factor without having to convert to a different form.

Example 2

Simplify $\sqrt[24]{a^6b^9c^{15}}$.

For this radical, notice that each exponent has the common factor 3.

The solution is to divide each exponent by 3, which yields $\sqrt[8]{a^2b^3c^5}$.

When encountering problems where the index of the radicals do not match,convert each radical to individual exponents and use the properties of exponents to combine and then reduce the radicals.

Example 3

Simplify $\sqrt[3]{4x^2y} \cdot \sqrt[4]{8xy^3}$.

First, convert each radical to a complete exponential form.

This looks like	$(4x^2y)^{\frac{1}{3}}(8xy^3)^{\frac{1}{4}}$
Multiply all exponents	$4^{\frac{1}{3}}x^{2\cdot\frac{1}{3}}y^{\frac{1}{3}}8^{\frac{1}{4}}x^{\frac{1}{4}}y^{3\cdot\frac{1}{4}}$
This yields	$4^{\frac{1}{3}}x^{\frac{2}{3}}y^{\frac{1}{3}}8^{\frac{1}{4}}x^{\frac{1}{4}}y^{\frac{3}{4}}$
Combining like variables leaves	$4^{\frac{1}{3}}8^{\frac{1}{4}}x^{\frac{2}{3}}x^{\frac{1}{4}}y^{\frac{1}{3}}y^{\frac{3}{4}}$
(Note:	$4^{\frac{1}{3}}8^{\frac{1}{4}} = 2^{2\cdot\frac{1}{3}}2^{3\cdot\frac{1}{4}} = 2^{\frac{2}{3}}2^{\frac{3}{4}})$
Accounting for this yields	$2^{\frac{2}{3}}2^{\frac{3}{4}}x^{\frac{2}{3}}x^{\frac{1}{4}}y^{\frac{1}{3}}y^{\frac{3}{4}}$
Reducing this yields	$2^{\frac{2}{3}+\frac{3}{4}}x^{\frac{2}{3}+\frac{1}{4}}y^{\frac{1}{3}+\frac{3}{4}}$
Which further reduces to	$2^{\frac{17}{12}}x^{\frac{11}{12}}y^{\frac{13}{12}}$
Reduce this	$2y\cdot 2^{\frac{5}{12}}x^{\frac{11}{12}}y^{\frac{1}{12}}$
Convert this back into a radical	$2y(2^5x^{11}y)^{\frac{1}{12}}$
Which leaves	$2y\sqrt[12]{2^5x^{11}y}$

The strategy of converting all radicals to exponents works for increasingly complex radicals.

Example 4

Simplify $\sqrt{3x(y+z)}\cdot\sqrt[3]{9x(y+z)^2}$.

First, convert each radical to a complete exponential form.

This looks like	$3^{\frac{1}{2}}x^{\frac{1}{2}}(y+z)^{\frac{1}{2}}9^{\frac{1}{3}}x^{\frac{1}{3}}(y+z)^{\frac{2}{3}}$
(Note:	$9^{\frac{1}{3}}=3^{\frac{2}{3}}$)
Combining like variables leaves	$3^{\frac{1}{2}}3^{\frac{2}{3}}x^{\frac{1}{2}}x^{\frac{1}{3}}(y+z)^{\frac{1}{2}}(y+z)^{\frac{2}{3}}$
Reducing this yields	$3^{\frac{1}{2}+\frac{2}{3}}x^{\frac{1}{2}+\frac{1}{3}}(y+z)^{\frac{1}{2}+\frac{2}{3}}$
Which further reduces to	$3^{\frac{7}{6}}x^{\frac{5}{6}}(y+z)^{\frac{7}{6}}$
Reduce this	$3(y+z)3^{\frac{1}{6}}x^{\frac{5}{6}}(y+z)^{\frac{1}{6}}$
Convert this back into a radical	$3(y+z)[3x^5(y+z)]^{\frac{1}{6}}$
Which leaves	$3(y+z)\sqrt[6]{3x^5(y+z)}$

Questions

Reduce the following radicals. Leave as fractional exponents.

1. $\sqrt[8]{16x^4y^6}$
2. $\sqrt[4]{9x^2y^6}$
3. $\sqrt[12]{64x^4y^6z^8}$
4. $\sqrt[8]{\dfrac{25x^3}{16x^5}}$
5. $\sqrt[6]{\dfrac{16x}{9y^4}}$
6. $\sqrt[15]{x^9y^{12}z^6}$
7. $\sqrt[12]{x^6y^9}$
8. $\sqrt[10]{64x^8y^4}$
9. $\sqrt[8]{x^6y^4z^2}$
10. $\sqrt[4]{25y^2}$
11. $\sqrt[9]{8x^3y^6}$
12. $\sqrt[16]{81x^8y^{12}}$

Combine the following radicals. Leave as fractional exponents.

13. $\sqrt[3]{5}\sqrt{5}$
14. $\sqrt[3]{7}\sqrt[4]{7}$
15. $\sqrt{x}\sqrt[3]{7x}$
16. $\sqrt[3]{y}\sqrt[5]{3y}$
17. $\sqrt{x}\sqrt[3]{x^2}$
18. $\sqrt[4]{3x}\sqrt{x^4}$
19. $\sqrt[5]{x^2y}\sqrt{x^2}$
20. $\sqrt{ab}\sqrt[5]{2a^2b^2}$
21. $\sqrt[4]{xy^2}\sqrt[3]{x^2y}$
22. $\sqrt[5]{3a^2b^3}\sqrt[4]{9a^2b}$
23. $\sqrt[4]{a^2bc^2}\sqrt[5]{a^2b^3c}$
24. $\sqrt[6]{x^2yz^3}\sqrt[5]{x^2yz^2}$

Answers to odd questions

1. $(2^4x^4y^6)^{\frac{1}{8}} \Rightarrow 2^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{3}{4}}$
3. $(2^6x^4y^6z^8)^{\frac{1}{12}} \Rightarrow 2^{\frac{1}{2}}x^{\frac{1}{3}}y^{\frac{1}{2}}z^{\frac{2}{3}}$
5. $\left(\dfrac{2^4x}{3^2y^4}\right)^{\frac{1}{6}} \Rightarrow \dfrac{2^{\frac{2}{3}}x^{\frac{1}{6}}}{3^{\frac{1}{3}}y^{\frac{2}{3}}}$
7. $(x^6y^9)^{\frac{1}{12}} \Rightarrow x^{\frac{1}{2}}y^{\frac{3}{4}}$
9. $(x^6y^4z^2)^{\frac{1}{8}} \Rightarrow x^{\frac{3}{4}}y^{\frac{1}{2}}z^{\frac{1}{4}}$
11. $(2^3x^3y^6)^{\frac{1}{9}} \Rightarrow 2^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{2}{3}}$
13. $5^{\frac{1}{3}} \cdot 5^{\frac{1}{2}} \Rightarrow 5^{\frac{3}{6}} \cdot 5^{\frac{2}{6}} \Rightarrow 5^{\frac{5}{6}}$
15. $x^{\frac{1}{2}} \cdot 7^{\frac{1}{3}}x^{\frac{1}{3}} \Rightarrow 7^{\frac{1}{3}}x^{\frac{5}{6}}$
17. $x^{\frac{1}{2}} \cdot x^{\frac{2}{3}} \Rightarrow x^{\frac{7}{6}}$
19. $x^{\frac{2}{5}}y^{\frac{1}{5}}x^{\frac{2}{2}} \Rightarrow x^{\frac{7}{5}}y^{\frac{1}{5}}$
21. $x^{\frac{1}{4}}y^{\frac{2}{4}} \cdot x^{\frac{2}{3}}y^{\frac{1}{3}} \Rightarrow x^{\frac{11}{12}}y^{\frac{5}{6}}$
23. $a^{\frac{2}{4}}b^{\frac{1}{4}}c^{\frac{2}{4}}a^{\frac{2}{5}}b^{\frac{3}{5}}c^{\frac{1}{5}} \Rightarrow a^{\frac{9}{10}}b^{\frac{17}{20}}c^{\frac{7}{10}}$

CHAPTER 8.9 SOLVING RADICAL EQUATIONS

In this section, radical equations that need to solved are discussed. The strategy is relatively simple: isolate the radical on one side of the equation and all variables will remain on the other side. Once this is done, square, cube, or raise each side to a power that removes the radical. For example:

For $\sqrt{5x+3}=1,$ the solution can be found by squaring both sides $\quad(\sqrt{5x+3})^2=(1)^2$

For $\sqrt[3]{5x+3}=1,$ the solution can be found by cubing both sides $\quad(\sqrt[3]{5x+3})^3=(1)^3$

For $\sqrt[4]{5x+3}=1,$ the solution can be found by using the fourth power $\quad(\sqrt[4]{5x+3})^4=(1)^4$

For $\sqrt[5]{5x+3}=1,$ the solution can be found by using the fifth power $\quad(\sqrt[5]{5x+3})^5=(1)^5$

Once the radical is removed, then solve the resulting equation. Consider how this strategy is used in the following example.

Example 1

Solve for x in $\sqrt{2x+4}-6=0$.

First, isolate the radical:

$$\begin{array}{rrrcr}\sqrt{2x+4}&-&6&=&0\\&+&6&=&+6\\\sqrt{2x+4}&&&=&6\end{array}$$

This leaves:

$$\sqrt{2x+4}=6$$

Squaring both sides:

$$(\sqrt{2x+4})^2=(6)^2$$

This results in:

$$\begin{array}{rrrrr}
2x & + & 4 & = & 36 \\
 & - & 4 & & -4 \\
 & & 2x & = & 32 \\
 & & & & \\
 & & x & = & 16
\end{array}$$

This same strategy works for equations having indices larger than 2.

Example 2

Solve for x in $\sqrt[3]{3-3x}=\sqrt[3]{2x-7}$.

For this problem, there are two radicals equalling each other, and all that is required is to cube each to cancel the radicals out.

Because there is an odd index, there will have two solutions to this equation: one positive and one negative.

Positive Solution

$$\begin{array}{rrrrrrr}
 & 3 & - & 3x & = & 2x & - & 7 \\
- & 3 & - & 2x & & -2x & - & 3 \\
 & & & -5x & = & -10 & & \\
 & & & x & = & 2 & &
\end{array}$$

Negative Solution

$$\begin{array}{rrrrrrrr} & 3 & - & 3x & = & -(2x & - & 7) \\ & 3 & - & 3x & = & -2x & + & 7 \\ - & 3 & + & 2x & & +2x & - & 3 \\ & & & -x & = & 4 & & \\ & & & x & = & -4 & & \end{array}$$

The strategy used above to isolate and solve for the radicals works the same for radicals in inequalities, except that you will now have to square, cube or use a larger power on each of the terms. For example:

Example 3

Solve for x in $3 < \sqrt{3x+9} \leq 6$.

For this problem, there are three terms to square out.

This looks like:

$$(3)^2 < (\sqrt{3x+9})^2 \leq (6)^2$$

Which results in:

$$\begin{array}{rcrrrcr} 9 & < & 3x & + & 9 & \leq & 36 \\ -9 & & & - & 9 & & -9 \\ \dfrac{0}{3} & < & & \dfrac{3x}{3} & & \leq & \dfrac{27}{3} \\ \\ 0 & < & & x & & \leq & 9 \end{array}$$

For all cases of radical equations, check answers to see if they work. There may be variations of these radical equations in higher levels of math, but the strategy will always be similar in that you will always work to square the radicals out.

Questions

1. $\sqrt{2x+3}-3=0$
2. $\sqrt{5x+1}-4=0$
3. $\sqrt{6x-5}-x=0$
4. $\sqrt{7x+8}=x$
5. $\sqrt{3+x}=\sqrt{6x+13}$
6. $\sqrt{x-1}=\sqrt{7-x}$
7. $\sqrt[3]{3-3x}=\sqrt[3]{2x-5}$
8. $\sqrt[4]{3x-2}=\sqrt[4]{x+4}$
9. $\sqrt{x+7}\geq 2$
10. $\sqrt{x-2}\leq 4$
11. $3<\sqrt{3x+6}\leq 6$
12. $0<\sqrt{x+5}<5$

Answers to odd questions

1. $$\begin{array}{rrrrr} \sqrt{2x+3} & - & 3 & = & 0 \\ & + & 3 & & +3 \\ (\sqrt{2x+3})^2 & & & = & (3)^2 \\ \\ 2x & + & 3 & = & 9 \\ & - & 3 & & -3 \\ & & \dfrac{2x}{2} & = & \dfrac{6}{2} \\ \\ & & x & = & 3 \end{array}$$

3.

$$\begin{array}{rrrrrrrrr}
\sqrt{6x-5} & - & x & = & 0 & & & & \\
 & + & x & & +x & & & & \\
(\sqrt{6x-5})^2 & & & = & (x)^2 & & & & \\
 & & & & & & & & \\
6x & - & 5 & = & x^2 & & & & \\
-6x & + & 5 & & & - & 6x & + & 5 \\
 & & 0 & = & x^2 & - & 6x & + & 5 \\
 & & 0 & = & (x & - & 5)(x & - & 1) \\
 & & & & & & & & \\
 & & x & = & 5, & 1 & & &
\end{array}$$

5. $(\sqrt{3+x})^2 = (\sqrt{6x+13})^2$

$$\begin{array}{rrrrrrr}
3 & + & x & = & 6x & + & 13 \\
-3 & - & 6x & & -6x & - & 3 \\
 & & \dfrac{-5x}{-5} & = & \dfrac{10}{-5} & & \\
 & & & & & & \\
 & & x & = & -2 & &
\end{array}$$

7. $(\sqrt[3]{3-3x})^3 = (\sqrt[3]{2x-5})^3$

$$\begin{array}{rrrrrrr}
3 & - & 3x & = & 2x & - & 5 \\
-3 & - & 2x & & -2x & - & 3 \\
 & & \dfrac{-5x}{-5} & = & \dfrac{-8}{-5} & & \\
 & & & & & & \\
 & & x & = & \dfrac{8}{5} & &
\end{array}$$

9. $(\sqrt{x+7})^2 \geq (2)^2$

$$\begin{array}{rrrrr}
x & + & 7 & \geq & 4 \\
 & - & 7 & & -7 \\
 & & x & \geq & -3
\end{array}$$

11. $(3)^2 < (\sqrt{3x+6})^2 \leq (6)^2$

$$\begin{array}{rrrrrrr}
9 & < & 3x & + & 6 & \leq & 36 \\
-6 & & & - & 6 & & -6 \\
\dfrac{3}{3} & < & & \dfrac{3x}{3} & & \leq & \dfrac{30}{3} \\
 & & & & & & \\
1 & < & & x & & \leq & 10
\end{array}$$

CHAPTER 9: INTRODUCTION TO EXPONENTIAL AND LOGARITHMIC FUNCTIONS

CHAPTER 9.1: INTRODUCTION TO EXPONENTIAL AND LOGARITHMIC FUNCTIONS

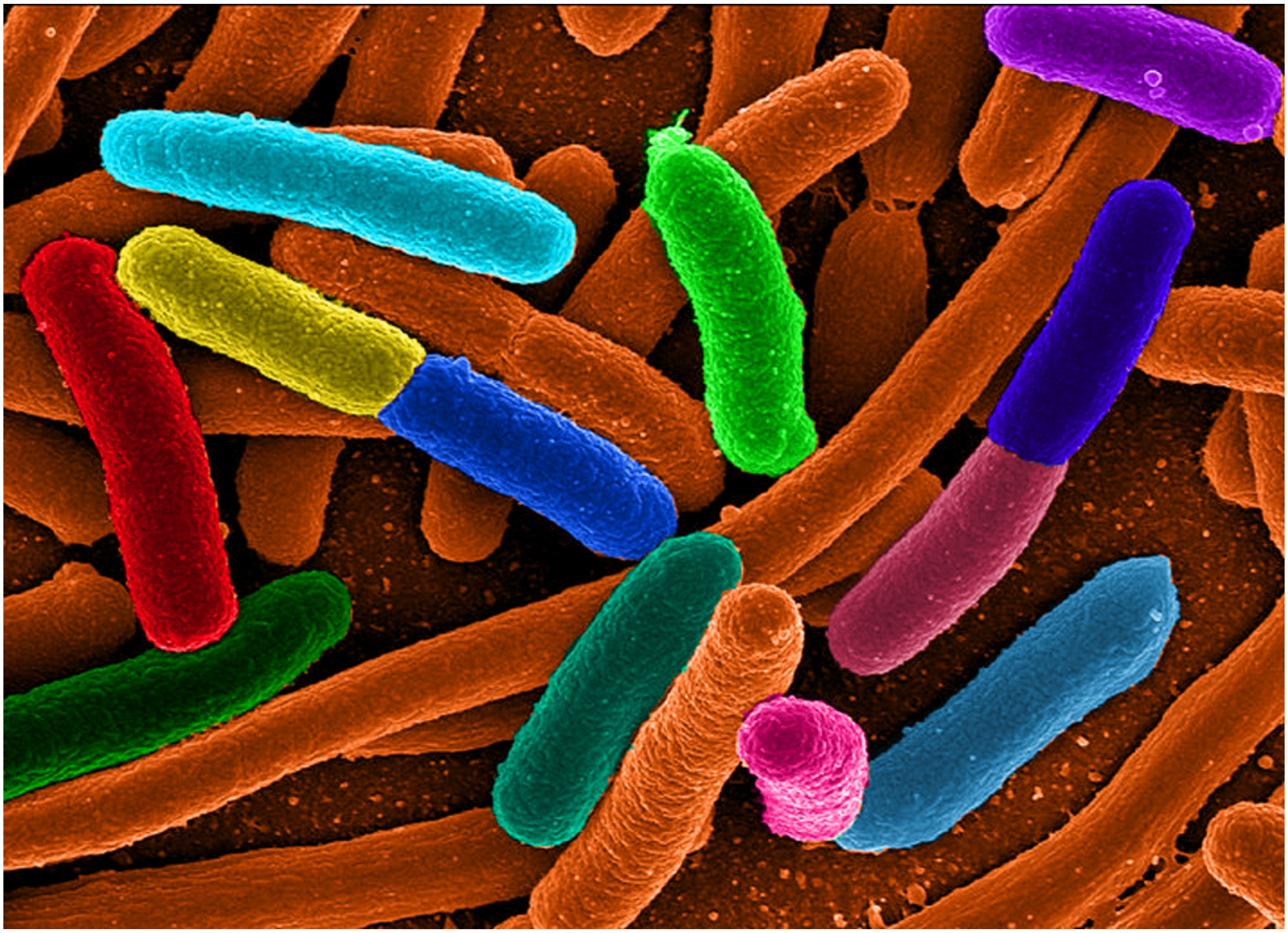


Figure 1. Electron micrograph of E.Coli bacteria (credit: "Mattosaurus," Wikimedia Commons)

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years.[1]

For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. (Figure) shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

1. Todar, PhD, Kenneth. Todar's Online Textbook of Bacteriology. http://textbookofbacteriology.net/growth_3.html.

CHAPTER 9.2: EXPONENTIAL FUNCTIONS

Learning Objectives

In this section, you will:

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about 1.2 each year[1] . If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

Identifying Exponential Functions

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f\left(x\right)=3x+4$, the slope tells us the output increases by 3 each time the input increases by 1. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009

1. http://www.worldometers.info/world-population/. Accessed February 24, 2014.

to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2021.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- Exponential decay refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See (Figure).

x	$f(x)=2^x$	$g(x)=2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

From (Figure) we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Apparently, the difference between “the same percentage” and “the same amount” is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

The general form of the exponential function is $f(x) = ab^x$, where a is any nonzero number, b is a positive real number not equal to 1.

- If $b > 1$, the function grows at a rate proportional to its size.
- If $0 < b < 1$, the function decays at a rate proportional to its size.

Let's look at the function $f(x) = 2^x$ from our example. We will create a table ((Figure)) to determine the corresponding outputs over an interval in the domain from -3 to 3.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Let us examine the graph of f by plotting the ordered pairs we observe on the table in (Figure), and then make a few observations.

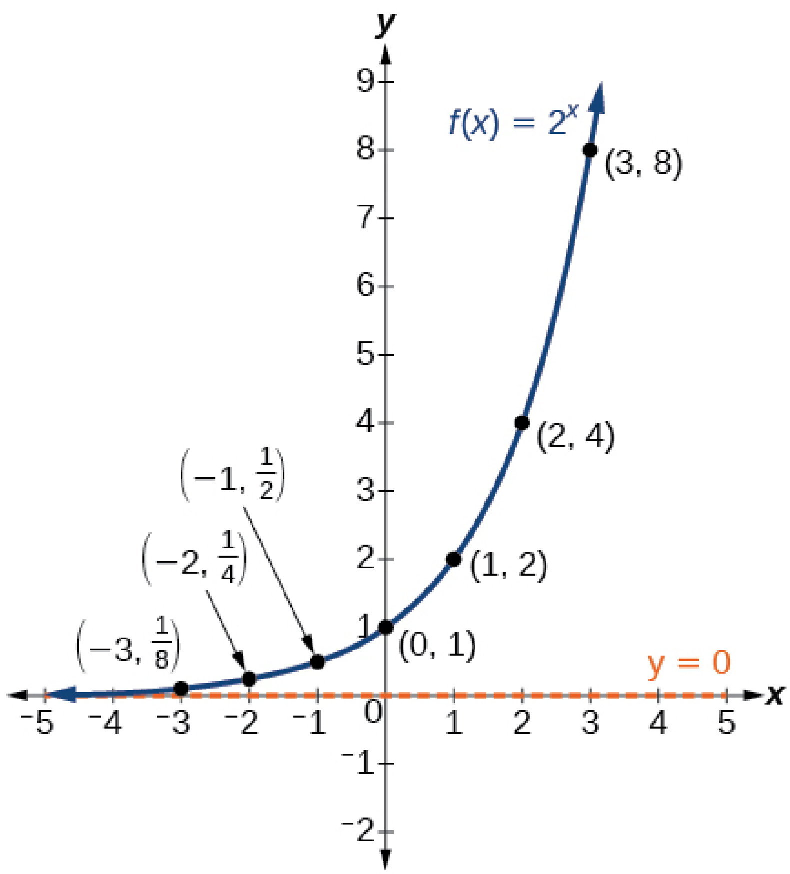


Figure 1.

Let's define the behavior of the graph of the exponential function $f\left(x\right)=2^{x}$ and highlight some its key characteristics.

- the domain is $\left(-\infty,\infty\right)$,
- the range is $\left(0,\infty\right)$,
- as $x\to\infty$, $f\left(x\right)\to\infty$,
- as $x\to-\infty$, $f\left(x\right)\to 0$,
- $f\left(x\right)$ is always increasing,
- the graph of $f\left(x\right)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y=0$ is the horizontal asymptote.
- the *y*-intercept is 1.

Exponential Function

For any real number x, an exponential function is a function with the form

$f\left(x\right)=ab^{x}$
where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b\ne 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a>0$.
- The range of f is all negative real numbers if $a<0$.
- The *y*-intercept is $\left(0,a\right)$, and the horizontal asymptote is $y=0$.

Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f\left(x\right)=4^{3\left(x-2\right)}$
- $g\left(x\right)=x^{3}$

- $h\left(x\right)=\left(\frac{1}{3}\right)^x$
- $j\left(x\right)=\left(-2\right)^x$

Show Solution

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g\left(x\right)=x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g\left(x\right)=x^3$ is a power function.

Recall that the base *b* of an exponential function is always a positive constant, and $b\neq 1.$ Thus, $j\left(x\right)=\left(-2\right)^x$ does not represent an exponential function because the base, $-2,$ is less than $0.$

Try It

Which of the following equations represent exponential functions?

- $f\left(x\right)=2x^2-3x+1$
- $g\left(x\right)=0.875^x$
- $h\left(x\right)=1.75x+2$
- $j\left(x\right)=1095.6^{-2x}$

Show Solution

$g\left(x\right)=0.875^x$ and $j\left(x\right)=1095.6^{-2x}$ represent exponential functions.

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b=-9$ and $x=\frac{1}{2}$. Then $f\left(x\right)=f\left(\frac{1}{2}\right)=(-9)^{\frac{1}{2}}=\sqrt{-9}$, which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1 :

- Let $b=1$. Then $f\left(x\right)=1^{x}=1$ for any value of x.

To evaluate an exponential function with the form $f\left(x\right)=b^{x}$, we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f\left(x\right)=2^{x}$. What is $f\left(3\right)$?

$$\begin{array}{lll} f\left(x\right) & =2^{x} & \\ f\left(3\right) & =2^{3} & \text{Substitute } x=3. \\ & =8 & \text{Evaluate the power.} \end{array}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f\left(x\right)=30(2)^{x}$. What is $f\left(3\right)$?

$$\begin{array}{lll} f\left(x\right) & =30(2)^{x} & \\ f\left(3\right) & =30(2)^{3} & \text{Substitute } x=3. \\ & =30\left(8\right) & \text{Simplify the power first.} \\ & =240 & \text{Multiply.} \end{array}$$

Note that if the order of operations were not followed, the result would be incorrect:

$$f\left(3\right)=30(2)^{3}\neq 60^{3}=216,000$$

Evaluating Exponential Functions

Let $f\left(x\right)=5(3)^{x+1}$. Evaluate $f\left(2\right)$ without using a calculator.

Show Solution

Follow the order of operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} f(x) &= 5(3)^{x+1} & \\ f(2) &= 5(3)^{2+1} & \text{Substitute } x = 2. \\ &= 5(3)^{3} & \text{Add the exponents.} \\ &= 5(27) & \text{Simplify the power.} \\ &= 135 & \text{Multiply.} \end{aligned}$$

Try It

Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Show Solution

5.5556

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Exponential Growth

A function that models exponential growth grows by a rate proportional to the amount present.

For any real number x and any positive real numbers a and b such that $b \neq 1$, an exponential growth function has the form

$f\left(x\right) = ab^x$
where

- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x .

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A\left(x\right) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B\left(x\right) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in (Figure).

Year, x	**Stores, Company A**	**Stores, Company B**
0	$100 + 50\left(0\right) = 100$	$100(1 + 0.5)^0 = 100$
1	$100 + 50\left(1\right) = 150$	$100(1 + 0.5)^1 = 150$
2	$100 + 50\left(2\right) = 200$	$100(1 + 0.5)^2 = 225$
3	$100 + 50\left(3\right) = 250$	$100(1 + 0.5)^3 = 337.5$
x	$A\left(x\right) = 100 + 50x$	$B\left(x\right) = 100(1 + 0.5)^x$

The graphs comparing the number of stores for each company over a five-year period are shown in (Figure**)**. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

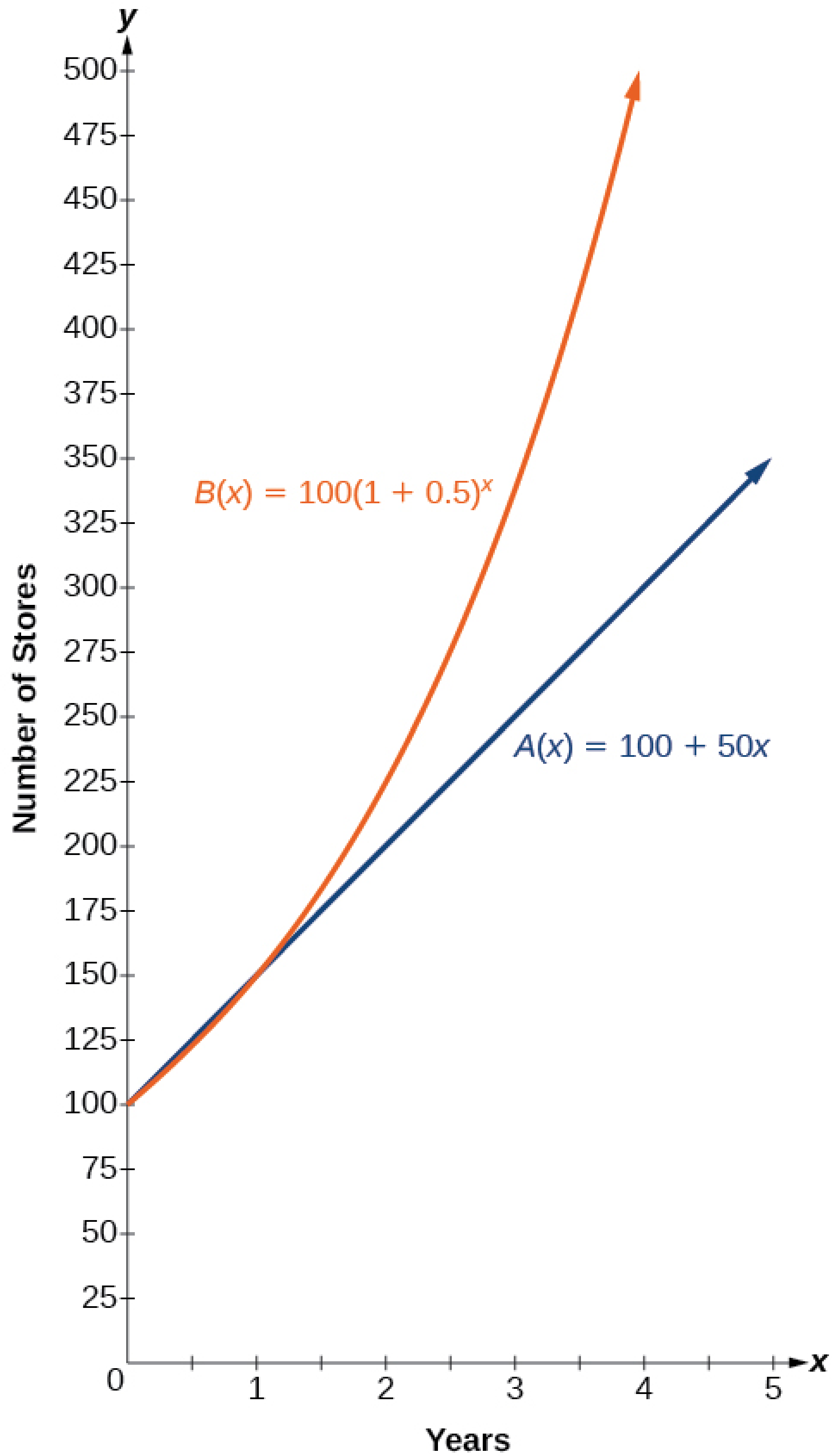


Figure 2. The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$, and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$, where 100 is the initial value, 1.5 is called the *base*, and x is called the *exponent*.

Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2 This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since $2013.$ To the nearest thousandth, what will the population of India be in $2031?$

Show Solution

To estimate the population in 2031, we evaluate the models for $t = 18,$ because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

Try It

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6 This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since $2013.$ To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in (Figure)?

Show Solution

About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people.

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b, and evaluate the function.

How To

Given two data points, write an exponential model.

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f\left(x\right)=a(b)^{x}$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f\left(x\right)=a(b)^{x}$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f\left(x\right)=a(b)^{x}$.

Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population (N) of deer over time t.

Show Solution

We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80.$ We can now substitute the second point into the equation $N(t) = 80b^t$ to find b :

$$\begin{array}{lll} N(t) & = 80b^t & \\ 180 & = 80b^6 & \text{Substitute using point } (6, 180). \\ \frac{9}{4} & = b^6 & \text{Divide and write in lowest terms.} \\ b & = \left(\frac{9}{4}\right)^{\frac{1}{6}} & \text{Isolate } b \text{ using properties of exponents.} \\ b & \approx 1.1447 & \text{Round to 4 decimal places.} \end{array}$$

NOTE: *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t) = 80(1.1447)^t.$ (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in (Figure) passes through the initial points given in the problem, $(0,\ 80)$ and $(6,\ 180)$. We can also see that the domain for the function is $[0, \infty)$, and the range for the function is $[80, \infty)$.

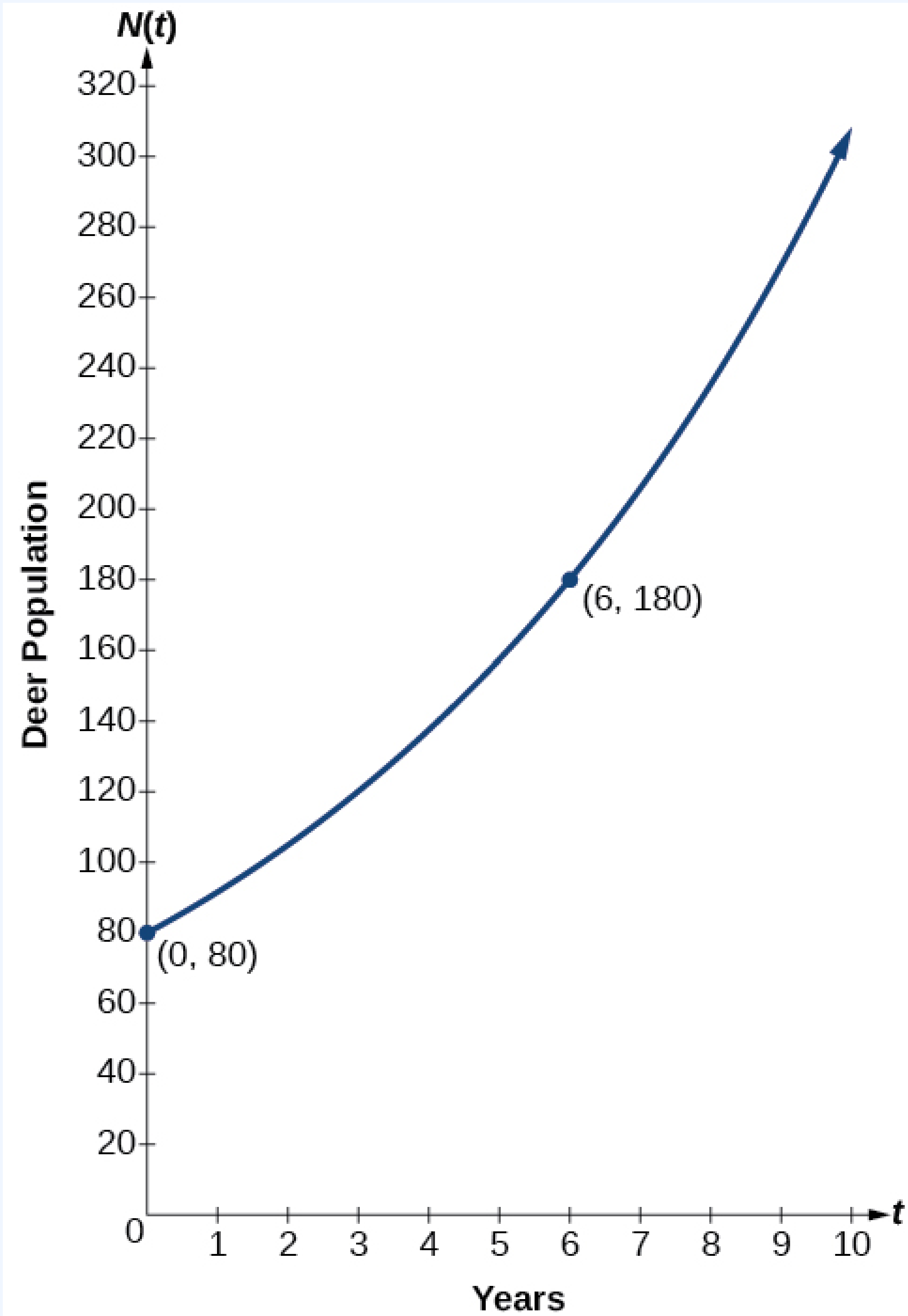

N(t)
Deer Population
320
300
280
260
240
220
200
180
160
140
120
100
80
60
40
20
0
(6, 180)
(0, 80)
1
2
3
4
5
6
7
8
9
10
t
Years

Figure 3. Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, t years after 2006

Try It

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Show Solution

$(0, 129)$ and $(2, 236)$; $\quad N(t) = 129(1.3526)^t$

Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Show Solution

Because we don't have the initial value, we substitute both points into an equation of the form $f\left(x\right)=a{b}^{x}$, and then solve the system for a and b.

- Substituting $\left(-2,6\right)$ gives $6=a{b}^{-2}$
- Substituting $\left(2,1\right)$ gives $1=a{b}^{2}$

Use the first equation to solve for a in terms of b :

$6=ab^{-2}$	
$\frac{6}{b^{-2}}=a$	Divide.
$a=6b^{2}$	Use properties of exponents to rewrite the denominator.

Substitute a in the second equation, and solve for b :

$1=ab^{2}$	
$1=6b^{2}b^{2}=6b^{4}$	Substitute a.
$b=\left(\frac{1}{6}\right)^{\frac{1}{4}}$	Use properties of exponents to isolate b.
$b\approx 0.6389$	Round 4 decimal places.

Use the value of b in the first equation to solve for the value of a :

$$a=6b^{2}\approx 6(0.6389)^{2}\approx 2.4492$$

Thus, the equation is $f\left(x\right)=2.4492{\left(0.6389\right)}^{x}$.

We can graph our model to check our work. Notice that the graph in (Figure) passes through the initial points given in the problem, $\left(-2,\text{ }6\right)$ and $\left(2,\text{ }1\right)$. The graph is an example of an exponential decay function.

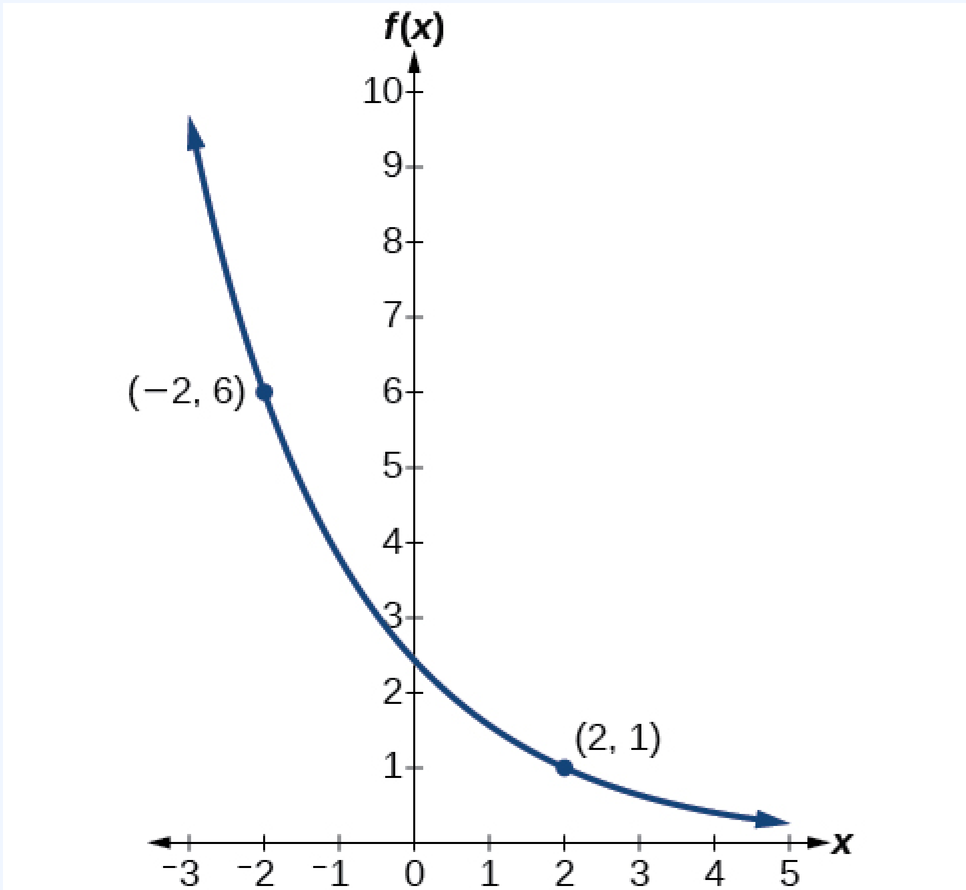


Figure 4. The graph of $f\ (x) = 2.4492(0.6389)^x$ models exponential decay.

Try It

Given the two points $(1,3)$ and $(2,4.5)$, find the equation of the exponential function that passes through these two points.

Show Solution

$f(x) = 2(1.5)^x$

Do two points always determine a unique exponential function?

Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x, which in many real world cases involves time.

How To

Given the graph of an exponential function, write its equation.

1. First, identify two points on the graph. Choose the *y*-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the *y*-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x) = a(b)^x$.

Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in (Figure).

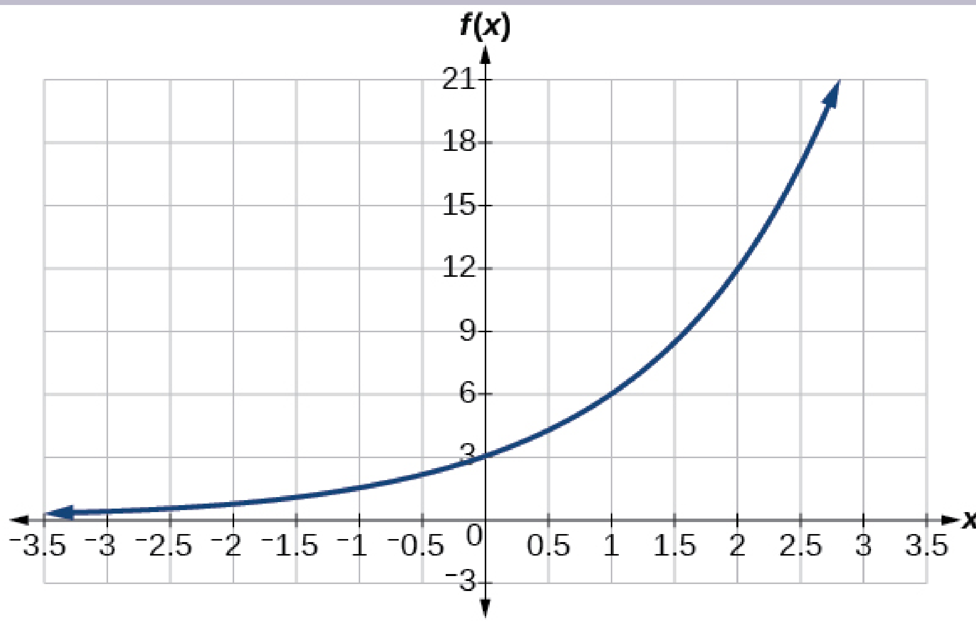


Figure 5.

Show Solution

We can choose the *y*-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3.$ Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$$\begin{array}{ll} y = ab^x & \text{Write the general form of an exponential equation.} \\ y = 3b^x & \text{Substitute the initial value 3 for } a. \\ 12 = 3b^2 & \text{Substitute in 12 for } y \text{ and 2 for } x. \\ 4 = b^2 & \text{Divide by 3.} \\ b = 2 & \text{Take the square root.} \end{array}$$

Because we restrict ourselves to positive values of b, we will use $b = 2.$ Substitute a and b into the standard form to yield the equation $f\left(x\right) = 3(2)^x.$

Try It

Find an equation for the exponential function graphed in (Figure).

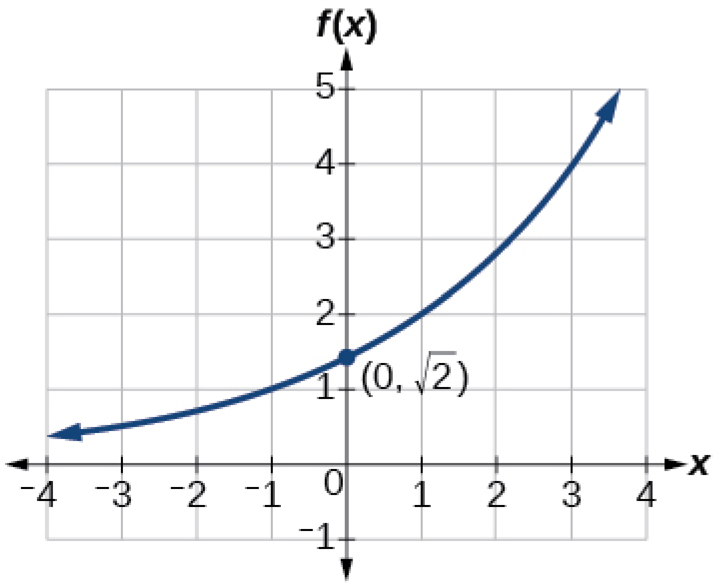


Figure 6.

Show Solution

$f\left(x\right)=\sqrt{2}{\left(\sqrt{2}\right)}^{x}$. Answers may vary due to round-off error. The answer should be very close to $1.4142{\left(1.4142\right)}^{x}$.

How To

Given two points on the curve of an exponential function, use a graphing calculator to find the equation.

1. Press **[STAT].**
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
6. The screen displays the values of *a* and *b* in the exponential equation $y = a \cdot b^x$.

Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2, 24.8)$ and $(5, 198.4)$.

Show Solution

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit…],** and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the *x*-coordinates, 2 and 5. Do the same in the **L2** column for the *y*-coordinates, 24.8 and 198.4.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a = 6.2$ and $b = 2$ will be displayed. The exponential equation is $y = 6.2 \cdot 2^x$.

Try It

Use a graphing calculator to find the exponential equation that includes the points (3, 75.98) and (6, 481.07).

Show Solution

$y \approx 12 \cdot 1.85^x$

Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use compound interest. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The annual percentage rate (APR) of an account, also called the nominal rate, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, APR r, and number of compounding periods in a year n :

$$A\left(t\right)=P{\left(1+\frac{r}{n}\right)}^{nt}$$

For example, observe (Figure), which shows the result of investing $1,000 at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Frequency	**Value after 1 year**
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

The Compound Interest Formula

Compound interest can be calculated using the formula

$A\left(t\right)=P{\left(1+\frac{r}{n}\right)}^{nt}$

where

- $A\left(t\right)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Calculating Compound Interest

If we invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Show Solution

Because we are starting with $3,000, $P=3000.$ Our interest rate is 3%, so $r\;=\;0.03.$ Because we are compounding quarterly, we are compounding 4 times per year, so $n=4.$ We want to know the value of the account in 10 years, so we are looking for $A\left(10\right)$, the value when $t\;=\;10.$

$$\begin{array}{lll} A\left(t\right) & =P{\left(1+\frac{r}{n}\right)}^{nt} & \text{Use the compound interest formula.} \\ A\left(10\right) & =3000{\left(1+\frac{0.03}{4}\right)}^{4\cdot 10} & \text{Substitute using given values.} \\ & \approx \$4045.05 & \text{Round to two decimal places.} \end{array}$$

The account will be worth about $4,045.05 in 10 years.

Try It

An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Show Solution

about $3,644,675.88

Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Show Solution

The nominal interest rate is 6%, so $r = 0.06.$ Interest is compounded twice a year, so $k = 2.$

We want to find the initial investment, $P,$ needed so that the value of the account will be worth $40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for $P.$

$$\begin{array}{lll} A\left(t\right) & = P\left(1+\frac{r}{n}\right)^{nt} & \text{Use the compound interest formula.} \\ 40,000 & = P\left(1+\frac{0.06}{2}\right)^{2(18)} & \text{Substitute using given values } A,\ r, n, \text{ and } t. \\ 40,000 & = P(1.03)^{36} & \text{Simplify.} \\ \frac{40{,}000}{(1.03)^{36}} & = P & \text{Isolate } P. \\ P & \approx \$13,801 & \text{Divide and round to the nearest dollar.} \end{array}$$

Lily will need to invest $13,801 to have $40,000 in 18 years.]

Try It

Refer to (Figure). To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Show Solution

\$13,693

Evaluating Functions with Base *e*

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. (Figure) shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of \$1 invested at 100% interest for 1 year, compounded at various frequencies, listed in (Figure).

Frequency	$A\left(t\right)=\left(1+\frac{1}{n}\right)^{n}$	**Value**
Annually	$\left(1+\frac{1}{1}\right)^{1}$	\$2
Semiannually	$\left(1+\frac{1}{2}\right)^{2}$	\$2.25
Quarterly	$\left(1+\frac{1}{4}\right)^{4}$	\$2.441406
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	\$2.613035
Daily	$\left(1+\frac{1}{365}\right)^{365}$	\$2.714567
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	\$2.718127
Once per minute	$\left(1+\frac{1}{525600}\right)^{525600}$	\$2.718279
Once per second	$\left(1+\frac{1}{31536000}\right)^{31536000}$	\$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

The Number *e*

The letter *e* represents the irrational number

$\left(1+\frac{1}{n}\right)^n$, as n $\text{increases without bound}$

The letter *e* is used as a base for many real-world exponential models. To work with base *e*, we use the approximation, $e \approx 2.718282.$ The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Using a Calculator to Find Powers of *e*

Calculate $e^{3.14}$. Round to five decimal places.

Show Solution

On a calculator, press the button labeled $[e^x]$. The window shows $[e^{(}\ ]$. Type 3.14 and then close parenthesis, [)]. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387.$

Caution: Many scientific calculators have an "Exp" button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Try It

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Show Solution
$e^{-0.5} \approx 0.60653$

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, *e* is used as the base for exponential functions. Exponential models that use e as the base are called *continuous growth or decay models*. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

The Continuous Growth/Decay Formula

For all real numbers t, and all positive numbers a and r, continuous growth or decay is represented by the formula

$A\left(t\right) = ae^{rt}$
where

- a is the initial value,
- r is the continuous growth rate per unit time,
- and t is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$A(t) = Pe^{rt}$
where

- P is the principal or the initial invested,
- r is the growth or interest rate per unit time,
- and t is the period or term of the investment.

How To

Given the initial value, rate of growth or decay, and time t, solve a continuous growth or decay function.

1. Use the information in the problem to determine a , the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
   a. If the problem refers to continuous growth, then $r > 0$.
   b. If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

Calculating Continuous Growth

A person invested $1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Show Solution

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10.$ The initial investment was $1,000, so $P = 1000.$ We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A\left(t\right) &= Pe^{rt} && \text{Use the continuous compounding formula.} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P, r, \text{ and } t. \\ &\approx 1105.17 && \text{Use a calculator to approximate.} \end{aligned}$$

The account is worth $1,105.17 after one year.

Try It

A person invests $100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Show Solution

$3,659,823.44

Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Show Solution

Since the substance is decaying, the rate, 17.3 , is negative. So, $r \quad = \quad -0.173.$ The initial amount of radon-222 was 100 mg, so $a = 100.$ We use the continuous decay formula to find the value after $t = 3$ days:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula.} \\ &= 100e^{-0.173(3)} && \text{Substitute known values for } a, r, \text{ and } t. \\ &\approx 59.5115 && \text{Use a calculator to approximate.} \end{aligned}$$

So 59.5115 mg of radon-222 will remain.

Try It

Using the data in (Figure), how much radon-222 will remain after one year?

Show Solution

3.77E-26 (This is calculator notation for the number written as 3.7710^{-26} in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

Key Equations

definition of the exponential function	$f\left(x\right)=b^x$, where $b>0, b\ne 1$
definition of exponential growth	$f\left(x\right)=ab^x,$ where $a>0, b>0, b\ne 1$
compound interest formula	$A\left(t\right)=P\left(1+\frac{r}{n}\right)^{nt},$ where $A\left(t\right)$ is the account value at time t t is the number of years P is the initial investment, often called the principal r is the annual percentage rate (APR), or nominal rate n is the number of compounding periods in one year
continuous growth formula	$A\left(t\right)=ae^{rt},$ where t is the number of unit time periods of growth a is the starting amount (in the continuous compounding formula a is replaced with P, the principal) e is the mathematical constant, $e\approx 2.718282$

Key Concepts

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent. See (Figure).
- A function is evaluated by solving at a specific value. See (Figure) and (Figure).
- An exponential model can be found when the growth rate and initial value are known. See (Figure).
- An exponential model can be found when the two data points from the model are known. See (Figure).
- An exponential model can be found using two data points from the graph of the model. See (Figure).
- An exponential model can be found using two data points from the graph and a calculator. See (Figure).
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See (Figure).
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See (Figure).

- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282.$
- Scientific and graphing calculators have the key $[e^x]$ or $[\exp(x)]$ for calculating powers of $e.$ See (Figure).
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See (Figure) and (Figure).

Section Exercises

Verbal

1. Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

Show Solution
Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

2. Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

3. The Oxford Dictionary defines the word *nominal* as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.”[2] Develop a reasonable argument for why the term *nominal rate* is used to describe the annual percentage rate of an investment account that compounds interest.

2. Oxford Dictionary. http://oxforddictionaries.com/us/definition/american_english/nomina.

Show Solution

When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of *nominal*.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

4. The average annual population increase of a pack of wolves is 25.

5. A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.

Show Solution

exponential; the population decreases by a proportional rate.

6. The value of a coin collection has increased by 3.25 annually over the last 20 years.

7. For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

Show Solution

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function. .

8. The height of a projectile at time t is represented by the function $h\left(t\right)=-4.9t^2+18t+40.$

For the following exercises, consider this scenario: For each year t, the population of a forest of trees is represented by the function $A\left(t\right)=115{(1.025)}^{t}$. In a neighboring forest, the population of the same type of tree is represented by the function $B\left(t\right)=82{(1.029)}^{t}$. (Round answers to the nearest whole number.)

9. Which forest's population is growing at a faster rate?

Show Solution
The forest represented by the function $B\left(t\right)=82{(1.029)}^{t}$.

10. Which forest had a greater number of trees initially? By how many?

11. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

Show Solution
After $t=20$ years, forest A will have 43 more trees than forest B.

12. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?

13. Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

Show Solution
Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population

will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

14. $y = 300(1-t)^5$

15. $y = 220(1.06)^x$

Show Solution
exponential growth; The growth factor, $1.06,$ is greater than $1.$

16. $y = 16.5(1.025)^{\frac{1}{x}}$

17. $y = 11,701(0.97)^t$

Show Solution
exponential decay; The decay factor, $0.97,$ is between 0 and $1.$

For the following exercises, find the formula for an exponential function that passes through the two points given.

18. $(0, 6)$ and $(3, 750)$

19. $(0, 2000)$ and $(2, 20)$

Show Solution

$f(x) = 2000(0.1)^x$

20. $\left(-1, \frac{3}{2}\right)$ and $(3, 24)$

21. $(-2, 6)$ and $(3, 1)$

Show Solution

$f(x) = \left(\frac{1}{6}\right)^{-\frac{3}{5}}\left(\frac{1}{6}\right)^{\frac{x}{5}} \approx 2.93(0.699)^x$

22. $(3, 1)$ and $(5, 4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

23.

x	1	2	3	4
$f(x)$	70	40	10	-20

Show Solution

Linear

24.

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

25.

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

Show Solution
Neither

26.

x	1	2	3	4
$f(x)$	10	20	40	80

27.

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

Show Solution
Linear

For the following exercises, use the compound interest formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

28. After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?

29. What was the initial deposit made to the account in the previous exercise?

Show Solution

$10,250$

30. How many years had the account from the previous exercise been accumulating interest?

31. An account is opened with an initial deposit of \$6,500 and earns 3.6 interest compounded semi-annually. What will the account be worth in 20 years?

Show Solution

$13,268.58$

32. How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

33. Solve the compound interest formula for the principal, P .

Show Solution

$P = A(t) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$

34. Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $14,472.74$ after earning 5.5 interest compounded monthly for 5 years. (Round to the nearest dollar.)

35. How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

Show Solution

$4,572.56$

36. Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

37. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $9,000 and was worth $13,373.53 after 10 years.

Show Solution

4

38. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $5,500, and was worth $38,455 after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

39. $y = 3742(e)^{0.75t}$

Show Solution

continuous growth; the growth rate is greater than 0.

40. $y = 150(e)^{\frac{3.25}{t}}$

41. $y = 2.25(e)^{-2t}$

Show Solution
continuous decay; the growth rate is less than $0.$

42. Suppose an investment account is opened with an initial deposit of $12,000$ earning 7.2 interest compounded continuously. How much will the account be worth after 30 years?

43. How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

Show Solution
669.42

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

44. $f\left(x\right)=2(5)^{x}$, for $f\left(-3\right)$

45. $f\left(x\right)=-4^{2x+3}$, for $f\left(-1\right)$

Show Solution
$f\left(-1\right)=-4$

46. $f\left(x\right)=e^{x}$, for $f\left(3\right)$

47. $f\left(x\right)=-2e^{x-1}$, for $f\left(-1\right)$

Show Solution

$f\left(-1\right) \approx -0.2707$

48. $f\left(x\right) = 2.7(4)^{-x+1} + 1.5$, for $f\left(-2\right)$

49. $f\left(x\right) = 1.2e^{2x} - 0.3$, for $f\left(3\right)$

Show Solution

$f\left(3\right) \approx 483.8146$

50. $f\left(x\right) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$, for $f\left(2\right)$

Technology

For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.

51. $(0, 3)$ and $(3, 375)$

Show Solution

$y = 3 \cdot 5^{x}$

52. $(3, 222.62)$ and $(10, 77.456)$

53. $(20, 29.495)$ and $(150, 730.89)$

Show Solution

$y \approx 18 \cdot 1.025^x$

54. $(5, 2.909)$ and $(13, 0.005)$

55. $(11, 310.035)$ and $(25, 356.3652)$

Show Solution

$y \approx 0.2 \cdot 1.95^x$

Extensions

56. The *annual percentage yield* (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $\text{APY} = \left(1 + \frac{r}{12}\right)^{12} - 1.$

57. Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I\left(n\right)$ for the APY of any account that compounds n times per year.

Show Solution

$\text{APY} = \frac{A(t)-a}{a} = \frac{a\left(1+\frac{r}{365}\right)^{365(1)}-a}{a} = \frac{a\left[\left(1+\frac{r}{365}\right)^{365}-1\right]}{a} = \left(1 + \frac{r}{365}\right)^{365} - 1;$
$I\left(n\right) = \left(1 + \frac{r}{n}\right)^{n} - 1$

58. Recall that an exponential function is any equation written in the form $f\left(x\right) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1$. Any positive number b can be written as $b = e^n$ for some value of n . Use this fact to rewrite the formula for an exponential function that uses the number e as a base.

59. In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f\left(x\right) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f\left(x\right) = a(e)^{-nx}$ for some positive number n .

Show Solution

Let f be the exponential decay function $f\left(x\right) = a \cdot \left(\frac{1}{b}\right)^x$ such that $b > 1$. Then for some number $n > 0$,

$f\left(x\right) = a \cdot \left(\frac{1}{b}\right)^x = a\left(b^{-1}\right)^x = a\left(\left(e^n\right)^{-1}\right)^x = a\left(e^{-n}\right)^x = a(e)^{-nx}.$

60. The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A\left(t\right) = a(e)^{rt}$, where a is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time t can be calculated with the formula $I\left(t\right) = e^{rt} - 1.$

Real-World Applications

61. The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were 23,900 fox counted in the area. What is the fox population predicted to be in the year 2020?

Show Solution

$47,622$ fox

62. A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

63. In the year 1985, a house was valued at $110,000. By the year 2005, the value had appreciated to $145,000. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

Show Solution
$1.39\ 155,368.09$

64. A car was valued at \$38,000 in the year 2007. By 2013, the value had depreciated to \$11,000 If the car's value continues to drop by the same percentage, what will it be worth by 2017?

65. Jamal wants to save \$54,000 for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

Show Solution
$35,838.76$

66. Kyoko has \$10,000 that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have \$15,000 by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (*Hint*: solve the compound interest formula for the interest rate.)

67. Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was \$13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

Show Solution
$82,247.78; 449.75$

68. An investment account with an annual interest rate of 7% was opened with an initial deposit of \$4,000 Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

Glossary

annual percentage rate (APR)
: the yearly interest rate earned by an investment account, also called *nominal rate*

compound interest
: interest earned on the total balance, not just the principal

exponential growth
: a model that grows by a rate proportional to the amount present

nominal rate
: the yearly interest rate earned by an investment account, also called *annual percentage rate*

CHAPTER 9.3: GRAPHS OF EXPONENTIAL FUNCTIONS

Learning Objectives

- Graph exponential functions.
- Graph exponential functions using transformations.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f\left(x\right)={b}^{x}$ whose base is greater than one. We'll use the function $f\left(x\right)={2}^{x}$. Observe how the output values in (Figure) change as the input increases by 1.

x	-3	-2	-1	0	1	2	3
$f\left(x\right)={2}^{x}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f\left(x\right)=a{b}^{x}$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

(Figure) shows the exponential growth function $f\left(x\right)=2^x$.

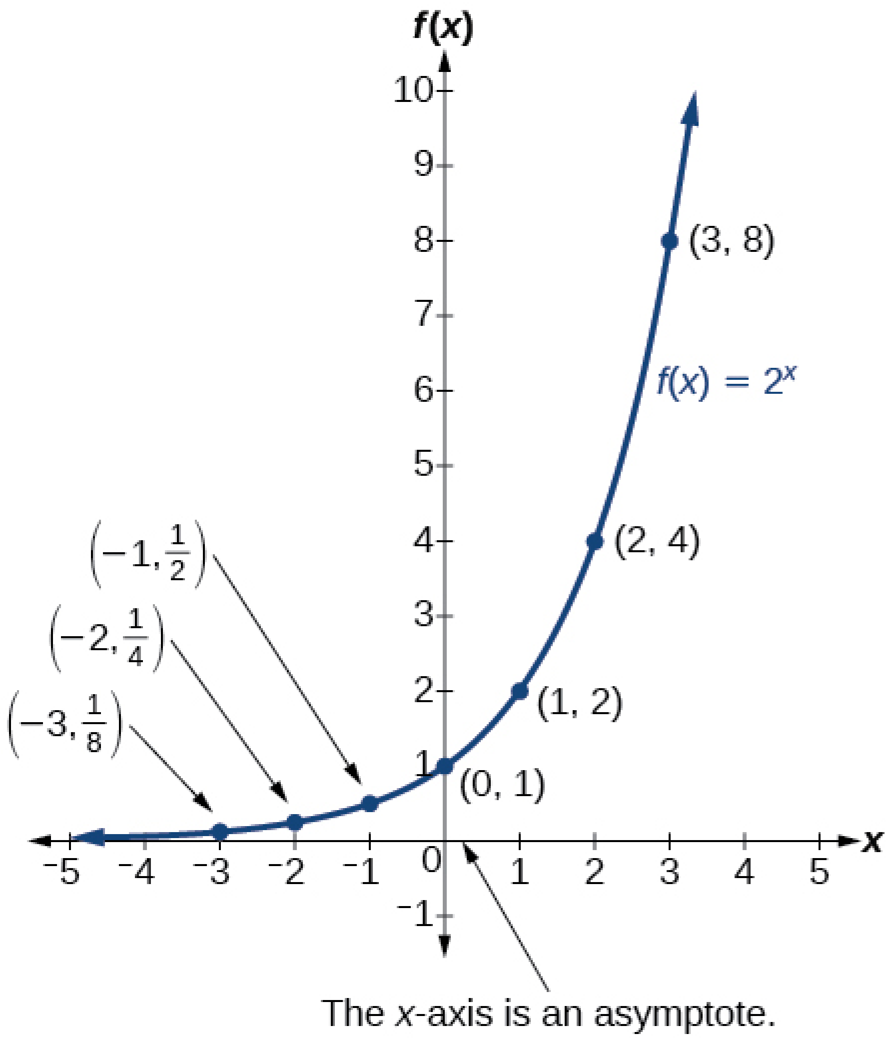


Figure 1. Notice that the graph gets close to the x-axis, but never touches it.

The domain of $f\left(x\right)=2^x$ is all real numbers, the range is $\left(0,\infty\right)$, and the horizontal asymptote is $y=0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f\left(x\right)=b^x$ whose base is between zero and one. We'll use the function $g\left(x\right)=\left(\frac{1}{2}\right)^x$. Observe how the output values in (Figure) change as the input increases by 1.

x	-3	-2	-1	0	1	2	3
$g(x)=\left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

(Figure) shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

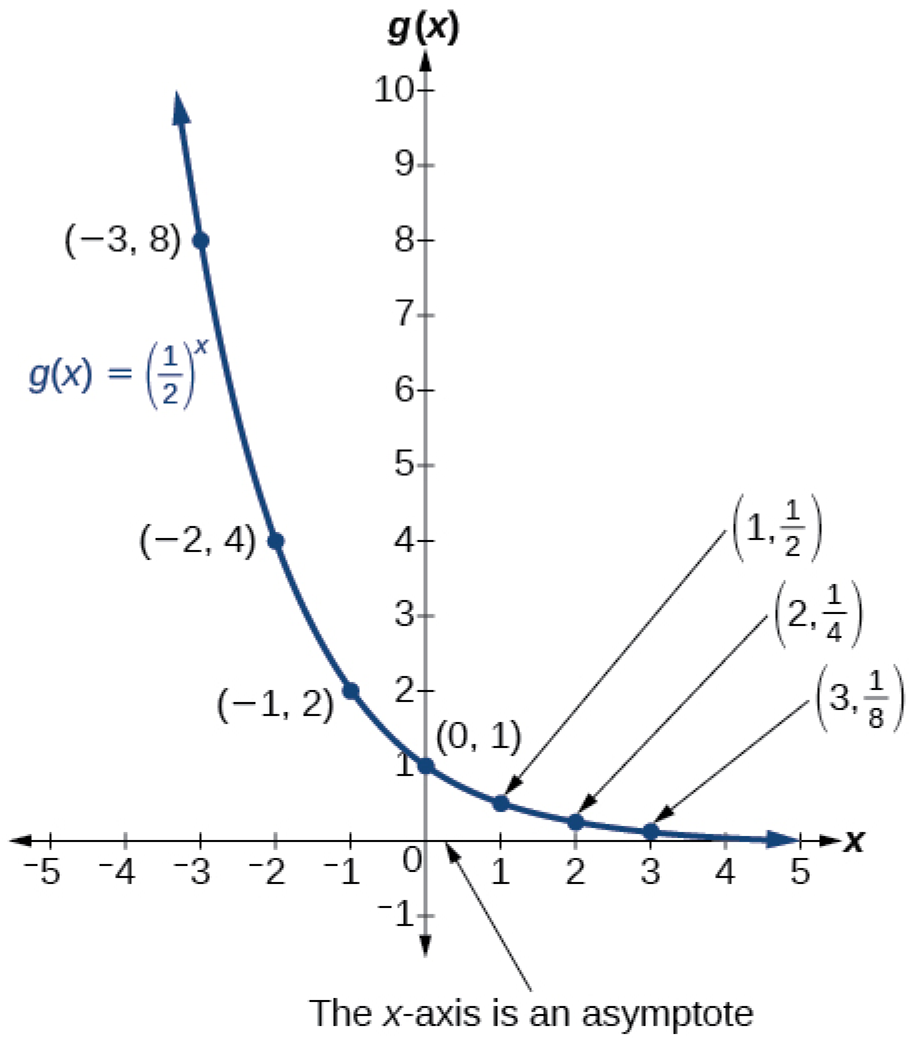


Figure 2.

The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

Characteristics of the Graph of the Parent Function *f*(*x*) = *b*^*x*

An exponential function with the form $f\left(x\right)={b}^{x}, b>0, b\ne 1,$ has these characteristics:

- one-to-one function
- horizontal asymptote: $y=0$
- domain: $\left(-\infty , \infty \right)$
- range: $\left(0,\infty \right)$
- *x*-intercept: none
- *y*-intercept: $\left(0,1\right)$
- increasing if $b>1$
- decreasing if $b<1$

(Figure) compares the graphs of exponential growth and decay functions.

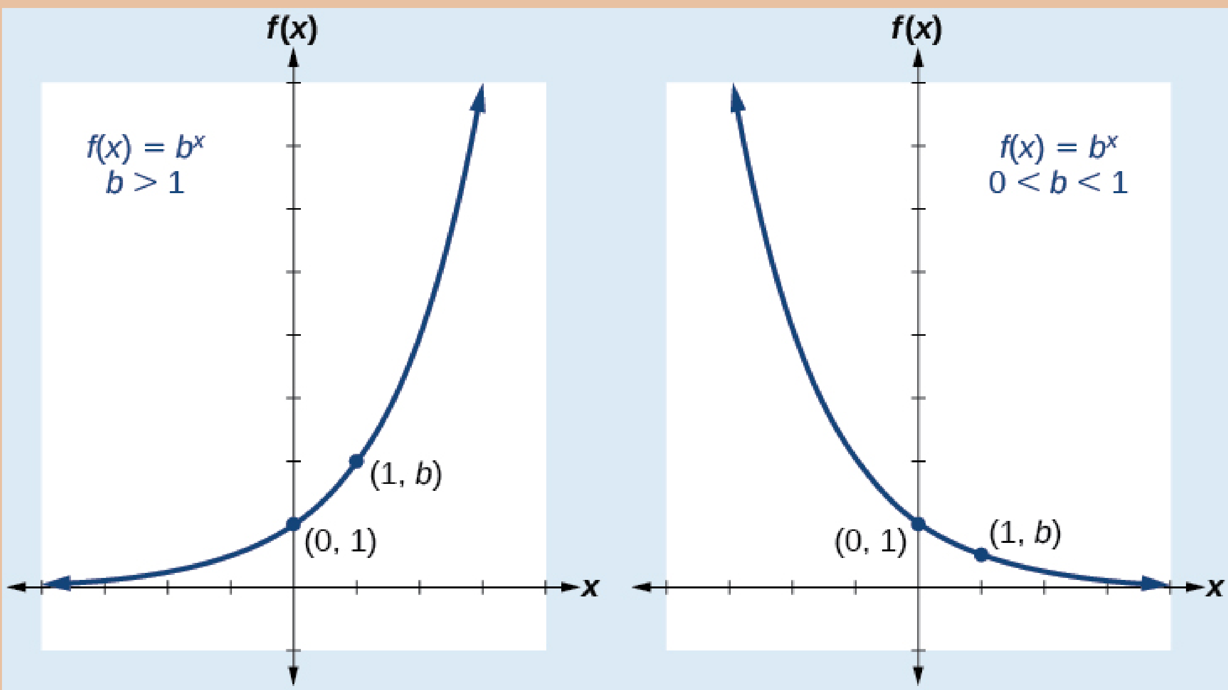


Figure 3.

How To

Given an exponential function of the form $f(x) = b^x$, graph the function.

1. Create a table of points.
2. Plot at least 3 point from the table, including the *y*-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $(-\infty, \infty)$, the range, $(0, \infty)$, and the horizontal asymptote, $y = 0$.

Sketching the Graph of an Exponential Function of the Form *f*(*x*) = *b*^{x}

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Show Solution

Before graphing, identify the behavior and create a table of points for the graph.

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in (Figure).

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	0.0625	0.015625

- Plot the *y*-intercept, $(0, 1)$, along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in (Figure).

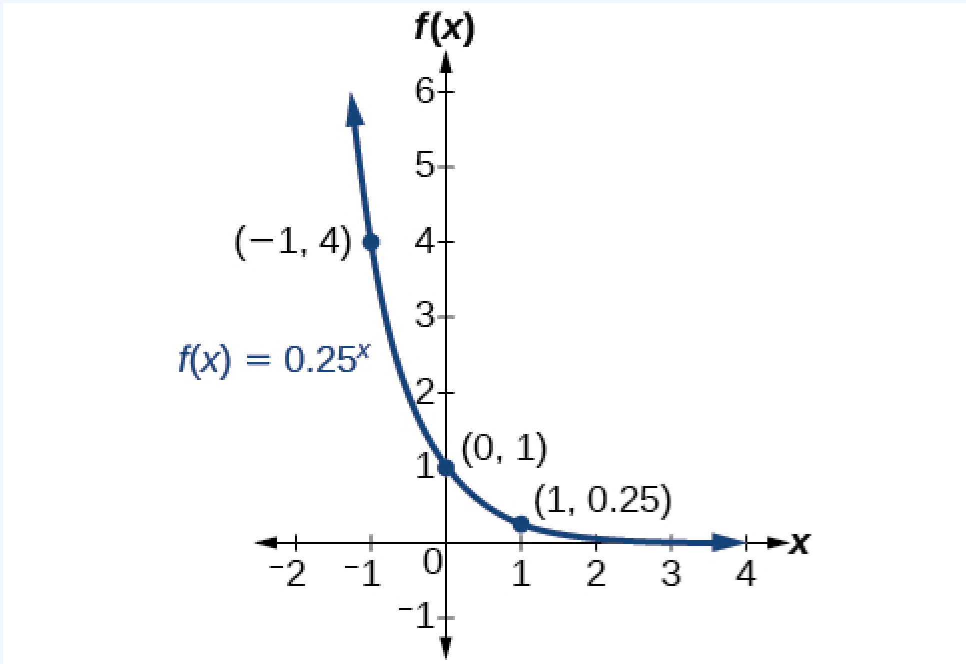


Figure 4.

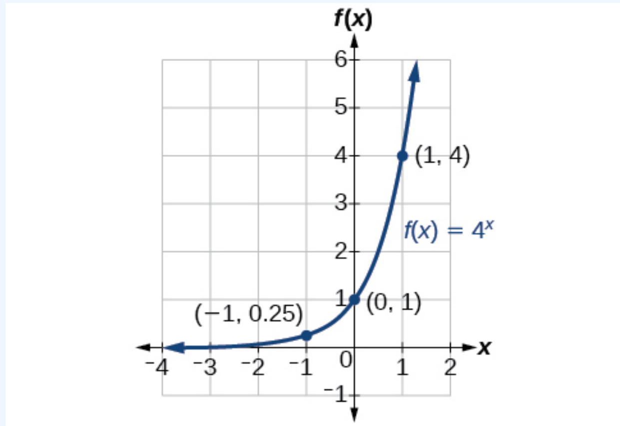
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Try It

Sketch the graph of $f\,(x) = 4^x$. State the domain, range, and asymptote.

Show Solution

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f\left(x\right)=b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the parent function $f\left(x\right)=b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f\left(x\right)=2^x$, we can then graph two vertical shifts alongside it, using $d=3:$ the upward shift, $g\left(x\right)=2^x+3$ and the downward shift, $h\left(x\right)=2^x-3$. Both vertical shifts are shown in (Figure).

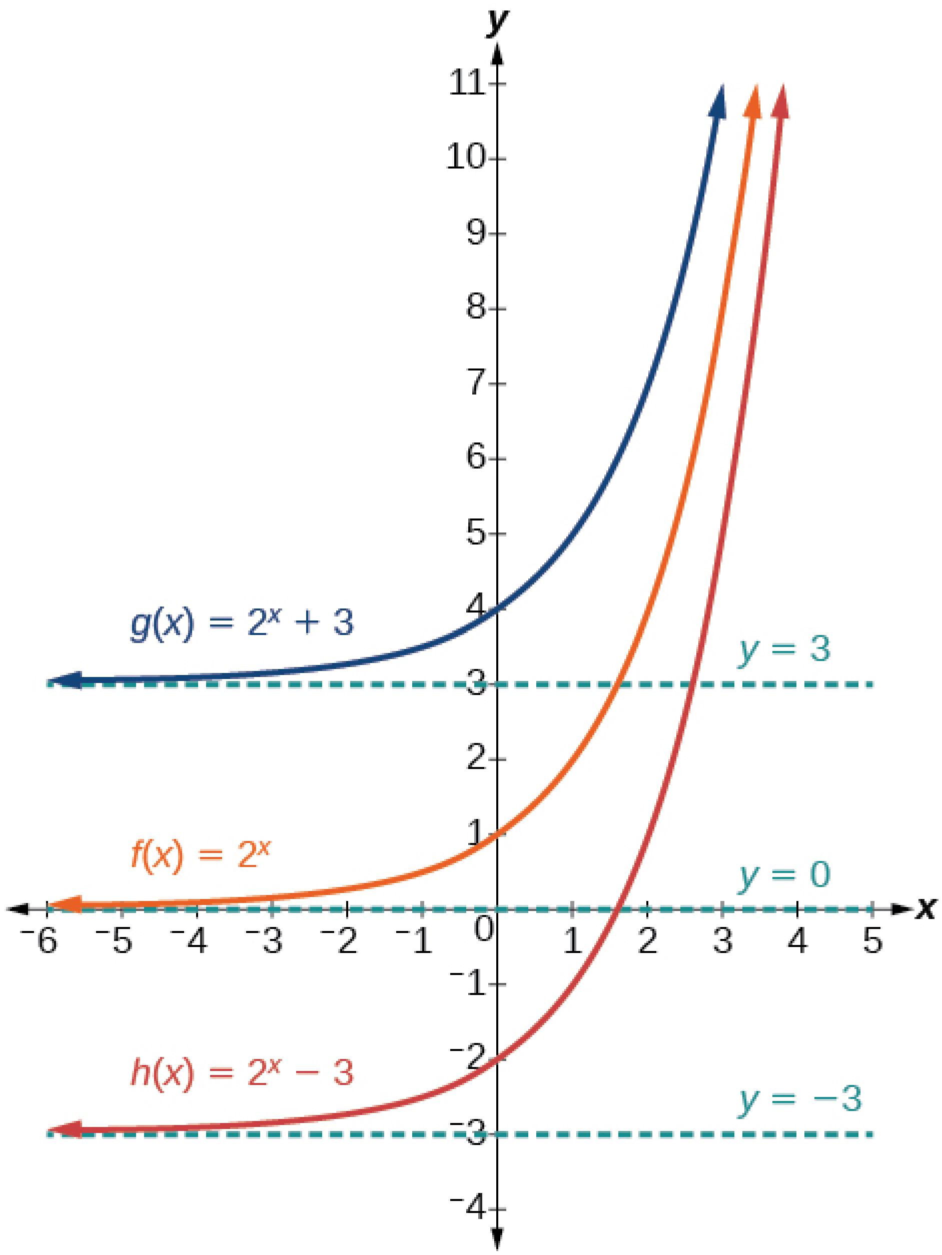


Figure 5.

Observe the results of shifting $f\left(x\right)=2^x$ vertically:

- The domain, $\left(-\infty,\infty\right)$ remains unchanged.
- When the function is shifted up 3 units to $g\left(x\right)=2^x+3$:
  - The *y*-intercept shifts up 3 units to $\left(0,4\right)$.
  - The asymptote shifts up 3 units to $y=3$.
  - The range becomes $\left(3,\infty\right)$.
- When the function is shifted down 3 units to $h\left(x\right)=2^x-3$:

- The y-intercept shifts down 3 units to $(0, -2)$.
- The asymptote also shifts down 3 units to $y = -3$.
- The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the parent function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. Both horizontal shifts are shown in (Figure).

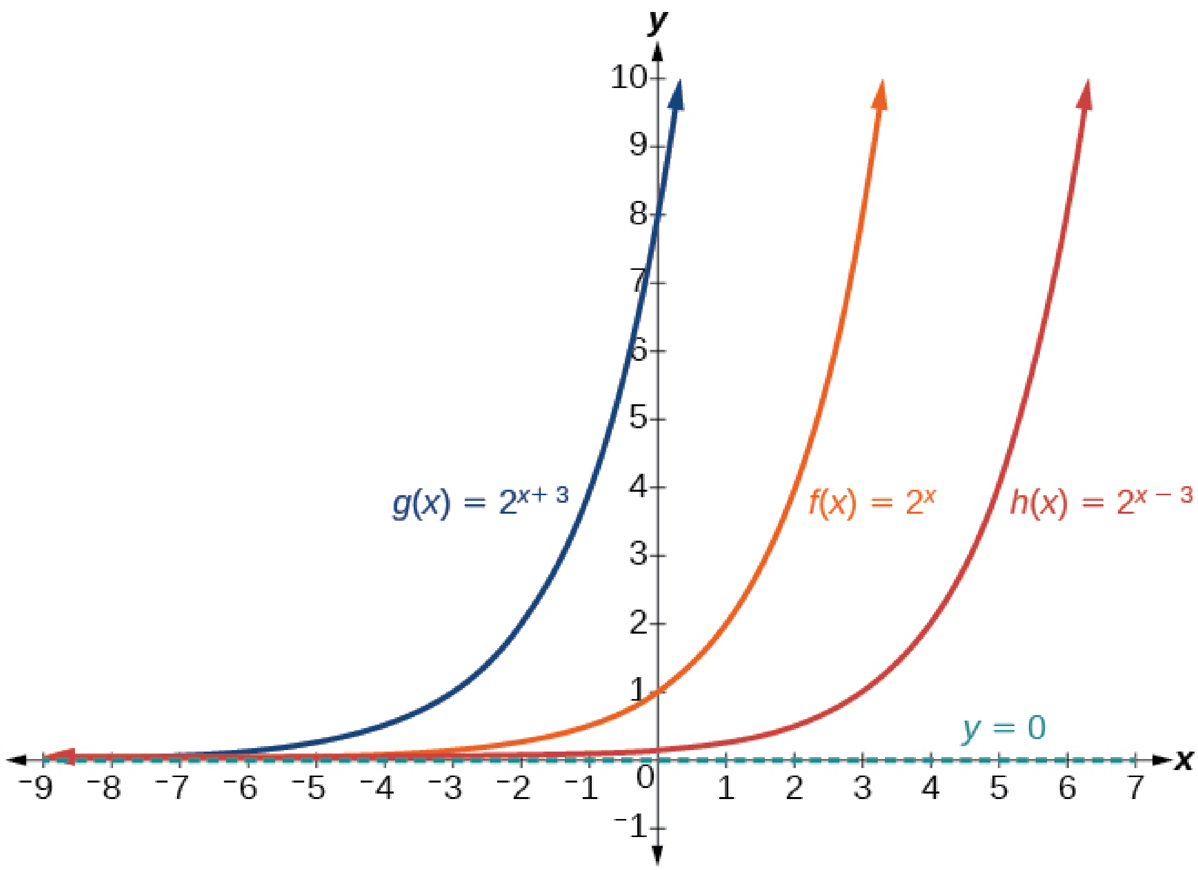


Figure 6.

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$, remains unchanged.
- The asymptote, $y = 0$, remains unchanged.
- The y-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the y-intercept becomes $(0, 8)$. This

is because $2^{x+3} = (8)\,2^x$, so the initial value of the function is 8.

- When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the *y*-intercept becomes $\left(0, \frac{1}{8}\right)$. Again, see that $2^{x-3} = \left(\frac{1}{8}\right)2^x$, so the initial value of the function is $\frac{1}{8}$.

Shifts of the Parent Function $f(x) = b^x$

For any constants c and d, the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically d units, in the *same* direction of the sign of d.
- horizontally c units, in the *opposite* direction of the sign of c.
- The *y*-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes (d, ∞).
- The domain, $(-\infty, \infty)$, remains unchanged.

How To

Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation.

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $(-\infty, \infty)$, the range, (d, ∞), and the horizontal asymptote $y = d$.

Graphing a Shift of an Exponential Function

Graph $f\left(x\right)=2^{x+1}-3.$ State the domain, range, and asymptote.

Show Solution

We have an exponential equation of the form $f\left(x\right)=b^{x+c}+d,$ with $b=2,$ $c=1,$ and $d=-3.$

Draw the horizontal asymptote $y=d$, so draw $y=-3.$

Identify the shift as $\left(-c,d\right)$, so the shift is $\left(-1,-3\right)$.

Shift the graph of $f\left(x\right)=b^{x}$ left 1 units and down 3 units.

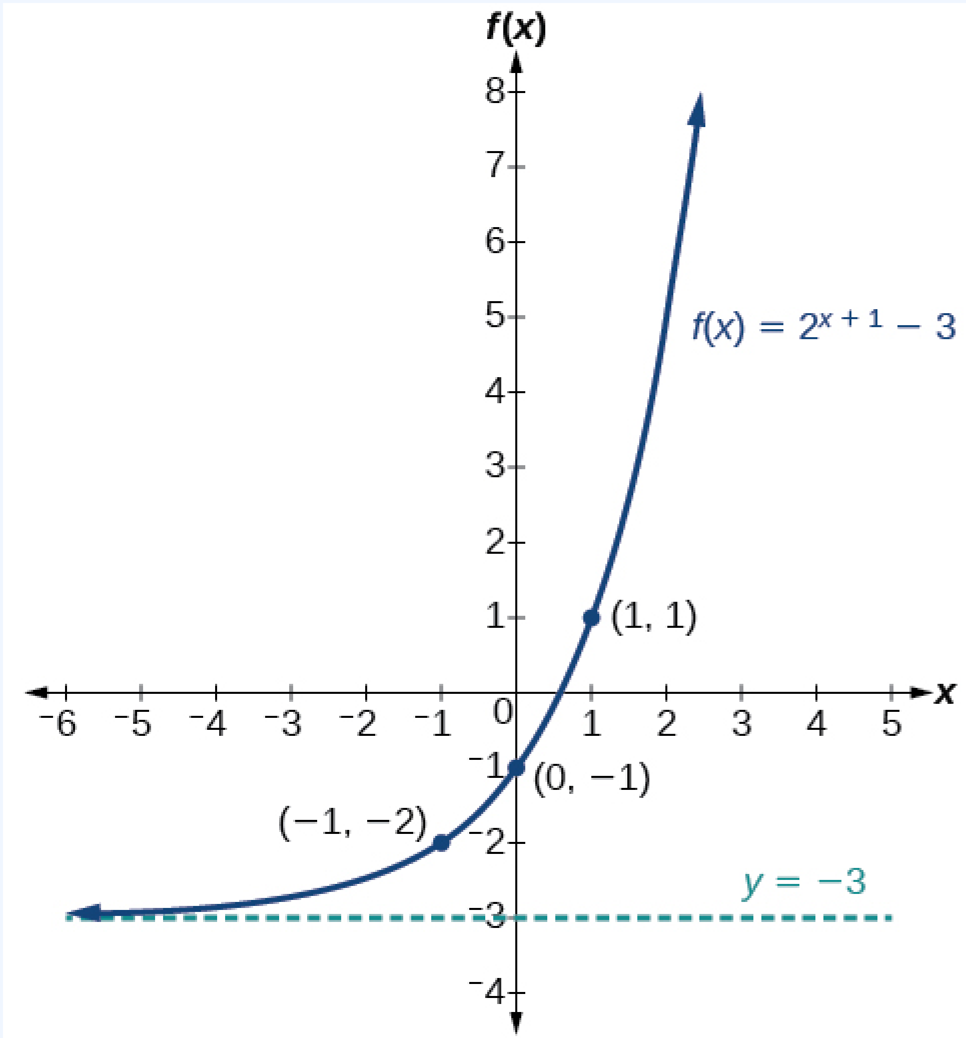


Figure 7.

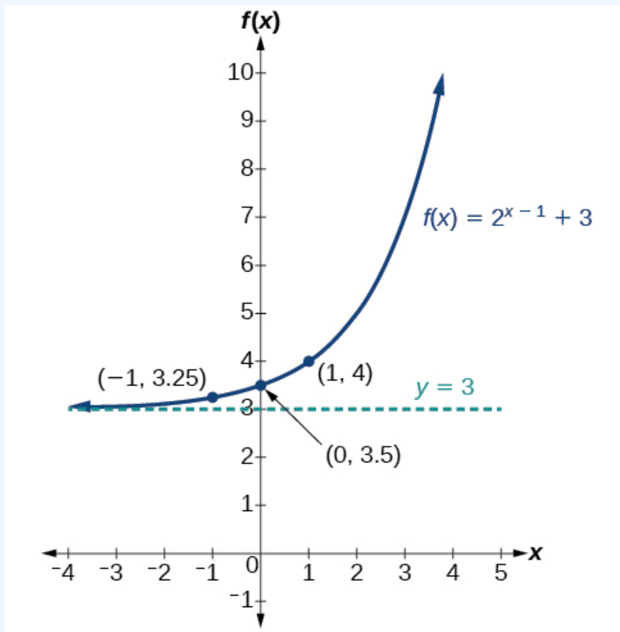
The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

Try It

Graph $f\left(x\right)=2^{x-1}+3.$ State domain, range, and asymptote.

Show Solution

The domain is $\left(-\infty,\infty\right)$; the range is $\left(3,\infty\right)$; the horizontal asymptote is $y=3.$



How To

Given an equation of the form $f\left(x\right)=b^{x+c}+d$ for $x,$ use a graphing calculator to approximate the solution.

- Press **[Y=]**. Enter the given exponential equation in the line headed "**Y_1=**".

- Enter the given value for $f\left(x\right)$ in the line headed "**Y_2=**".
- Press **[WINDOW]**. Adjust the *y*-axis so that it includes the value entered for "**Y_2=**".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value of $f\left(x\right)$.
- To find the value of $x,$ we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of *x* for the indicated value of the function.

Approximating the Solution of an Exponential Equation

Solve $42=1.2{\left(5\right)}^{x}+2.8$ graphically. Round to the nearest thousandth.

Show Solution

Press **[Y=]** and enter $1.2{\left(5\right)}^{x}+2.8$ next to **Y_1**=. Then enter 42 next to **Y2=**. For a window, use the values –3 to 3 for x and –5 to 55 for $y.$ Press **[GRAPH]**. The graphs should intersect somewhere near $x=2.$

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 2.1661943. (Your answer may be different if you use a different window or use a different value for **Guess?**) To the nearest thousandth, $x\approx 2.166.$

Try It

Solve $4=7.85{\left(1.15\right)}^{x}-2.27$ graphically. Round to the nearest thousandth.

Show Solution

$x \approx -1.608$

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the parent function $f\left(x\right)={b}^{x}$ by a constant $|a|>0$. For example, if we begin by graphing the parent function $f\left(x\right)={2}^{x}$, we can then graph the stretch, using $a=3$, to get $g\left(x\right)=3{\left(2\right)}^{x}$ as shown on the left in (Figure), and the compression, using $a=\frac{1}{3}$, to get $h\left(x\right)=\frac{1}{3}{\left(2\right)}^{x}$ as shown on the right in (Figure).

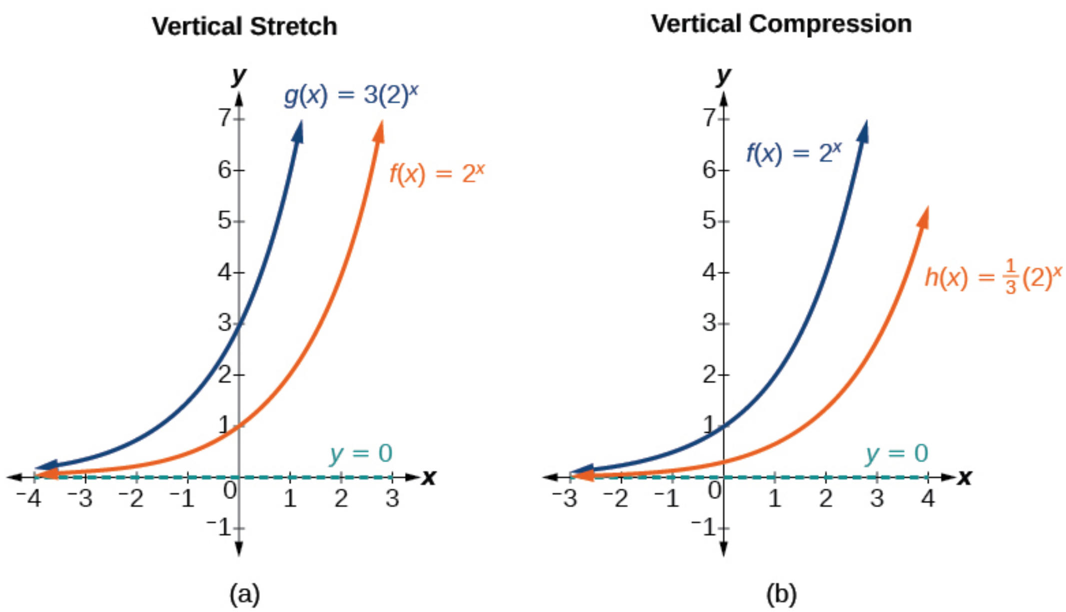


Figure 8. (a) $g\left(x\right)=3(2)^{x}$ stretches the graph of $f\left(x\right)=2^{x}$ vertically by a factor of 3. (b) $h\left(x\right)=\frac{1}{3}(2)^{x}$ compresses the graph of $f\left(x\right)=2^{x}$ vertically by a factor of $\frac{1}{3}$.

Stretches and Compressions of the Parent Function *f(x)* = *b*^{x}

For any factor $a>0$, the function $f\left(x\right)=a(b)^{x}$

- is stretched vertically by a factor of a if $|a|>1$.
- is compressed vertically by a factor of a if $|a|<1$.
- has a *y*-intercept of $\left(0,a\right)$.
- has a horizontal asymptote at $y=0,$ a range of $\left(0,\infty\right)$, and a domain of $\left(-\infty,\infty\right)$, which are unchanged from the parent function.

Graphing the Stretch of an Exponential Function

Sketch a graph of $f\left(x\right)=4\left(\frac{1}{2}\right)^{x}$. State the domain, range, and asymptote.

Show Solution

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the *x*-axis as x increases.
- Since $a = 4,$ the graph of $f\left(x\right) = \left(\frac{1}{2}\right)^{x}$ will be stretched by a factor of $4.$
- Create a table of points as shown in (Figure).

x	-3	-2	-1	0	1	2	3
$f\left(x\right) = 4\left(\frac{1}{2}\right)^{x}$	32	16	8	4	2	1	0.5

- Plot the *y*-intercept, $\left(0, 4\right)$, along with two other points. We can use $\left(-1, 8\right)$ and $\left(1, 2\right)$.

Draw a smooth curve connecting the points, as shown in (Figure).

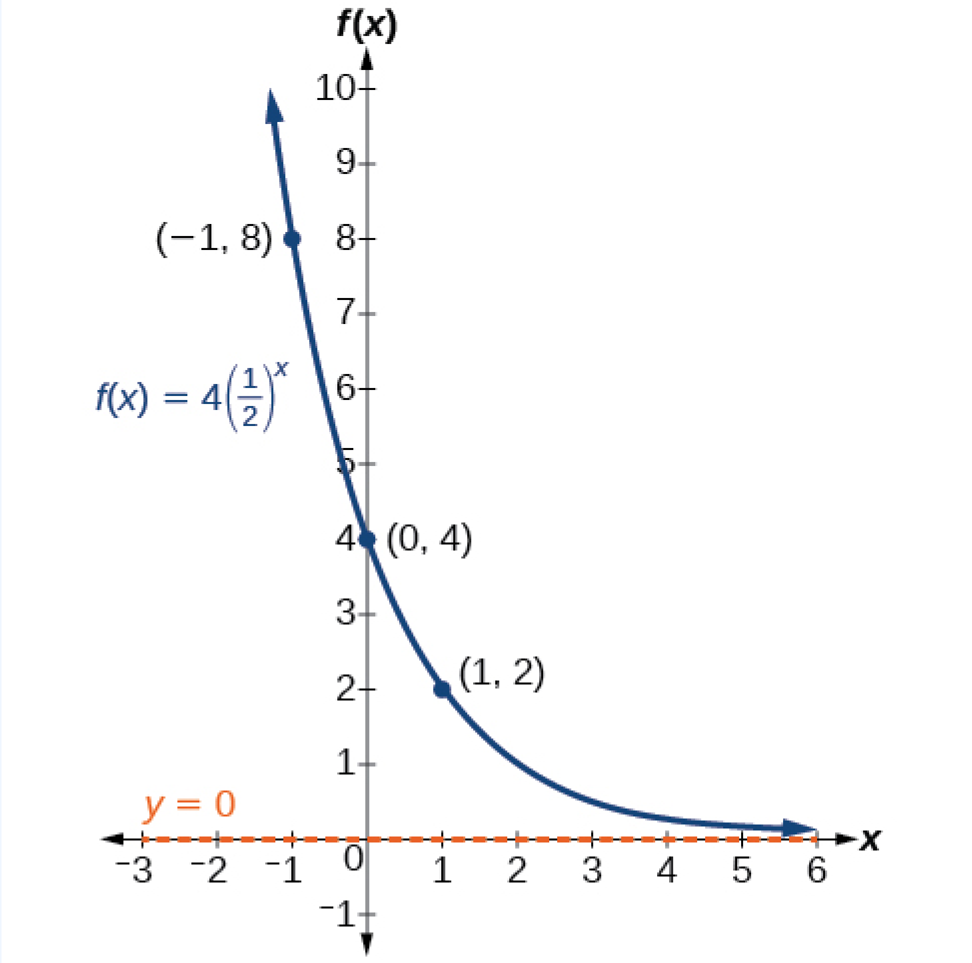


Figure 9.

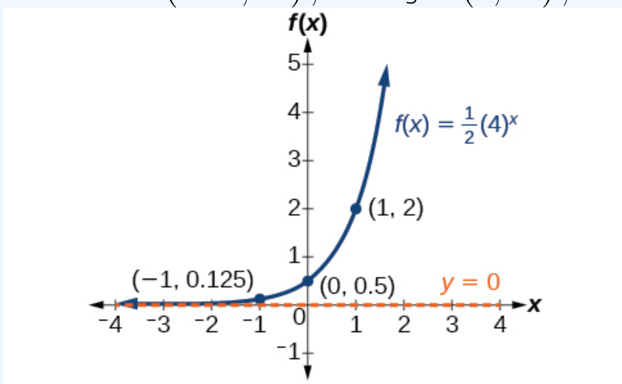
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Try It

Sketch the graph of $f\left(x\right) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Show Solution

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the *y*-axis. When we multiply the parent function $f(x) = b^x$ by -1, we get a reflection about the *x*-axis. When we multiply the input by -1, we get a reflection about the *y*-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the *x*-axis, $g(x) = -2^x$, is shown on the left side of (Figure), and the reflection about the *y*-axis $h(x) = 2^{-x}$, is shown on the right side of (Figure).

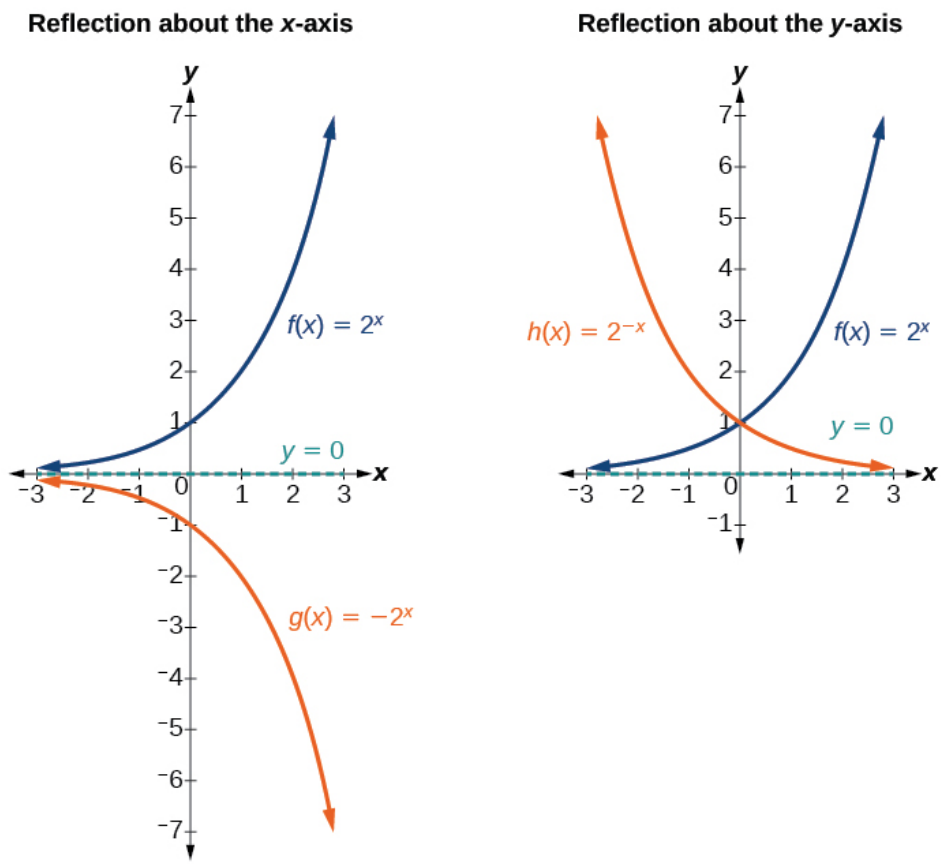


Figure 10. (a) $g\left(x\right)=-{2}^{x}$ reflects the graph of $f\left(x\right)={2}^{x}$ about the x-axis. (b) $g\left(x\right)={2}^{-x}$ reflects the graph of $f\left(x\right)={2}^{x}$ about the y-axis.

Reflections of the Parent Function *f*(*x*) = *b*^*x*

The function $f\left(x\right)=-{b}^{x}$

- reflects the parent function $f\left(x\right)={b}^{x}$ about the *x*-axis.
- has a *y*-intercept of $\left(0,-1\right)$.
- has a range of $\left(-\infty ,0\right)$
- has a horizontal asymptote at $y=0$ and domain of $\left(-\infty ,\infty \right)$, which are unchanged from the parent function.

The function $f\left(x\right)={b}^{-x}$

- reflects the parent function $f\left(x\right)={b}^{x}$ about the *y*-axis.
- has a *y*-intercept of $\left(0,1\right)$, a horizontal asymptote at $y=0$, a range of $\left(0,\infty \right)$, and a

domain of $(-\infty, \infty)$, which are unchanged from the parent function.

Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = \left(\frac{1}{4}\right)^x$ about the *x*-axis. State its domain, range, and asymptote.

Show Solution

Since we want to reflect the parent function $f(x) = \left(\frac{1}{4}\right)^x$ about the *x*-axis, we multiply $f(x)$ by -1 to get, $g(x) = -\left(\frac{1}{4}\right)^x$. Next we create a table of points as in (Figure).

x	-3	-2	-1	0	1	2	3
$g(x) = -\left(\frac{1}{4}\right)^x$	-64	-16	-4	-1	-0.25	-0.0625	-0.0156

Plot the *y*-intercept, $(0, -1)$, along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

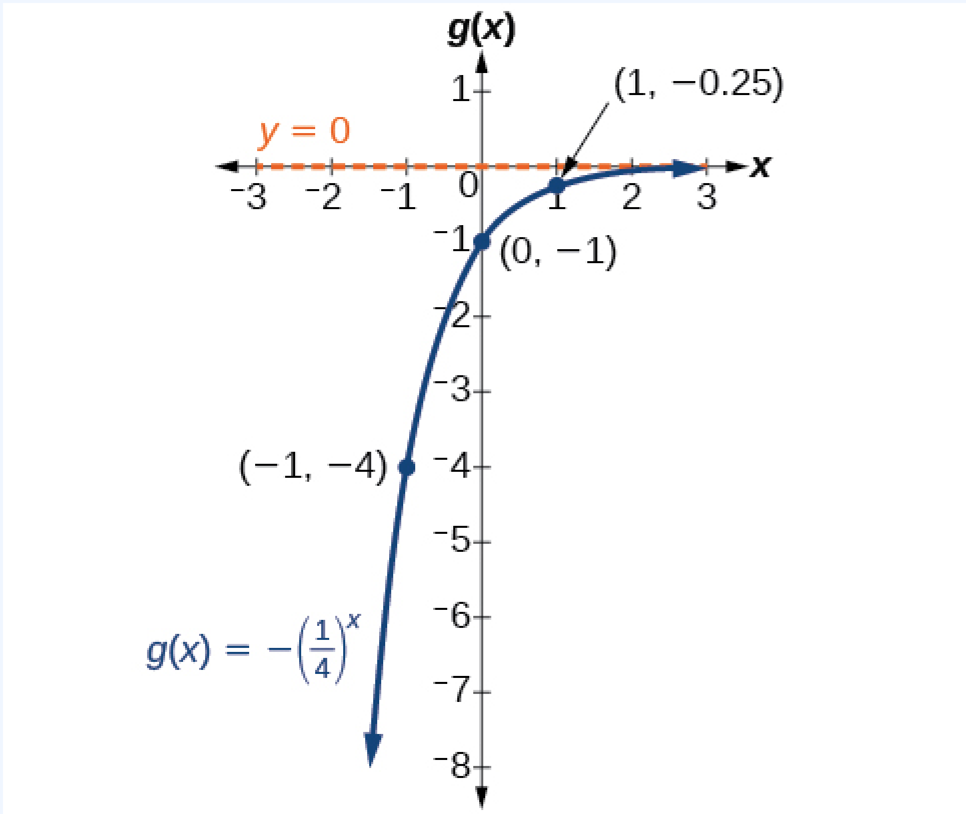


Figure 11.

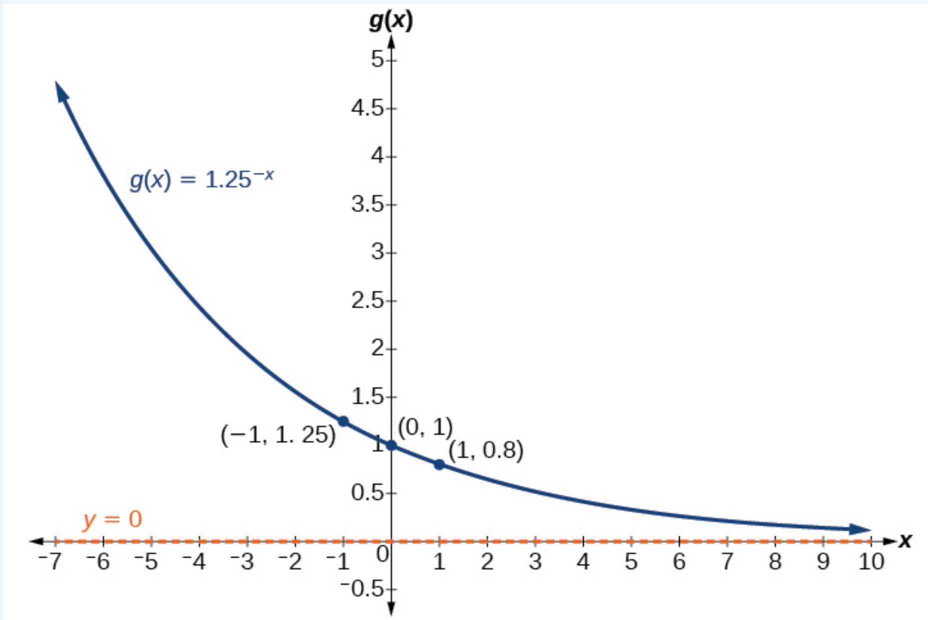
The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

Try It

Find and graph the equation for a function, $g\left(x\right)$, that reflects $f\left(x\right) = 1.25^x$ about the *y*-axis. State its domain, range, and asymptote.

Show Solution

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in (Figure) to arrive at the general equation for translating exponential functions.

1, and notes the following changes: the reflected function is decreasing as x moves from 0 to infinity, the asymptote remains x=0, the x-intercept remains (1, 0), the key point changes to (b^(-1), 1), the domain remains (0, infinity), and the range remains (-infinity, infinity). The second column shows the left shift of the equation g(x)=log_b(x) when b>1, and notes the following changes: the reflected function is decreasing as x moves from 0 to infinity, the asymptote remains x=0, the x-intercept changes to (-1, 0), the key point changes to (-b, 1), the domain changes to (-infinity, 0), and the range remains (-infinity, infinity).”>

Translations of the Parent Function $f(x) = b^x$	
Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$f(x) = b^{x+c} + d$
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $0 < \lvert a \rvert < 1$	$f(x) = ab^x$
Reflect about the *x*-axis	$f(x) = -b^x$
Reflect about the *y*-axis	$f(x) = b^{-x} = \left(\frac{1}{b}\right)^x$
General equation for all translations	$f(x) = ab^{x+c} + d$

Translations of Exponential Functions

A translation of an exponential function has the form

$f(x) = ab^{x+c} + d$

Where the parent function, $y = b^x$, $b > 1$, is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- shifted vertically d units.
- reflected about the *x*-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is vertically stretched by a factor of 2 , reflected across the *y*-axis, and then shifted up 4 units.

Show Solution

We want to find an equation of the general form $f(x) = ab^{x+c} + d.$ We use the description provided to find $a, b, c,$ and $d.$

- We are given the parent function $f(x) = e^x$, so $b = e.$
- The function is stretched by a factor of 2 , so $a = 2.$
- The function is reflected about the *y*-axis. We replace x with $-x$ to get: $e^{-x}.$
- The graph is shifted vertically 4 units, so $d = 4.$

Substituting in the general form we get,

$$\begin{aligned} f(x) &= ab^{x+c} + d \\ &= 2e^{-x+0} + 4 \\ &= 2e^{-x} + 4 \end{aligned}$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4.$

Try It

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down 2 units.

Show Solution

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

Access this online resource for additional instruction and practice with graphing exponential functions.

- Graph Exponential Functions

Key Equations

General Form for the Translation of the Parent Function $f(x) = b^x$	$f(x) = ab^{x+c} + d$

Key Concepts

- The graph of the function $f(x) = b^x$ has a *y*-intercept at $(0, 1)$, domain $(-\infty, \infty)$, range $(0, \infty)$, and horizontal asymptote $y = 0$. See (Figure).
- If $b > 1$, the function is increasing. The left tail of the graph will approach the asymptote $y = 0$, and the right tail will increase without bound.
- If $0 < b < 1$, the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- The equation $f(x) = b^x + d$ represents a vertical shift of the parent function $f(x) = b^x$.
- The equation $f(x) = b^{x+c}$ represents a horizontal shift of the parent function $f(x) = b^x$. See (Figure).
- Approximate solutions of the equation $f(x) = b^{x+c} + d$ can be found using a graphing calculator. See (Figure).
- The equation $f(x) = ab^x$, where $a > 0$, represents a vertical stretch if $|a| > 1$ or compression if $0 < |a| < 1$ of the parent function $f(x) = b^x$. See (Figure).

- When the parent function $f\left(x\right)=b^{x}$ is multiplied by $-1,$ the result, $f\left(x\right)=-b^{x}$, is a reflection about the *x*-axis. When the input is multiplied by $-1,$ the result, $f\left(x\right)=b^{-x}$, is a reflection about the *y*-axis. See (Figure).
- All translations of the exponential function can be summarized by the general equation $f\left(x\right)=ab^{x+c}+d.$ See (Figure).
- Using the general equation $f\left(x\right)=ab^{x+c}+d,$ we can write the equation of a function given its description. See (Figure).

Section Exercises

Verbal

1. What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

Show Solution

An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

2. What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Algebraic

3. The graph of $f\left(x\right)={3}^{x}$ is reflected about the *y*-axis and stretched vertically by a factor of $4.$ What is the equation of the new function, $g\left(x\right)?$ State its *y*-intercept, domain, and range.

Show Solution

$g\left(x\right)=4{\left(3\right)}^{-x}$; *y*-intercept: $\left(0,4\right)$; Domain: all real numbers; Range: all real numbers greater than 0.

4. The graph of $f\left(x\right)={\left(\frac{1}{2}\right)}^{-x}$ is reflected about the *y*-axis and compressed vertically by a factor of $\frac{1}{5}$. What is the equation of the new function, $g\left(x\right)$? State its *y*-intercept, domain, and range.

5. The graph of $f\left(x\right)={10}^{x}$ is reflected about the *x*-axis and shifted upward 7 units. What is the equation of the new function, $g\left(x\right)$? State its *y*-intercept, domain, and range.

Show Solution

$g\left(x\right)=-{10}^{x}+7$; *y*-intercept: $\left(0,6\right)$; Domain: all real numbers; Range: all real numbers less than 7.

6. The graph of $f\left(x\right)={\left(1.68\right)}^{x}$ is shifted right 3 units, stretched vertically by a factor of 2, reflected about the *x*-axis, and then shifted downward 3 units. What is the equation of the new function, $g\left(x\right)$? State its *y*-intercept (to the nearest thousandth), domain, and range.

7. The graph of $f\left(x\right)=2{\left(\frac{1}{4}\right)}^{x-20}$ is shifted left 2 units, stretched vertically by a factor of 4, reflected about the *x*-axis, and then shifted downward 4 units. What is the equation of the new function, $g\left(x\right)$? State its *y*-intercept, domain, and range.

Show Solution

$g\left(x\right)=2{\left(\frac{1}{4}\right)}^{x}$; *y*-intercept: $\left(0,\ 2\right)$; Domain: all real numbers; Range: all real numbers greater than 0.

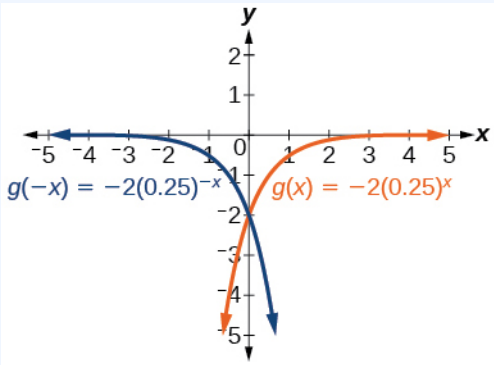
Graphical

For the following exercises, graph the function and its reflection about the *y*-axis on the same axes, and give the *y*-intercept.

8. $f\left(x\right)=3{\left(\frac{1}{2}\right)}^{x}$

9. $g\left(x\right)=-2{\left(0.25\right)}^{x}$

Show Solution



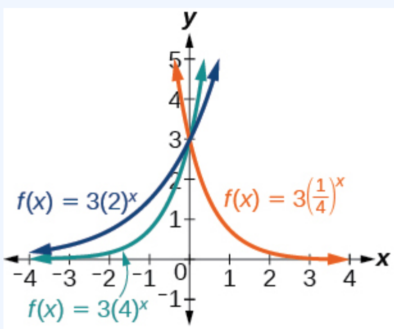
y-intercept: $\left(0,-2\right)$

10. $h\left(x\right)=6{\left(1.75\right)}^{-x}$

For the following exercises, graph each set of functions on the same axes.

11. $f\left(x\right)=3{\left(\frac{1}{4}\right)}^{x}$, $g\left(x\right)=3{\left(2\right)}^{x}$, and $h\left(x\right)=3{\left(4\right)}^{x}$

Show Solution



12. $f\left(x\right)=\frac{1}{4}{\left(3\right)}^{x}$, $g\left(x\right)=2{\left(3\right)}^{x}$, and $h\left(x\right)=4{\left(3\right)}^{x}$

For the following exercises, match each function with one of the graphs in (Figure).

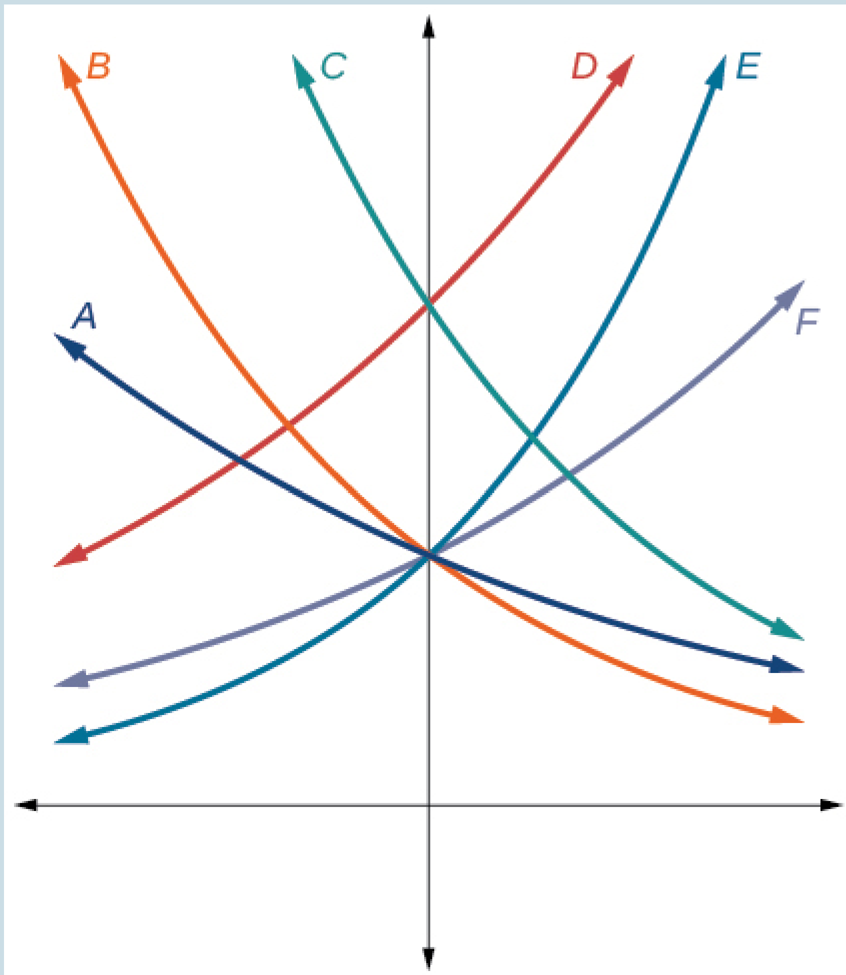


Figure 12.

13. $f\left(x\right)=2{\left(0.69\right)}^{x}$

Show Solution

B

14. $f\left(x\right)=2{\left(1.28\right)}^{x}$

15. $f\left(x\right)=2{\left(0.81\right)}^{x}$

Show Solution

A

16. $f\left(x\right)=4{\left(1.28\right)}^{x}$

17. $f\left(x\right)=2{\left(1.59\right)}^{x}$

Show Solution

E

18. $f\left(x\right)=4{\left(0.69\right)}^{x}$

For the following exercises, use the graphs shown in (Figure). All have the form $f\left(x\right)=a{b}^{x}$.

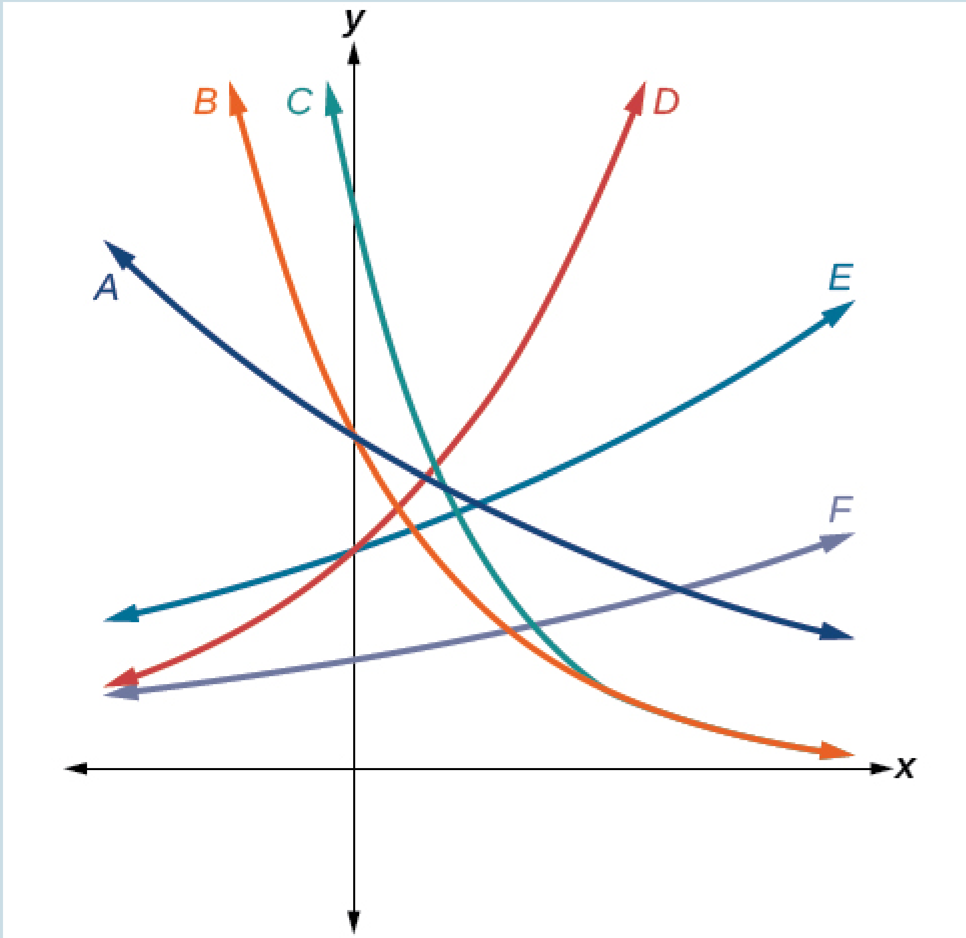


Figure 13.

19. Which graph has the largest value for b?

Show Solution
D

20. Which graph has the smallest value for b?

21. Which graph has the largest value for a?

Show Solution

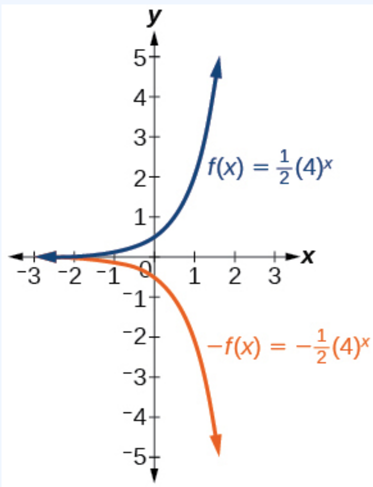
C

22. Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the *x*-axis on the same axes.

23. $f\left(x\right)=\frac{1}{2}(4)^{x}$

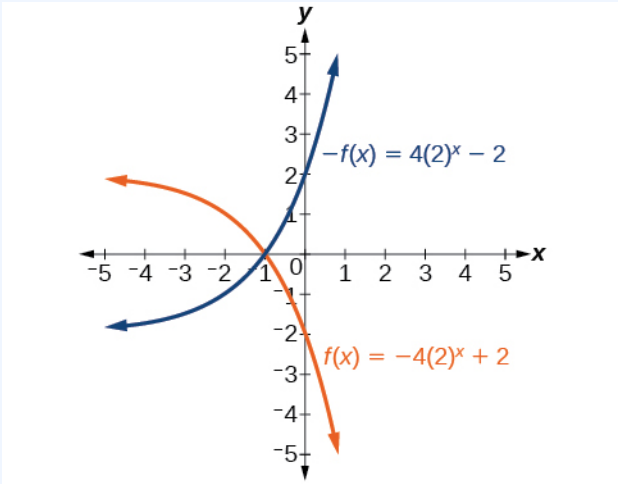
Show Solution



24. $f\left(x\right)=3(0.75)^{x}-1$

25. $f\left(x\right)=-4(2)^{x}+2$

Show Solution

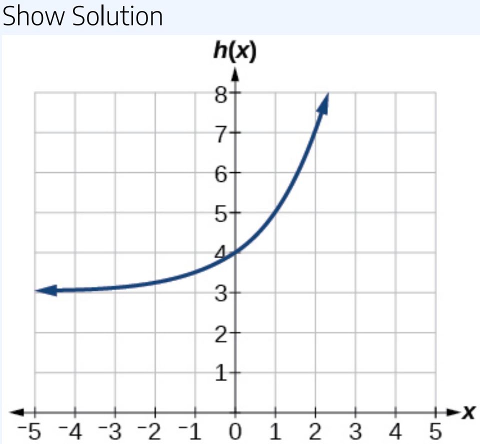


For the following exercises, graph the transformation of $f\left(x\right)=2^x$. Give the horizontal asymptote, the domain, and the range.

26. $f\left(x\right)=2^{-x}$

27. $h\left(x\right)=2^x+3$

Show Solution



Horizontal asymptote: $h\left(x\right)=3;$ Domain: all real numbers; Range: all real numbers strictly greater than $3.$

28. $f\left(x\right)=2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.

29. $f\left(x\right)=-5(4)^{x}-1$

Show Solution

As $x\to\infty$,
$f\left(x\right)\to-\infty$;

30. $f\left(x\right)=3\left(\frac{1}{2}\right)^{x}-2$

31. $f\left(x\right)=3(4)^{-x}+2$

Show Solution

As $x\to\infty$,
$f\left(x\right)\to2$;

For the following exercises, start with the graph of $f\left(x\right)=4^{x}$. Then write a function that results from the given transformation.

32. Shift $f\left(x\right)$ 4 units upward

33. Shift $f\left(x\right)$ 3 units downward

Show Solution

$f\left(x\right)=4^{x}-3$

36. Shift $f\left(x\right)$ 2 units left

37. Shift $f(x)$ 5 units right

Show Solution

$f(x) = 4^{x-5}$

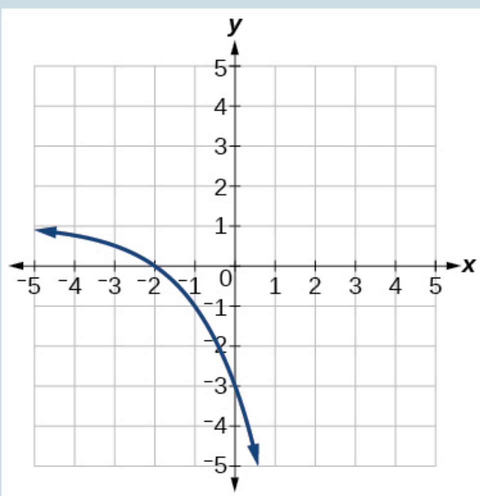
38. Reflect $f(x)$ about the *x*-axis

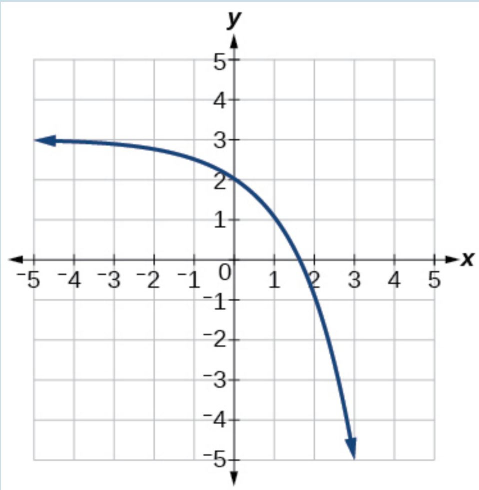
39. Reflect $f(x)$ about the *y*-axis

Show Solution

$f(x) = 4^{-x}$

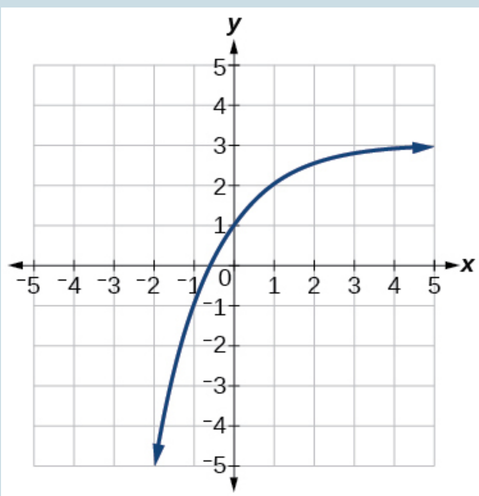
For the following exercises, each graph is a transformation of $y = 2^x$. Write an equation describing the transformation.

40. 

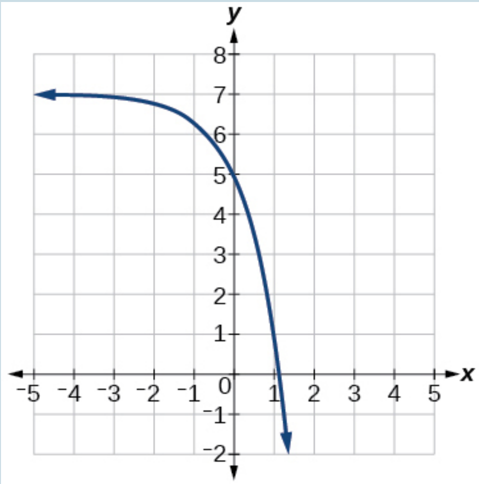
41. 

Show Solution

$y = -2^x + 3$

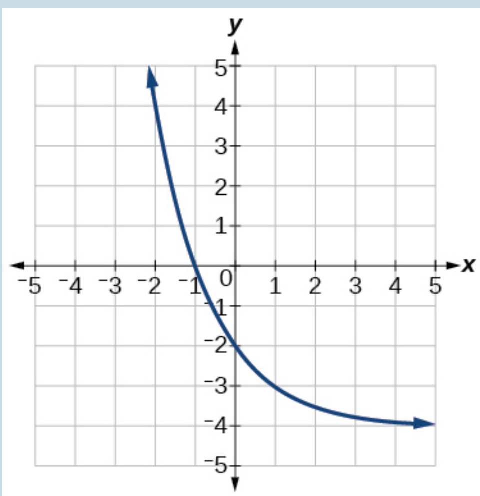
42. 

For the following exercises, find an exponential equation for the graph.

43. 

Show Solution

$y = -2(3)^x + 7$

44. 

Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

45. $g\left(x\right) = \frac{1}{3}(7)^{x-2}$ for $g\left(6\right)$.

Show Solution

$g\left(6\right)=800+\frac{1}{3}\approx 800.3333$

46. $f\left(x\right)=4{\left(2\right)}^{x-1}-2$ for $f\left(5\right).$

47. $h\left(x\right)=-\frac{1}{2}{\left(\frac{1}{2}\right)}^{x}+6$ for $h\left(-7\right).$

Show Solution

$h\left(-7\right)=-58$

Technology

For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.

48. $-50=-{\left(\frac{1}{2}\right)}^{-x}$

49. $116=\frac{1}{4}{\left(\frac{1}{8}\right)}^{x}$

Show Solution

$x\approx -2.953$

50. $12=2{\left(3\right)}^{x}+1$

51. $5=3{\left(\frac{1}{2}\right)}^{x-1}-2$

Show Solution

$x\approx -0.222$

52. $-30 = -4(2)^{x+2} + 2$

Extensions

53. Explore and discuss the graphs of $F\left(x\right) = (b)^{x}$ and $G\left(x\right) = \left(\frac{1}{b}\right)^{x}$. Then make a conjecture about the relationship between the graphs of the functions b^{x} and $\left(\frac{1}{b}\right)^{x}$ for any real number $b > 0.$

Show Solution

The graph of $G\left(x\right) = \left(\frac{1}{b}\right)^{x}$ is the refelction about the *y*-axis of the graph of $F\left(x\right) = b^{x}$; For any real number $b > 0$ and function $f\left(x\right) = b^{x}$, the graph of $\left(\frac{1}{b}\right)^{x}$ is the the reflection about the *y*-axis, $F\left(-x\right)$.

54. Prove the conjecture made in the previous exercise.

55. Explore and discuss the graphs of $f\left(x\right) = 4^{x}$, $g\left(x\right) = 4^{x-2}$, and $h\left(x\right) = \left(\frac{1}{16}\right)4^{x}$. Then make a conjecture about the relationship between the graphs of the functions b^{x} and $\left(\frac{1}{b^{n}}\right)b^{x}$ for any real number *n* and real number $b > 0.$

Show Solution

The graphs of $g\left(x\right)$ and $h\left(x\right)$ are the same and are a horizontal shift to the right of the graph of $f\left(x\right)$; For any real number *n*, real number $b > 0,$ and function $f\left(x\right) = b^{x}$, the graph of $\left(\frac{1}{b^{n}}\right)b^{x}$ is the horizontal shift $f\left(x - n\right)$.

56. Prove the conjecture made in the previous exercise.

CHAPTER 9.4: LOGARITHMIC FUNCTIONS

Learning Objectives

In this section, you will:

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

Figure 1. Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes[1] . One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings,[2] like those shown in (Figure). Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale[3] whereas the Japanese earthquake registered a 9.0.[4]

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10,000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in (Figure), to better estimate the solution.

1. http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/#summary. Accessed 3/4/2013.
2. http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#summary. Accessed 3/4/2013.
3. http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/. Accessed 3/4/2013.
4. http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/. Accessed 3/4/2013.

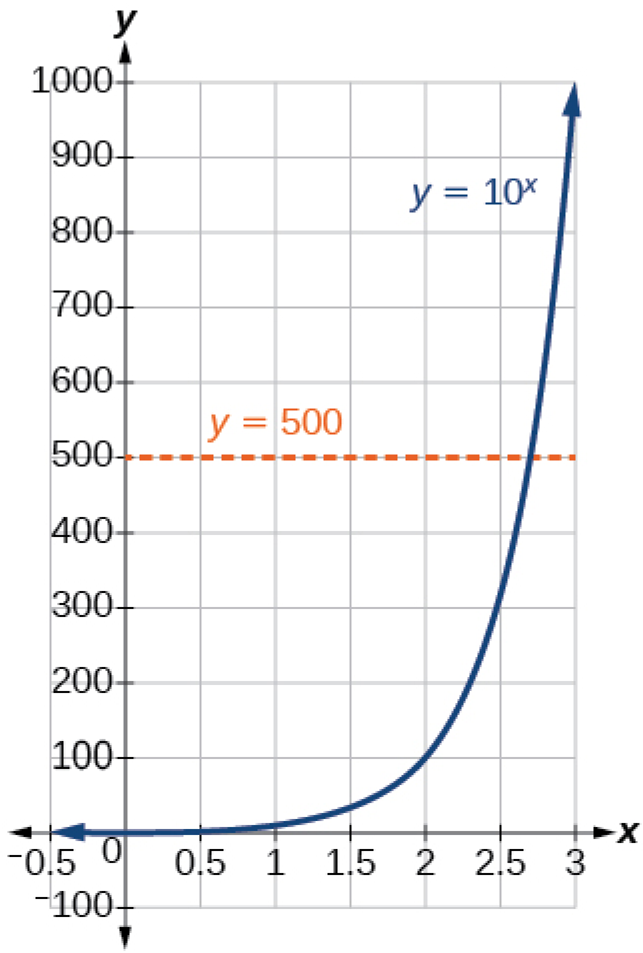


Figure 2.

Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in (Figure) passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, "The logarithm with base b of x is equal to y, " or, simplified, "log base b of x is y. " We can also say, " b raised to the power of y is x, " because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y b^y = x, b > 0, b \neq 1$$

Note that the base b is always positive.

=

$\log_b(x) = y$ Think b to the $y = x$

to

Because logarithm is a function, it is most correctly written as $\log_b\left(x\right)$, using parentheses to denote function evaluation, just as we would with $f\left(x\right)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:

=

$\log_b(c) = a$ means $b^a = c$

to

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b\left(x\right)$ and $y = b^x$ are inverse functions.

Definition of the Logarithmic Function

A logarithm base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1,$

$y = \log_b\left(x\right)$ is equivalent to $b^y = x$

where,

- we read $\log_b\left(x\right)$ as, "the logarithm with base b of x" or the "log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $\left(0, \infty\right)$.
- the range of the logarithm function with base b is $\left(-\infty, \infty\right)$.

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

How To

Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form.

1. Examine the equation $y = \log_b x$ and identify $b, y, \text{and} x.$
2. Rewrite $\log_b x = y$ as $b^y = x.$

Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$
b. $\log_3(9) = 2$

Show Solution
First, identify the values of $b, y, \text{and} x.$ Then, write the equation in the form $b^y = x.$

a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$
Here, $b = 6, y = \frac{1}{2}, \text{and} x = \sqrt{6}.$ Therefore, the equation $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ is equivalent to $6^{\frac{1}{2}} = \sqrt{6}.$

b. $\log_3(9) = 2$
Here, $b = 3, y = 2, \text{and} x = 9.$ Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9.$

Try It

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

Show Solution

a. $\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$
b. $\log_5(25) = 2$ is equivalent to $5^2 = 25$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b, exponent x, and output y. Then we write $x = \log_b(y)$.

Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \frac{1}{10{,}000}$

Show Solution

First, identify the values of $b, y, \text{and} x.$ Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$
   Here, $b = 2,$ $x = 3,$ and $y = 8.$ Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3.$
b. $5^2 = 25$
   Here, $b = 5,$ $x = 2,$ and $y = 25.$ Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2.$
c. $10^{-4} = \frac{1}{10{,}000}$
   Here, $b = 10,$ $x = -4,$ and $y = \frac{1}{10{,}000}.$ Therefore, the equation $10^{-4} = \frac{1}{10{,}000}$ is equivalent to $\log_{10}\left(\frac{1}{10{,}000}\right) = -4.$

Try It

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \frac{1}{2}$

Show Solution

a. $3^2 = 9$ is equivalent to $\log_3(9) = 2$
b. $5^3 = 125$ is equivalent to $\log_5(125) = 3$
c. $2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? " We know $2^2 = 4$ and $3^2 = 9$, so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2$.

How To

Given a logarithm of the form $y = \log_b(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of $b : b^y = x$.
2. Use previous knowledge of powers of b identify y by asking, "To what exponent should b be raised in order to get $x?$ "

Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Show Solution

First we rewrite the logarithm in exponential form: $4^y = 64.$ Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know

$$4^3 = 64$$

Therefore,

$$\log_4(64) = 3$$

Try It

Solve $y = \log_{121}(11)$ without using a calculator.

Show Solution

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Show Solution

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$? "

We know $3^3 = 27$, but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Try It

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Show Solution

$\log_2\left(\frac{1}{32}\right) = -5$

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**.

Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

Definition of the Common Logarithm

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x$$

We read $\log(x)$ as, "the logarithm with base 10 of x" or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

How To

Given a common logarithm of the form $y = \log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of $10 : 10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x?"

Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Show Solution

First we rewrite the logarithm in exponential form: $10^y = 1000.$ Next, we ask, “To what exponent must 10 be raised in order to get 1000?” We know

$10^3 = 1000$

Therefore, $\log(1000) = 3.$

Try It

Evaluate $y = \log(1,000,000)$.

Show Solution

$\log(1,000,000) = 6$

How To

Given a common logarithm with the form $y = \log(x)$, **evaluate it using a calculator.**

1. Press **[LOG]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Show Solution

- Press **[LOG]**.
- Enter 321, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065.$

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000.$ Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

$$\begin{array}{ccccc} 100 & < & 321 & < & 1000 \\ 2 & < & 2.5065 & < & 3 \end{array}$$

Try It

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Show Solution

$\log(123) \approx 2.0899$

Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Show Solution

We begin by rewriting the exponential equation in logarithmic form.

$$\begin{array}{ll} 10^x & = 500 \\ \log(500) & = x \qquad \text{Use the definition of the common log.} \end{array}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500, followed by **[ )]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699.$

The difference in magnitudes was about $2.699.$

Try It

The amount of energy released from one earthquake was $8{,}500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Show Solution

The difference in magnitudes was about 3.929.

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$.

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Definition of the Natural Logarithm

A natural logarithm is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

How To

Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator.

1. Press **[LN]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Show Solution

- Press **[LN]**.
- Enter 500, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

Try It

Evaluate $\ln(-500)$.

Show Solution

It is not possible to take the logarithm of a negative number in the set of real numbers.

Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms

Key Equations

Definition of the logarithmic function	For $x>0, b>0, b\neq 1,$ < $y=\log_b(x)$ if and only if $b^y=x.$ </td>
Definition of the common logarithm	For $x>0, y=\log(x)$ if and only if $10^y=x.$
Definition of the natural logarithm	For $x>0, y=\ln(x)$ if and only if $e^y=x.$

Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See (Figure).
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See (Figure).
- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of $b.$ See (Figure) and (Figure).
- Common logarithms can be evaluated mentally using previous knowledge of powers of $10.$ See (Figure).
- When common logarithms cannot be evaluated mentally, a calculator can be used. See

(Figure).

- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See (Figure).
- Natural logarithms can be evaluated using a calculator (Figure).

Section Exercises

Verbal

1. What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1.$

Show Solution

A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, $y,$ in the expression b^y can also be written as the logarithm, $\log_b x,$ and the value of x is the result of raising b to the power of $y.$

2. How is the logarithmic function $f\left(x\right) = \log_b x$ related to the exponential function $g\left(x\right) = b^x$? What is the result of composing these two functions?

3. How can the logarithmic equation $\log_b x = y$ be solved for x using the properties of exponents?

Show Solution

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x,$ and then properties of exponents can be applied to solve for $x.$

4. Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

5. Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Show Solution
The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$, the notation used is $\ln(x)$.

Algebraic

For the following exercises, rewrite each equation in exponential form.

6. $\log_4(q) = m$

7. $\log_a(b) = c$

Show Solution
$a^c = b$

8. $\log_{16}(y) = x$

9. $\log_x(64) = y$

Show Solution
$x^y = 64$

10. $\log_y(x) = -11$

11. $\log_{15}(a) = b$

Show Solution
$15^b = a$

12. $\log_y(137) = x$

13. $\log_{13}(142) = a$

Show Solution
$13^a = 142$

14. $\log(v) = t$

15. $\ln(w) = n$

Show Solution
$e^n = w$

For the following exercises, rewrite each equation in logarithmic form.

16. $4^x = y$

17. $c^d = k$

Show Solution
$\log_c(k) = d$

18. $m^{-7} = n$

19. $19^x = y$

Show Solution
$\log_{19} y = x$

20. $x^{-\frac{10}{13}} = y$

21. $n^4 = 103$

Show Solution
$\log_n (103) = 4$

22. $\left(\frac{7}{5}\right)^m = n$

23. $y^x = \frac{39}{100}$

Show Solution
$\log_y \left(\frac{39}{100}\right) = x$

24. $10^a = b$

25. $e^k = h$

Show Solution
$\ln (h) = k$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

26. $\log_3(x) = 2$

27. $\log_2(x) = -3$

Show Solution
$x = 2^{-3} = \frac{1}{8}$

28. $\log_5(x) = 2$

29. $\log_3(x) = 3$

Show Solution
$x = 3^3 = 27$

30. $\log_2(x) = 6$

31. $\log_9(x) = \frac{1}{2}$

Show Solution
$x = 9^{\frac{1}{2}} = 3$

32. $\log_{18}(x) = 2$

33. $\log_6(x) = -3$

Show Solution

$x = 6^{-3} = \frac{1}{216}$

34. $\log(x) = 3$

35. $\ln(x) = 2$

Show Solution
$x = e^2$

For the following exercises, use the definition of common and natural logarithms to simplify.

36. $\log\left(100^8\right)$

37. $10^{\log(32)}$

Show Solution
32

38. $2\log(.0001)$

39. $e^{\ln(1.06)}$

Show Solution
1.06

40. $\ln\left(e^{-5.03}\right)$

41. $e^{\ln(10.125)} + 4$

Show Solution
14.125

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

42. $\log_3\left(\frac{1}{27}\right)$

43. $\log_6\left(\sqrt{6}\right)$

Show Solution
$\frac{1}{2}$

44. $\log_2\left(\frac{1}{8}\right)+4$

45. $6\log_8(4)$

Show Solution
4

For the following exercises, evaluate the common logarithmic expression without using a calculator.

46. $\log(10,000)$

47. $\log(0.001)$

Show Solution

-3

48. $\log(1) + 7$

49. $2\log\left(100^{-3}\right)$

Show Solution
-12

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

50. $\ln\left(e^{\frac{1}{3}}\right)$

51. $\ln(1)$

Show Solution
0

52. $\ln\left(e^{-0.225}\right) - 3$

53. $25\ln\left(e^{\frac{2}{5}}\right)$

Show Solution
10

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

54. $\log\left(0.04\right)$

55. $\ln\left(15\right)$

Show Solution
2.708

56. $\ln\left(\frac{4}{5}\right)$

57. $\log\left(\sqrt{2}\right)$

Show Solution
0.151

58. $\ln\left(\sqrt{2}\right)$

Extensions

59. Is $x=0$ in the domain of the function $f\left(x\right)=\log\left(x\right)$? If so, what is the value of the function when $x=0$? Verify the result.

Show Solution
No, the function has no defined value for $x=0.$ To verify, suppose $x=0$ is in the domain of the function $f\left(x\right)=\log\left(x\right).$ Then there is some number n such that $n=\log\left(0\right).$

Rewriting as an exponential equation gives: $10^n = 0,$ which is impossible since no such real number n exists. Therefore, $x = 0$ is *not* the domain of the function $f(x) = \log(x)$.

60. Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.

61. Is there a number x such that $\ln x = 2$? If so, what is that number? Verify the result.

Show Solution

Yes. Suppose there exists a real number x such that $\ln x = 2.$ Rewriting as an exponential equation gives $x = e^2$, which is a real number. To verify, let $x = e^2.$ Then, by definition, $\ln(x) = \ln(e^2) = 2.$

62. Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$? Verify the result.

63. Is the following true: $\frac{\ln\left(e^{1.725}\right)}{\ln(1)} = 1.725$? Verify the result.

Show Solution

No; $\ln(1) = 0,$ so $\frac{\ln\left(e^{1.725}\right)}{\ln(1)}$ is undefined.

Real-World Applications

64. The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

65. Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?

Show Solution

2

66. The intensity levels *I* of two earthquakes measured on a seismograph can be compared by the formula $\log \frac{I_1}{I_2} = M_1 - M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0.[5] How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

Glossary

common logarithm
: the exponent to which 10 must be raised to get x; $\log_{10}(x)$ is written simply as $\log(x)$.

logarithm
: the exponent to which b must be raised to get x; written $y = \log_b(x)$

natural logarithm
: the exponent to which the number e must be raised to get x; $\log_e(x)$ is written as $\ln(x)$.

5. http://earthquake.usgs.gov/earthquakes/world/historical.php. Accessed 3/4/2014.

CHAPTER 9.5: GRAPHS OF LOGARITHMIC FUNCTIONS

Learning Objectives

In this section, you will:

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

In Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5 compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? (Figure) shows this point on the logarithmic graph.

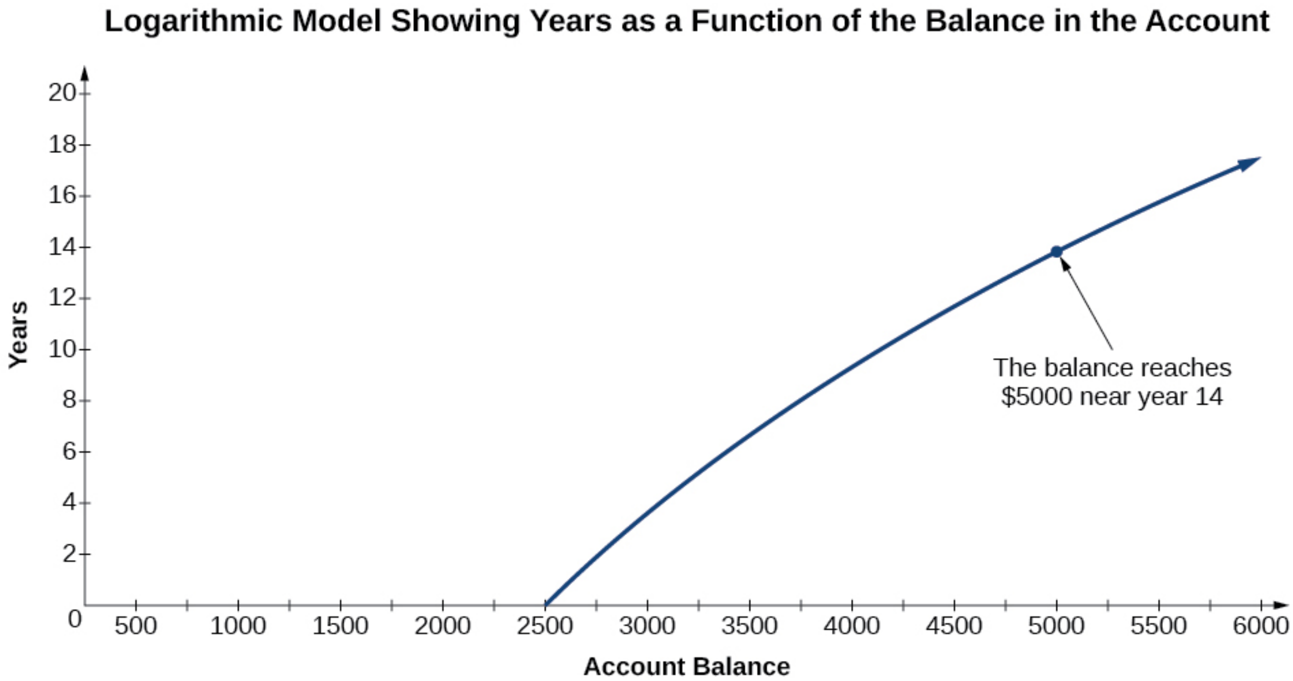


Figure 1.

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{array}{rl} 2x - 3 > 0 & \text{Show the argument greater than zero.} \\ 2x > 3 & \text{Add 3.} \\ x > 1.5 & \text{Divide by 2.} \end{array}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

How To

Given a logarithmic function, identify the domain.

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

Show Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$\begin{array}{ll} x+3>0 & \text{The input must be positive.} \\ \quad x>-3 & \text{Subtract 3.} \end{array}$$

The domain of $f\left(x\right)=\log_{2}\left(x+3\right)$ is $\left(-3,\infty\right)$.

Try It

What is the domain of $f\left(x\right)=\log_{5}\left(x-2\right)+1$?

Show Solution

$(2,\infty)$

Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f\left(x\right)=\log\left(5-2x\right)$?

Show Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $5-2x>0$. Solving this inequality,

$$\begin{array}{ll} 5-2x>0 & \text{The input must be positive.} \\ \quad -2x>-5 & \text{Subtract 5.} \\ \qquad x<\frac{5}{2} & \text{Divide by }-2\text{ and switch the inequality.} \end{array}$$

The domain of $f\left(x\right)=\log\left(5-2x\right)$ is $\left(-\infty,\frac{5}{2}\right)$.

Try It

What is the domain of $f\left(x\right)=\log\left(x-5\right)+2$?

Show Solution

$\left(5,\infty\right)$

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y=\log_b\left(x\right)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y=\log_b\left(x\right)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y=b^x$, their graphs will be reflections of each other across the line $y=x$. To illustrate this, we can observe the relationship between the input and output values of $y=2^x$ and its equivalent $x=\log_2\left(y\right)$ in (Figure).

x	-3	-2	-1	0	1	2	3
$2^x=y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2\left(y\right)=x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from (Figure), we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See (Figure).

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

As we'd expect, the *x*– and *y*-coordinates are reversed for the inverse functions. (Figure) shows the graph of f and g.

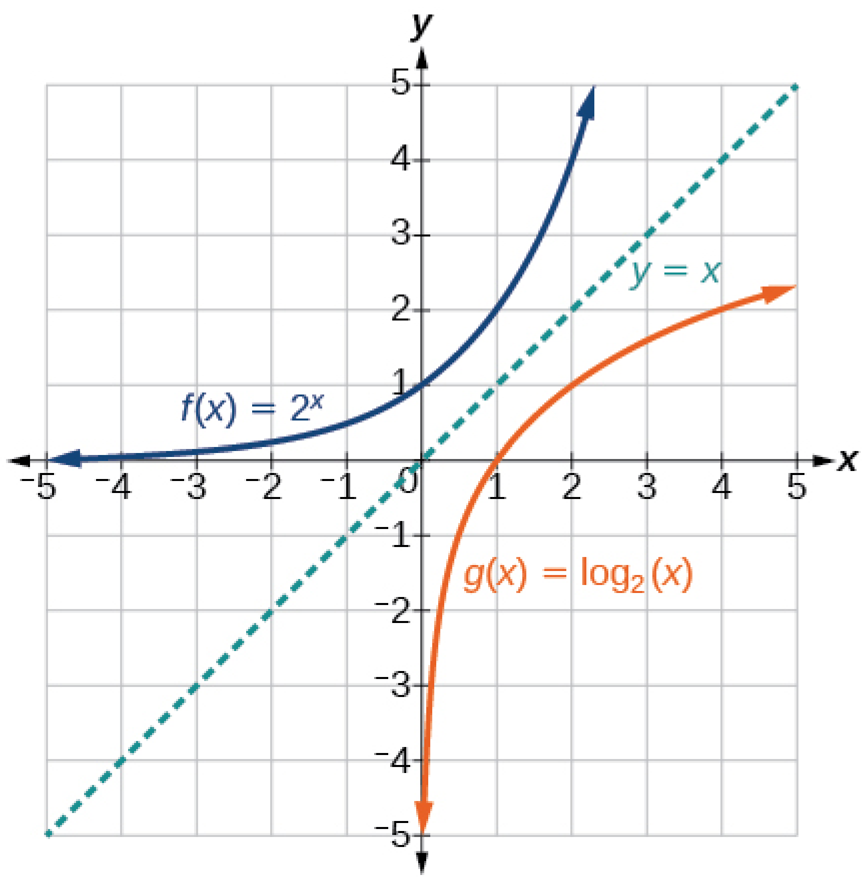


Figure 2. Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a *y*-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an *x*– intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

Characteristics of the Graph of the Parent Function, *f*(*x*) = log*b*(*x*)

For any real number x and constant $b > 0, b \neq 1,$ we can see the following characteristics in the graph of $f\left(x\right) = \log_b\left(x\right)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- *x*-intercept: $(1, 0)$ and key point $(b, 1)$
- *y*-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

See (Figure).

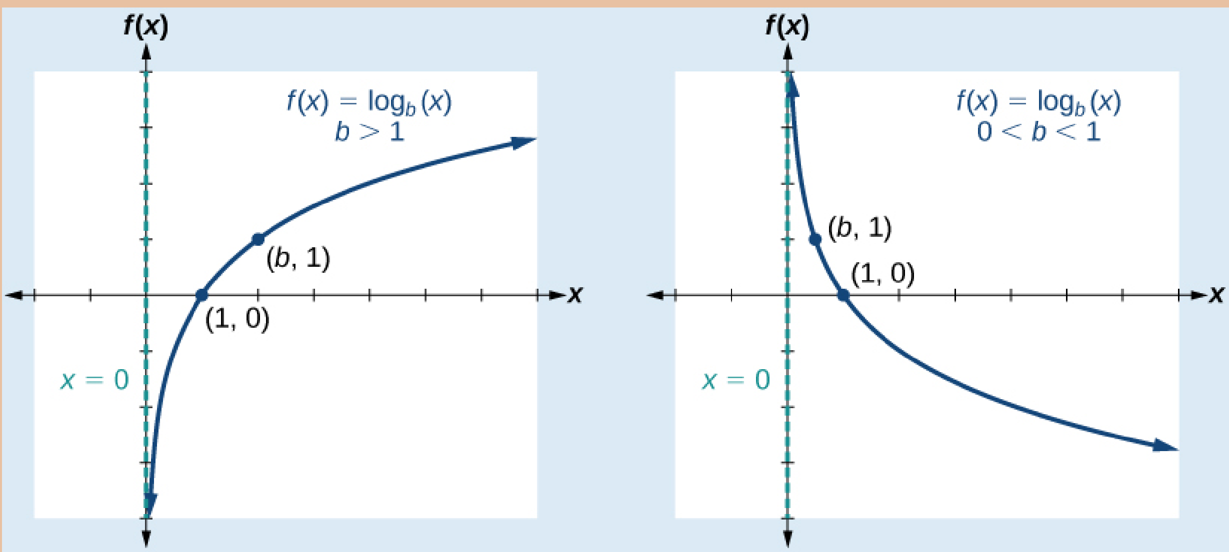


Figure 3.

(Figure) shows how changing the base b in $f\left(x\right) = \log_b\left(x\right)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note:* recall that the function $\ln\left(x\right)$ has base $e \approx 2.718.$)

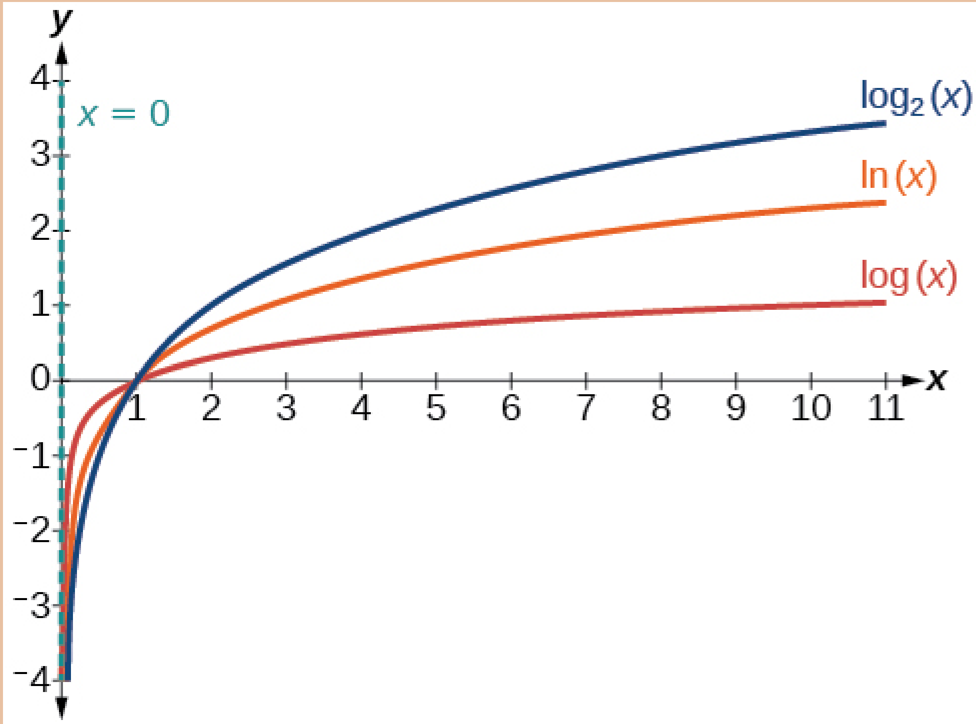


Figure 4. The graphs of three logarithmic functions with different bases, all greater than 1.

How To

Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

Graphing a Logarithmic Function with the Form *f*(*x*) = log$_b$(*x*).

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Show Solution

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0,$ and the right tail will increase slowly without bound.
- The *x*-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see (Figure)).

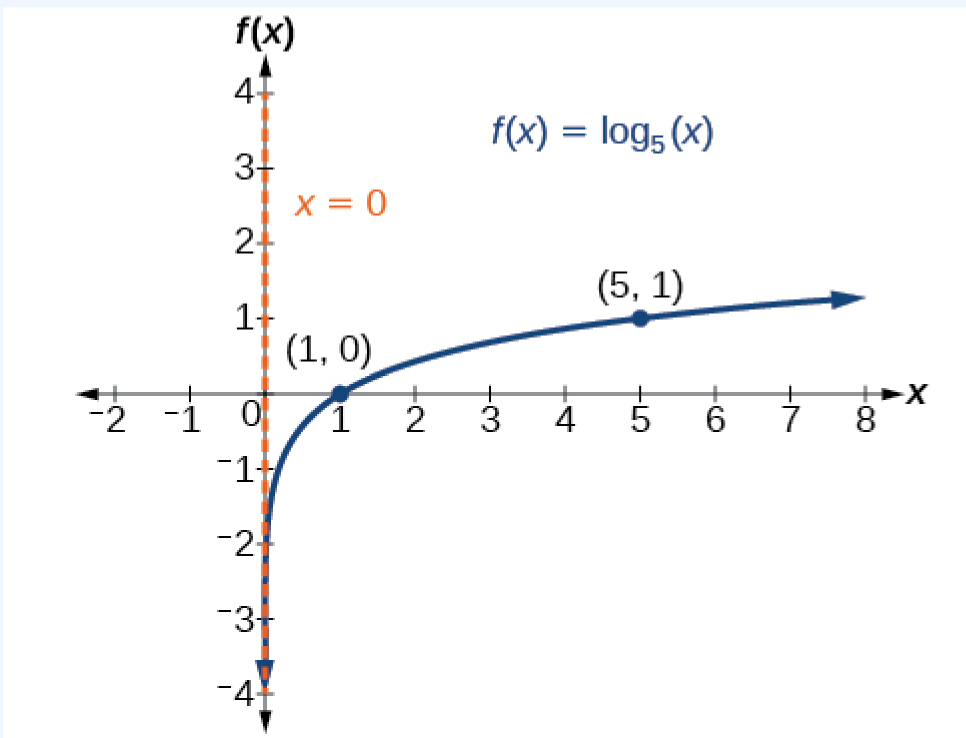


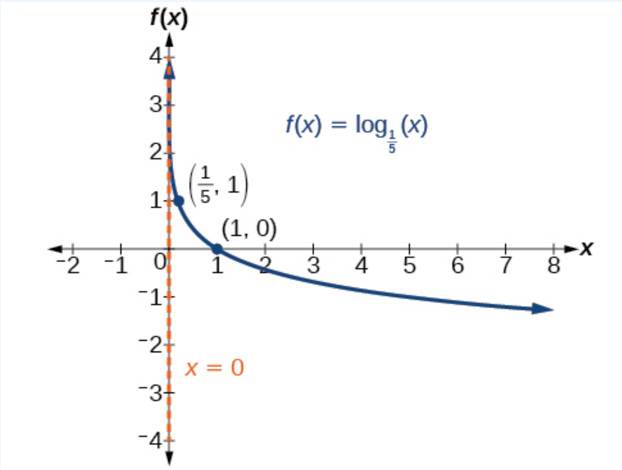
Figure 5.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0.$

Try It

Graph $f\left(x\right)=\log_{\frac{1}{5}}\left(x\right)$. State the domain, range, and asymptote.

Show Solution



The domain is $\left(0,\infty\right)$, the range is $\left(-\infty,\infty\right)$, and the vertical asymptote is $x=0$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y=\log_b\left(x\right)$ without loss of shape.

Graphing a Horizontal Shift of *f*(*x*) = log*b*(*x*)

When a constant c is added to the input of the parent function $f\left(x\right)=log_b\left(x\right)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f\left(x\right)=\log_b\left(x\right)$ and for $c>0$ alongside the shift left, $g\left(x\right)=\log_b\left(x+c\right)$, and the shift right, $h\left(x\right)=\log_b\left(x-c\right)$. See (Figure).

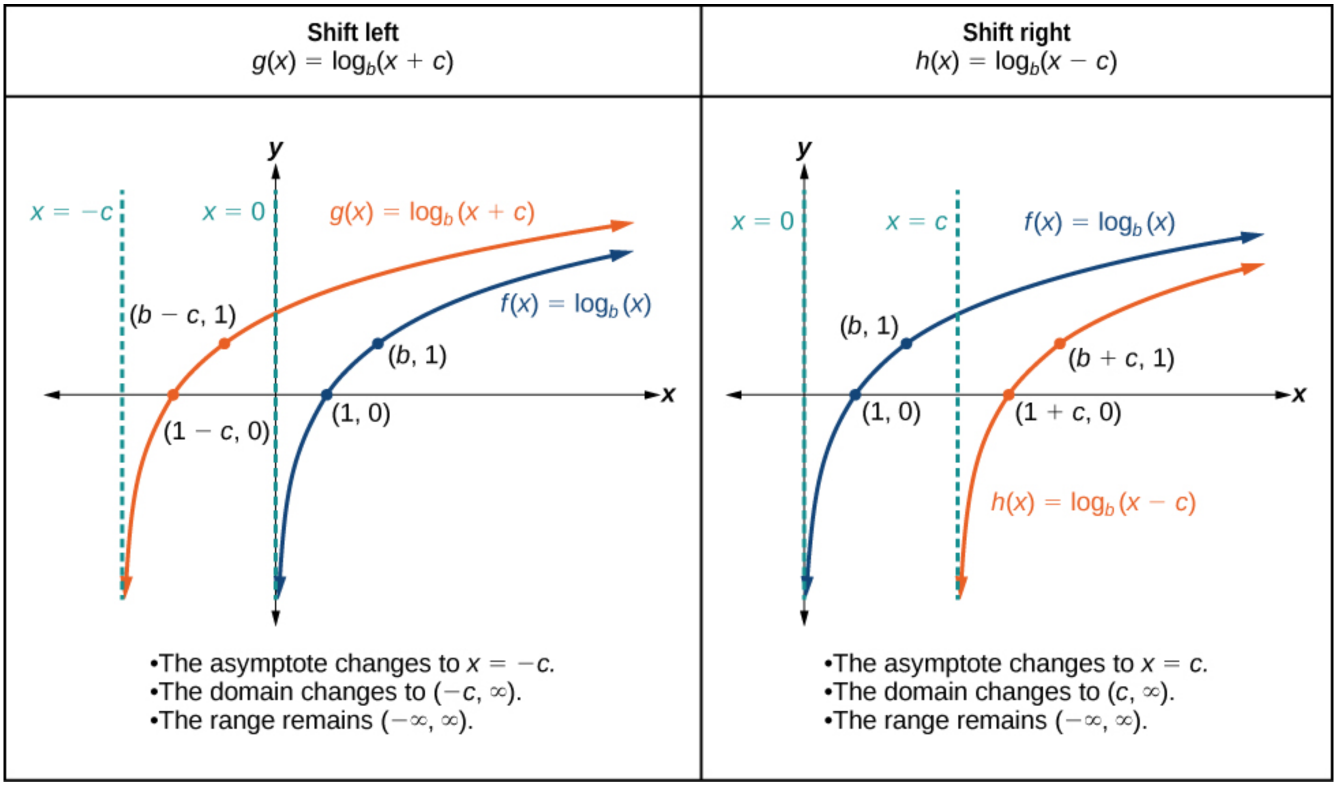


Figure 6.

Horizontal Shifts of the Parent Function *y* = log*b*(*x*)

For any constant c, the function $f\left(x\right)=\log_b\left(x+c\right)$

- shifts the parent function $y=\log_b\left(x\right)$ left c units if $c>0$.
- shifts the parent function $y=\log_b\left(x\right)$ right c units if $c<0$.
- has the vertical asymptote $x=-c$.
- has domain $\left(-c,\infty\right)$.
- has range $\left(-\infty,\infty\right)$.

How To

Given a logarithmic function with the form $f(x) = \log_b(x+c)$, **graph the translation.**

1. Identify the horizontal shift:
   a. If $c > 0$, shift the graph of $f(x) = \log_b(x)$ left c units.
   b. If $c < 0$, shift the graph of $f(x) = \log_b(x)$ right c units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

Graphing a Horizontal Shift of the Parent Function *y* = log*b*(*x*)

Sketch the horizontal shift $f(x) = \log_3(x-2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Show Solution

Since the function is $f(x) = \log_3(x-2)$, we notice $x + (-2) = x - 2$.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$, and $(5, 1)$.

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2.$

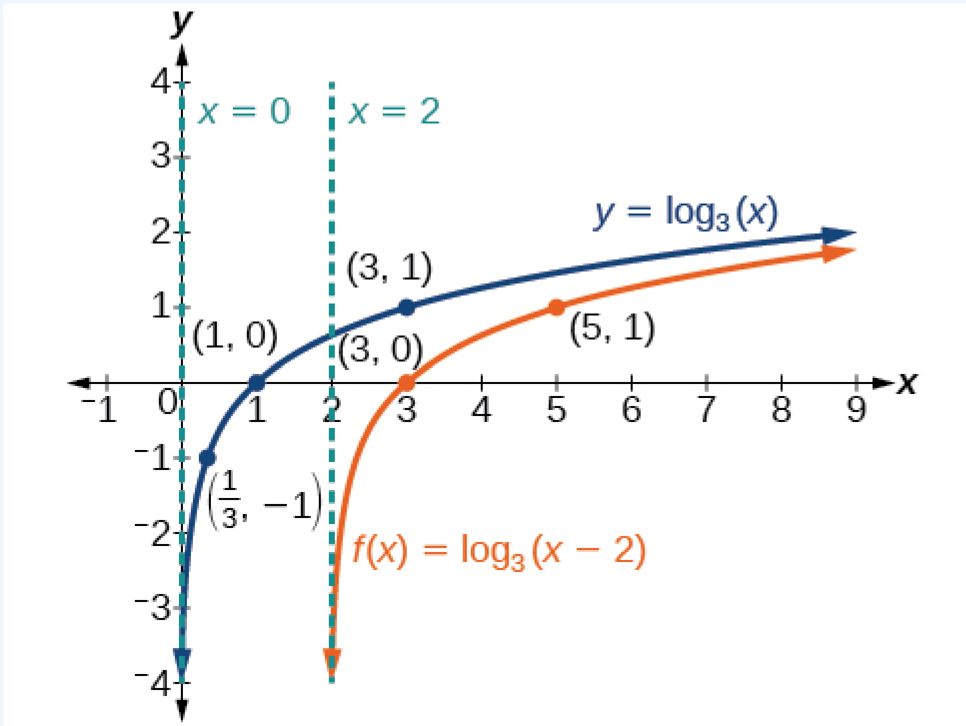
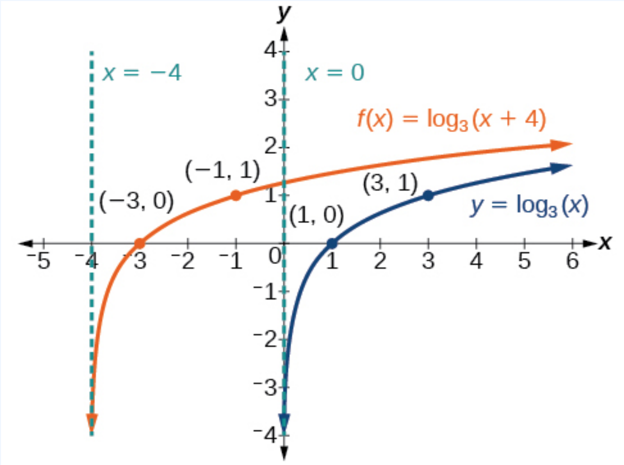


Figure 7.

Try It

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Show Solution



The domain is $(-4,\infty)$, the range $(-\infty,\infty)$, and the asymptote $x=-4.$

Graphing a Vertical Shift of *y* = log*b*(*x*)

When a constant d is added to the parent function $f\left(x\right)={\mathrm{log}}_{b}\left(x\right)$, the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f\left(x\right)={\mathrm{log}}_{b}\left(x\right)$ alongside the shift up, $g\left(x\right)={\mathrm{log}}_{b}\left(x\right)+d$ and the shift down, $h\left(x\right)={\mathrm{log}}_{b}\left(x\right)-d$. See (Figure).

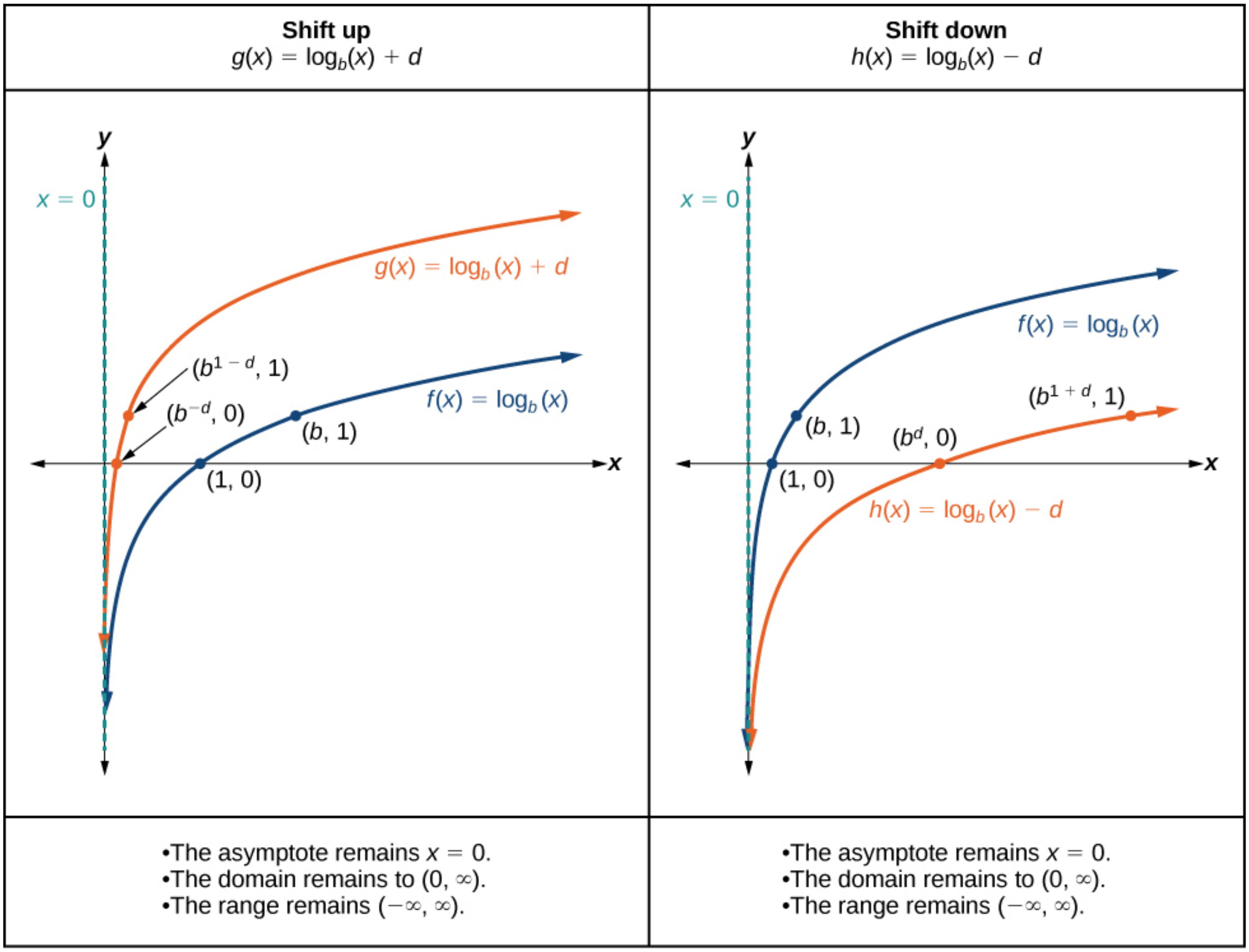


Figure 8.

Vertical Shifts of the Parent Function $y = \log_b(x)$

For any constant d, the function $f\left(x\right) = \log_b\left(x\right) + d$

- shifts the parent function $y = \log_b\left(x\right)$ up d units if $d > 0$.
- shifts the parent function $y = \log_b\left(x\right)$ down d units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $\left(0, \infty\right)$.
- has range $\left(-\infty, \infty\right)$.

How To

Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$ down d units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding d to the y coordinate.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing a Vertical Shift of the Parent Function *y* = log*b*(*x*)

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Show Solution

Since the function is $f(x) = \log_3(x) - 2$, we will notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by subtracting 2 from the *y* coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

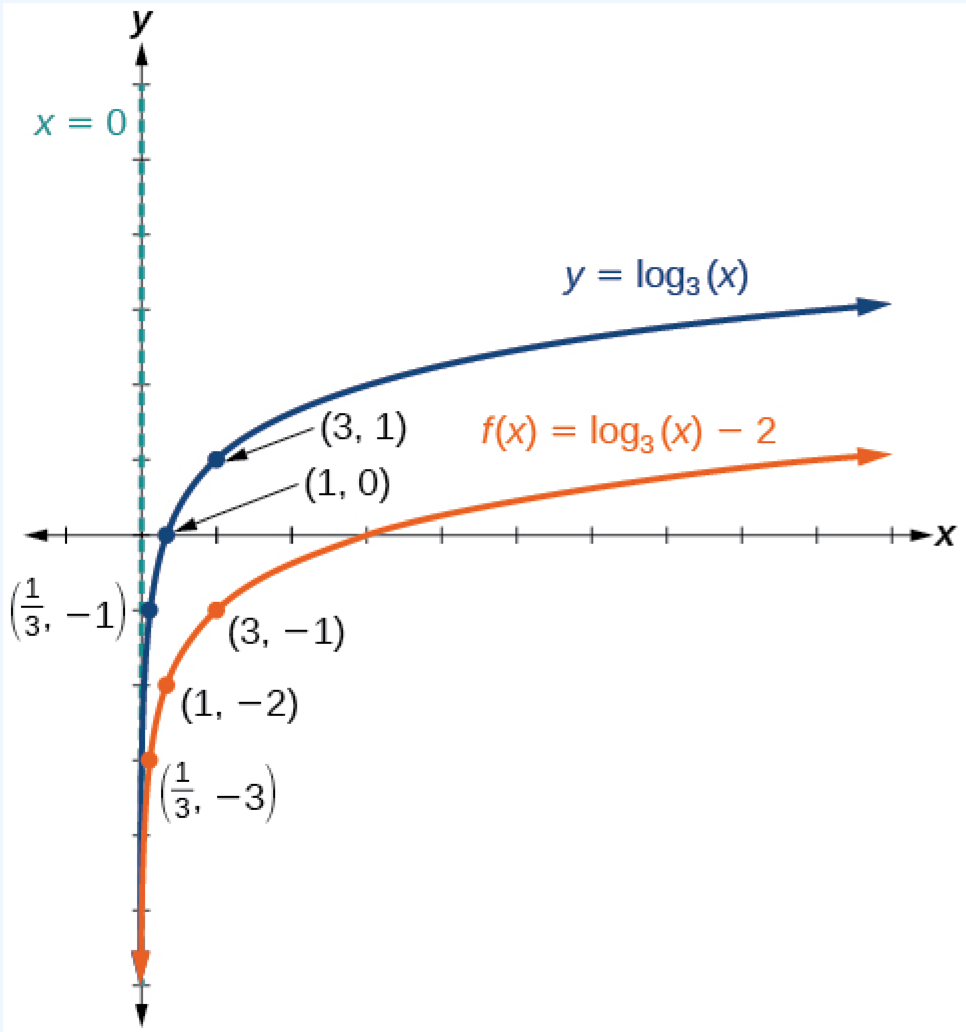


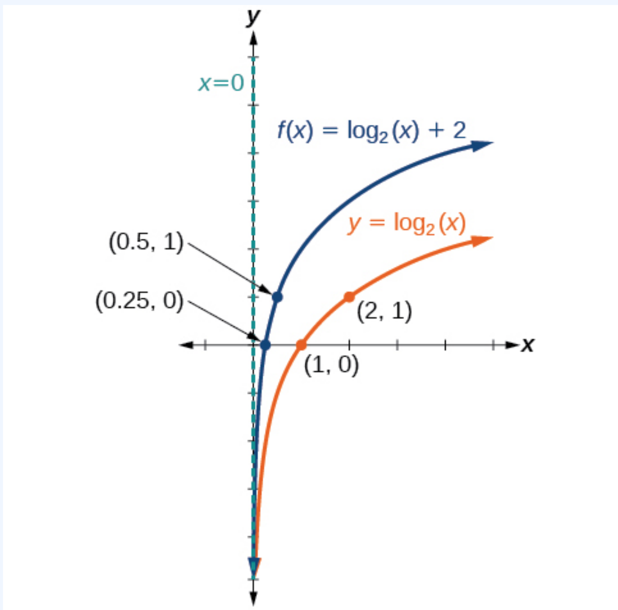
Figure 9.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Try It

Sketch a graph of $f\left(x\right)=\log_2\left(x\right)+2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Show Solution



The domain is $\left(0,\infty\right)$, the range is $\left(-\infty,\infty\right)$, and the vertical asymptote is $x=0.$

Graphing Stretches and Compressions of *y* = log*b*(*x*)

When the parent function $f\left(x\right)=\log_b\left(x\right)$ is multiplied by a constant $a>0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a>1$ and observe the general graph of the parent function $f\left(x\right)=\log_b\left(x\right)$ alongside the vertical stretch, $g\left(x\right)=a\log_b\left(x\right)$ and the vertical compression, $h\left(x\right)=\frac{1}{a}\log_b\left(x\right)$. See (Figure).

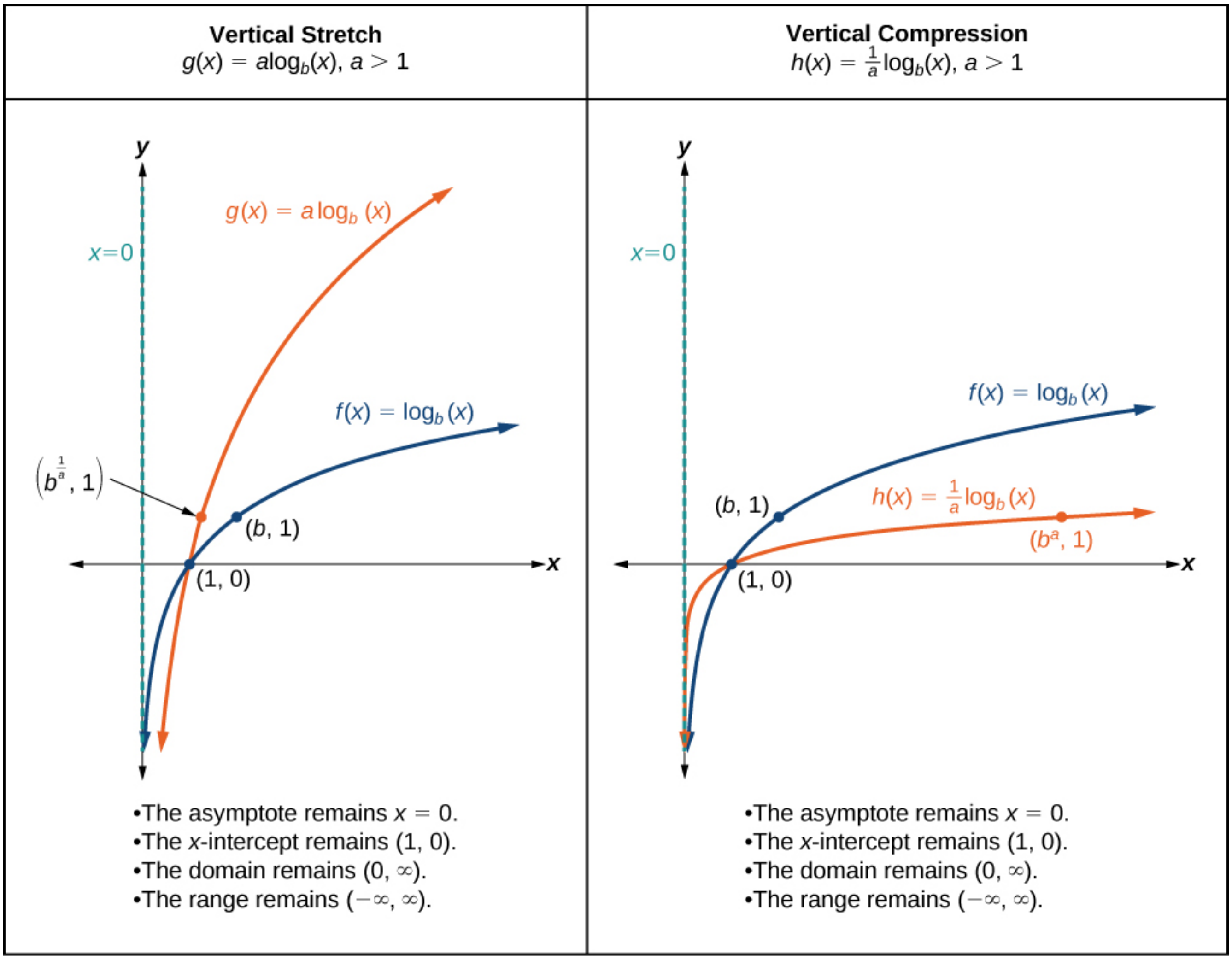


Figure 10.

Vertical Stretches and Compressions of the Parent Function *y* = log*b*(*x*)

For any constant $a > 1$, the function $f\left(x\right) = a\log_b\left(x\right)$

- stretches the parent function $y = \log_b\left(x\right)$ vertically by a factor of a if $a > 1$.
- compresses the parent function $y = \log_b\left(x\right)$ vertically by a factor of a if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the *x*-intercept $\left(1, 0\right)$.
- has domain $\left(0, \infty\right)$.
- has range $\left(-\infty, \infty\right)$.

Given a logarithmic function with the form $f\left(x\right)=a\log_b\left(x\right),a>0$, graph the translation.

1. Identify the vertical stretch or compressions:
   - If $|a|>1$, the graph of $f\left(x\right)=\log_b\left(x\right)$ is stretched by a factor of a units.
   - If $|a|<1$, the graph of $f\left(x\right)=\log_b\left(x\right)$ is compressed by a factor of a units.
2. Draw the vertical asymptote $x=0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying the y coordinates by a.
4. Label the three points.
5. The domain is $\left(0,\infty\right)$, the range is $\left(-\infty,\infty\right)$, and the vertical asymptote is $x=0$.

Graphing a Stretch or Compression of the Parent Function *y* = log*b*(*x*)

Sketch a graph of $f\left(x\right)=2\log_4\left(x\right)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Show Solution

Since the function is $f\left(x\right)=2\log_4\left(x\right)$, we will notice $a=2.$

This means we will stretch the function $f\left(x\right)=\log_4\left(x\right)$ by a factor of 2.

The vertical asymptote is $x=0.$

Consider the three key points from the parent function, $\left(\frac{1}{4},-1\right)$, $\left(1,0\right)$, and $\left(4,1\right)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4},-2\right)$, $\left(1,0\right)$, and $\left(4,2\right)$.

The domain is $\left(0,\ \infty\right)$, the range is $\left(-\infty,\infty\right)$, and the vertical asymptote is $x=0.$ See (Figure)**.**

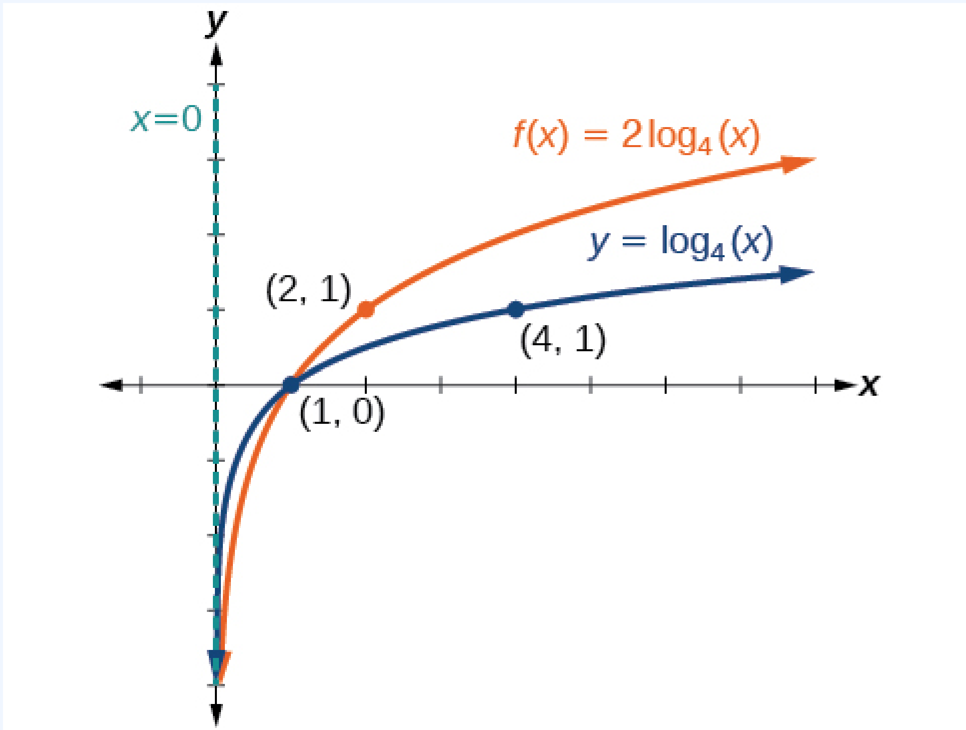


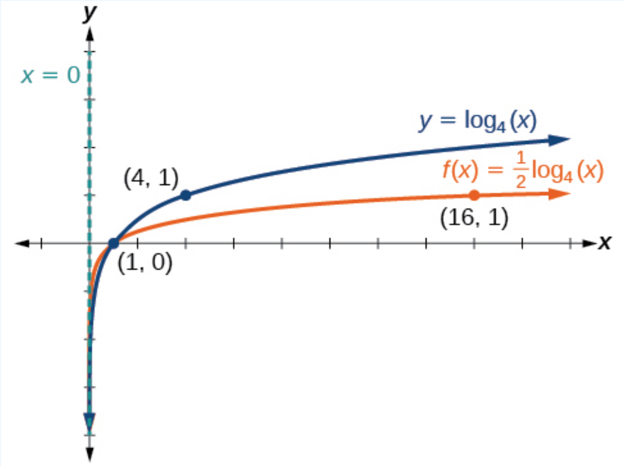
Figure 11.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Try It

Sketch a graph of $f\left(x\right) = \frac{1}{2}\log_4\left(x\right)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Show Solution



The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0.$

Combining a Shift and a Stretch

Sketch a graph of $f\left(x\right)=5\mathrm{log}\left(x+2\right)$. State the domain, range, and asymptote.

Show Solution

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in (Figure). The vertical asymptote will be shifted to $x=-2.$ The *x*-intercept will be $\left(-1,0\right)$. The domain will be $\left(-2,\infty\right)$. Two points will help give the shape of the graph: $\left(-1,0\right)$ and $\left(8,5\right)$. We chose $x=8$ as the *x*-coordinate of one point to graph because when $x=8,$ $x+2=10,$ the base of the common logarithm.

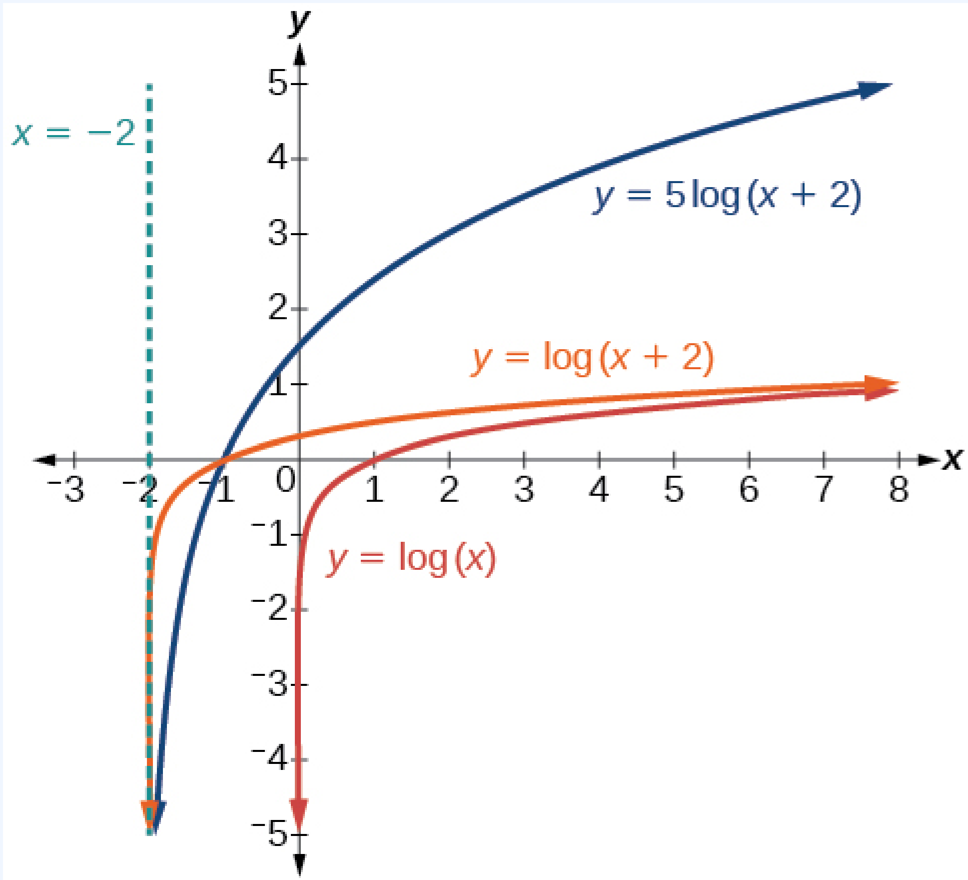


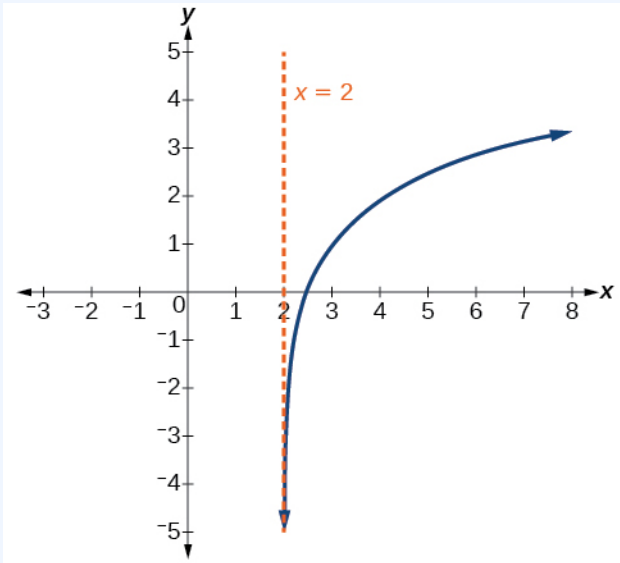
Figure 12.

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -2.$

Try It

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

Show Solution



The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2.$

Graphing Reflections of *f*(*x*) = log*b*(*x*)

When the parent function $f\left(x\right) = \log_b\left(x\right)$ is multiplied by -1, the result is a reflection about the *x*-axis. When the *input* is multiplied by -1, the result is a reflection about the *y*-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f\left(x\right) = \log_b\left(x\right)$ alongside the reflection about the *x*-axis, $g\left(x\right) = -\log_b\left(x\right)$ and the reflection about the *y*-axis, $h\left(x\right) = \log_b\left(-x\right)$.

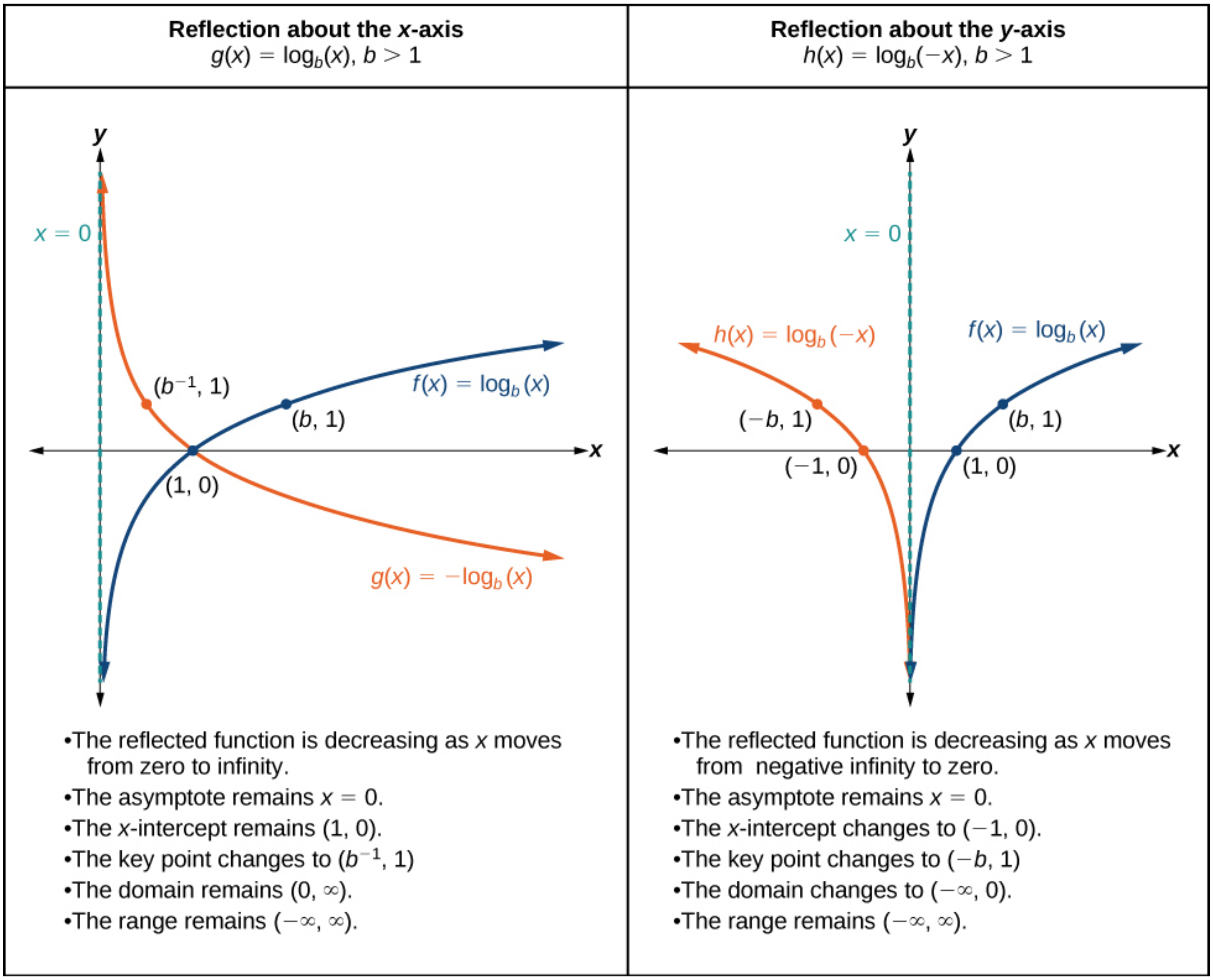


Figure 13.

Reflections of the Parent Function *y* = log*b*(*x*)

The function $f\left(x\right)=-\log_b\left(x\right)$

- reflects the parent function $y=\log_b\left(x\right)$ about the *x*-axis.
- has domain, $\left(0,\infty\right)$, range, $\left(-\infty,\infty\right)$, and vertical asymptote, $x=0$, which are unchanged from the parent function.

The function $f\left(x\right)=\log_b\left(-x\right)$

- reflects the parent function $y = \log_b(x)$ about the *y*-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
1. Draw the vertical asymptote, $x = 0$.	1. Draw the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.	2. Plot the *x*-intercept, $(1, 0)$.
3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *x*-axis.	3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *y*-axis.
4. Draw a smooth curve through the points.	4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.	5. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.

Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Show Solution

Before graphing $f\left(x\right)=\log\left(-x\right)$, identify the behavior and key points for the graph.

- Since $b=10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by $-1,$ f is a reflection of the parent graph about the *y*-axis. Thus, $f\left(x\right)=\log\left(-x\right)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x=0.$
- The *x*-intercept is $\left(-1,0\right)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

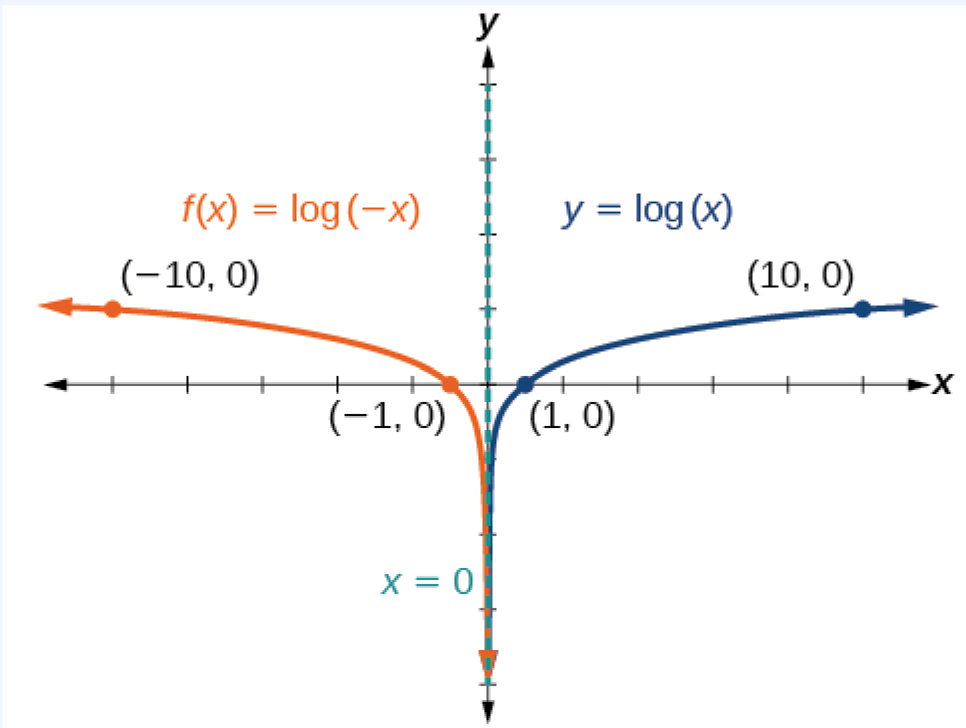


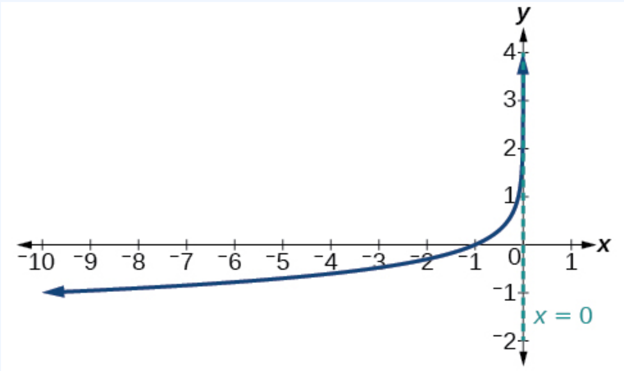
Figure 14.

The domain is $\left(-\infty,0\right)$, the range is $\left(-\infty,\infty\right)$, and the vertical asymptote is $x=0.$

Try It

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Show Solution



The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0.$

How To

Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as **Y_1=** and, if needed, **Y_2=**.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of x, for the point(s) of intersection.

Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x)+1=-2\ln(x-1)$ graphically. Round to the nearest thousandth.

Show Solution

Press **[Y=]** and enter $4\ln(x)+1$ next to **Y1**=. Then enter $-2\ln(x-1)$ next to **Y2=**. For a window, use the values 0 to 5 for x and –10 to 10 for $y.$ Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x=1.$

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 1.3385297. (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x\approx 1.339.$

Try It

Solve $5\log(x+2)=4-\log(x)$ graphically. Round to the nearest thousandth.

Show Solution

$x\approx 3.049$

Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in (Figure) to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $y = \log_b(x)$	
Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$y = \log_b(x + c) + d$
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $\lvert a \rvert < 1$	$y = a\log_b(x)$
Reflect about the *x*-axis	$y = -\log_b(x)$
Reflect about the *y*-axis	$y = \log_b(-x)$
General equation for all translations	$y = a\log_b(x + c) + d$

Translations of Logarithmic Functions

All translations of the parent logarithmic function, $y = \log_b(x)$, have the form

$f(x) = a\log_b(x + c) + d$
where the parent function, $y = \log_b(x)$, $b > 1$, is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the *x*-axis when $a < 0$.

For $f(x) = \log(-x)$, the graph of the parent function is reflected about the *y*-axis.

Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f\left(x\right)=-2{\mathrm{log}}_{3}\left(x+4\right)+5$?

Show Solution

The vertical asymptote is at $x=-4.$

Analysis

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x=-4.$

Try It

What is the vertical asymptote of $f\left(x\right)=3+\mathrm{ln}\left(x-1\right)$?

Show Solution

$x=1$

Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in (Figure).

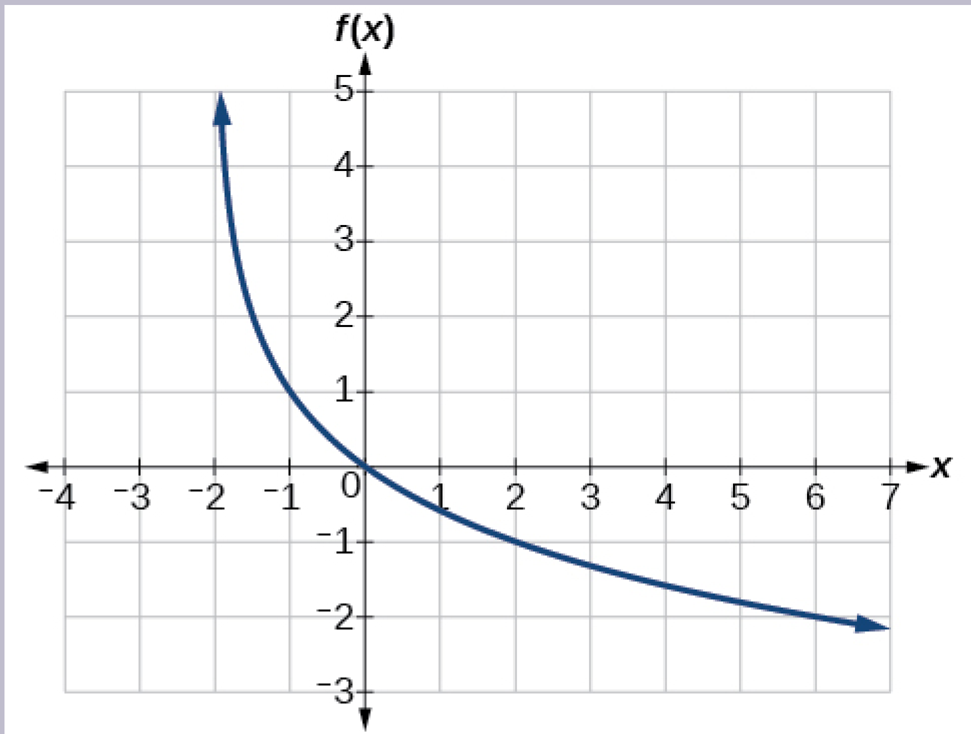


Figure 15.

Show Solution

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$$f\left(x\right) = -a\log\left(x + 2\right) + k$$

It appears the graph passes through the points $\left(-1, 1\right)$ and $\left(2, -1\right)$. Substituting $\left(-1, 1\right)$,

$$\begin{array}{ll} 1 = -a\log\left(-1 + 2\right) + k & \text{Substitute } \left(-1, 1\right). \\ 1 = -a\log\left(1\right) + k & \text{Arithmetic.} \\ 1 = k & \log(1) = 0. \end{array}$$

Next, substituting in $\left(2, -1\right)$,

$$\begin{array}{ll} -1 = -a\log\left(2 + 2\right) + 1 & \text{Plug in } \left(2, -1\right). \\ -2 = -a\log\left(4\right) & \text{Arithmetic.} \\ a = \frac{2}{\log(4)} & \text{Solve for } a. \end{array}$$

This gives us the equation $f\left(x\right) = -\frac{2}{\log(4)}\log\left(x + 2\right) + 1.$

Analysis

We can verify this answer by comparing the function values in (Figure) with the points on the graph in (Figure).

x	−1	0	1	2	3
$f(x)$	1	0	−0.58496	−1	−1.3219
x	4	5	6	7	8
$f(x)$	−1.5850	−1.8074	−2	−2.1699	−2.3219

Try It

Give the equation of the natural logarithm graphed in (Figure).

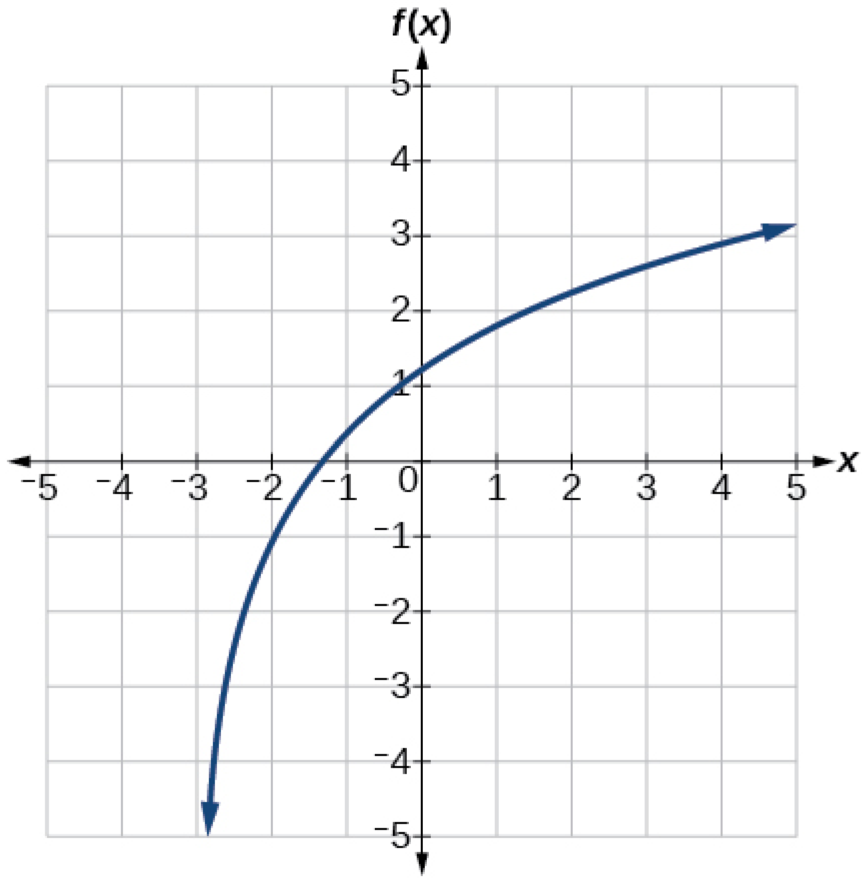


Figure 16.

Show Solution

$f\left(x\right)=2\ln\left(x+3\right)-1$

Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?

Yes, if we know the function is a general logarithmic function. For example, look at the graph in

(Figure). The graph approaches $x = -3$ *(or thereabouts) more and more closely, so* $x = -3$ *is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right,* $\{x \mid x > -3\}$ *. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as* $x \to -3^{+}, f(x) \to -\infty$ *and as* $x \to \infty, f(x) \to \infty.$

Access these online resources for additional instruction and practice with graphing logarithms.

- Graph an Exponential Function and Logarithmic Function
- Match Graphs with Exponential and Logarithmic Functions
- Find the Domain of Logarithmic Functions

Key Equations

General Form for the Translation of the Parent Logarithmic Function $f(x) = \log_b(x)$	$f(x) = a\log_b(x + c) + d$

Key Concepts

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. See (Figure) and (Figure)
- The graph of the parent function $f(x) = \log_b(x)$ has an *x*-intercept at $(1, 0)$, domain $(0, \infty)$, range $(-\infty, \infty)$, vertical asymptote $x = 0$, and
  - if $b > 1,$ the function is increasing.
  - if $0 < b < 1,$ the function is decreasing.

  See (Figure).
- The equation $f(x) = \log_b(x + c)$ shifts the parent function $y = \log_b(x)$ horizontally
  - left c units if $c > 0.$
  - right c units if $c < 0.$

See (Figure).

- The equation $f\left(x\right)={\mathrm{log}}_{b}\left(x\right)+d$ shifts the parent function $y={\mathrm{log}}_{b}\left(x\right)$ vertically
  - up d units if $d>0.$
  - down d units if $d<0.$

  See (Figure).
- For any constant $a>0,$ the equation $f\left(x\right)=a{\mathrm{log}}_{b}\left(x\right)$
  - stretches the parent function $y={\mathrm{log}}_{b}\left(x\right)$ vertically by a factor of a if $|a|>1.$
  - compresses the parent function $y={\mathrm{log}}_{b}\left(x\right)$ vertically by a factor of a if $|a|<1.$

  See (Figure) and (Figure).
- When the parent function $y={\mathrm{log}}_{b}\left(x\right)$ is multiplied by $-1,$ the result is a reflection about the *x*-axis. When the input is multiplied by $-1,$ the result is a reflection about the *y*-axis.
  - The equation $f\left(x\right)=-{\mathrm{log}}_{b}\left(x\right)$ represents a reflection of the parent function about the *x*-axis.
  - The equation $f\left(x\right)={\mathrm{log}}_{b}\left(-x\right)$ represents a reflection of the parent function about the *y*-axis.

  See (Figure).
  - A graphing calculator may be used to approximate solutions to some logarithmic equations See (Figure).
- All translations of the logarithmic function can be summarized by the general equation $f\left(x\right)=a{\mathrm{log}}_{b}\left(x+c\right)+d.$ See (Figure).
- Given an equation with the general form $f\left(x\right)=a{\mathrm{log}}_{b}\left(x+c\right)+d,$ we can identify the vertical asymptote $x=-c$ for the transformation. See (Figure).
- Using the general equation $f\left(x\right)=a{\mathrm{log}}_{b}\left(x+c\right)+d,$ we can write the equation of a logarithmic function given its graph. See (Figure).

Section Exercises

Verbal

1. The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

Show Solution

Since the functions are inverses, their graphs are mirror images about the line $y = x$. So for every point (a, b) on the graph of a logarithmic function, there is a corresponding point (b, a) on the graph of its inverse exponential function.

2. What type(s) of translation(s), if any, affect the range of a logarithmic function?

3. What type(s) of translation(s), if any, affect the domain of a logarithmic function?

Show Solution

Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

4. Consider the general logarithmic function $f\left(x\right) = \log_b\left(x\right)$. Why can't x be zero?

5. Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Show Solution

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

Algebraic

For the following exercises, state the domain and range of the function.

6. $f\left(x\right)=\log_3\left(x+4\right)$

7. $h\left(x\right)=\ln\left(\frac{1}{2}-x\right)$

Show Solution
Domain: $\left(-\infty,\frac{1}{2}\right)$; Range: $\left(-\infty,\infty\right)$

8. $g\left(x\right)=\log_5\left(2x+9\right)-2$

9. $h\left(x\right)=\ln\left(4x+17\right)-5$

Show Solution
Domain: $\left(-\frac{17}{4},\infty\right)$; Range: $\left(-\infty,\infty\right)$

10. $f\left(x\right)=\log_2\left(12-3x\right)-3$

For the following exercises, state the domain and the vertical asymptote of the function.

11. $f\left(x\right)=\log_b\left(x-5\right)$

Show Solution
Domain: $\left(5,\infty\right)$; Vertical asymptote: $x=5$

12. $g\left(x\right)=\ln\left(3-x\right)$

13. $f\left(x\right)=\log\left(3x+1\right)$

Show Solution

Domain: $\left(-\frac{1}{3},\infty\right)$; Vertical asymptote: $x=-\frac{1}{3}$

14. $f\left(x\right)=3\text{log}\left(-x\right)+2$

15. $g\left(x\right)=-\ln\left(3x+9\right)-7$

Show Solution

Domain: $\left(-3,\infty\right)$; Vertical asymptote: $x=-3$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

16. $f\left(x\right)=\ln\left(2-x\right)$

17. $f\left(x\right)=\log\left(x-\frac{3}{7}\right)$

Show Solution

Domain: $\left(\frac{3}{7},\infty\right)$;

18. $h\left(x\right)=-\text{log}\left(3x-4\right)+3$

19. $g\left(x\right)=\ln\left(2x+6\right)-5$

Show Solution

Domain: $\left(-3,\infty\right)$; Vertical asymptote: $x=-3$;

20. $f\left(x\right)=\log_{3}\left(15-5x\right)+6$

For the following exercises, state the domain, range, and *x*– and *y*-intercepts, if they exist. If they do not exist, write DNE.

21. $h\left(x\right)={\mathrm{log}}_{4}\left(x-1\right)+1$

Show Solution
Domain: $\left(1,\infty \right)$; Range: $\left(-\infty ,\infty \right)$; Vertical asymptote: $x=1$; *x*-intercept: $\left(\frac{5}{4},0\right)$; *y*-intercept: DNE

22. $f\left(x\right)=\mathrm{log}\left(5x+10\right)+3$

23. $g\left(x\right)=\mathrm{ln}\left(-x\right)-2$

Show Solution
Domain: $\left(-\infty ,0\right)$; Range: $\left(-\infty ,\infty \right)$; Vertical asymptote: $x=0$; *x*-intercept: $\left(-{e}^{2},0\right)$; *y*-intercept: DNE

24. $f\left(x\right)={\mathrm{log}}_{2}\left(x+2\right)-5$

25. $h\left(x\right)=3\mathrm{ln}\left(x\right)-9$

Show Solution
Domain: $\left(0,\infty \right)$; Range: $\left(-\infty ,\infty \right)$; Vertical asymptote: $x=0$; *x*-intercept: $\left({e}^{3},0\right)$; *y*-intercept: DNE

Graphical

For the following exercises, match each function in (Figure) with the letter corresponding to its graph.

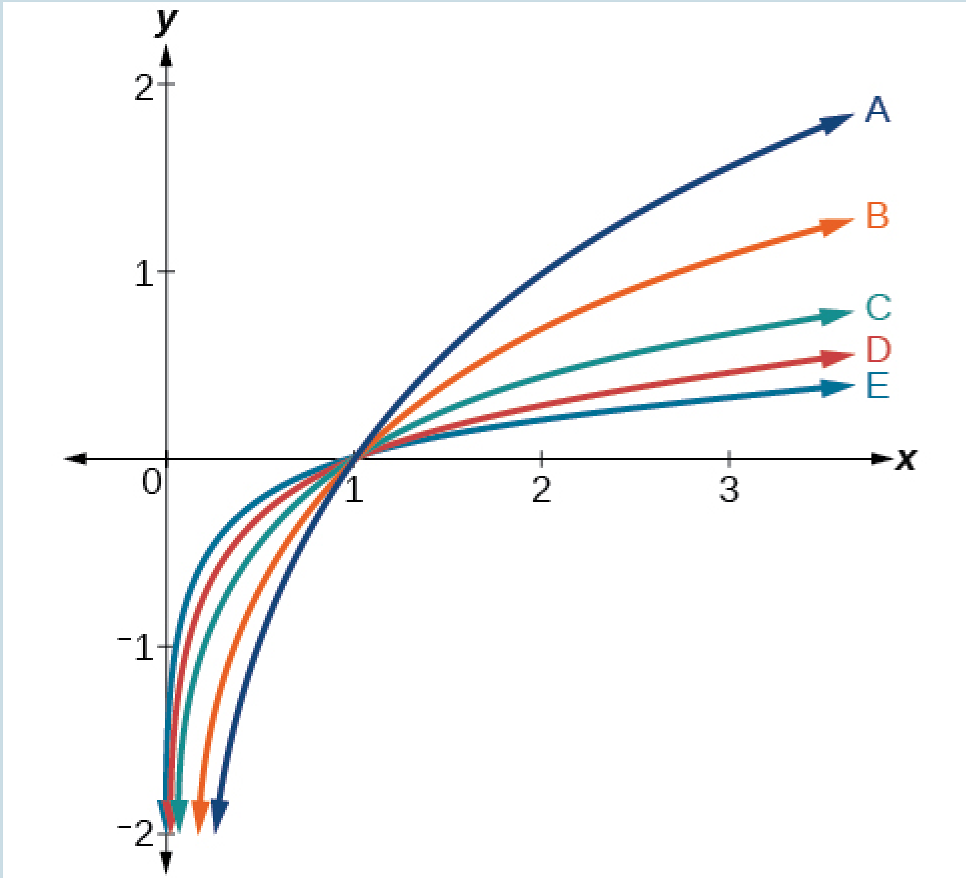


Figure 17.

26. $d\left(x\right)=\log\left(x\right)$

27. $f\left(x\right)=\ln\left(x\right)$

Show Solution
B

28. $g\left(x\right)={\log}_{2}\left(x\right)$

29. $h\left(x\right)={\log}_{5}\left(x\right)$

Show Solution
C

30. $j\left(x\right)=\log_{25}\left(x\right)$

For the following exercises, match each function in (Figure) with the letter corresponding to its graph.

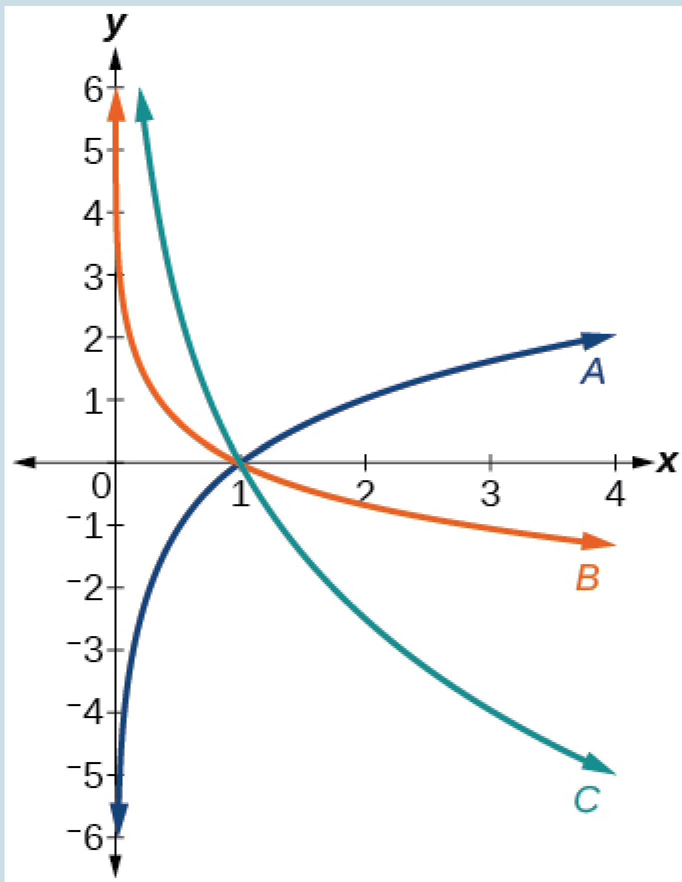


Figure 18.

31. $f\left(x\right)=\log_{\frac{1}{3}}\left(x\right)$

Show Solution

B

32. $g(x) = \log_2(x)$

33. $h(x) = \log_{\frac{3}{4}}(x)$

Show Solution

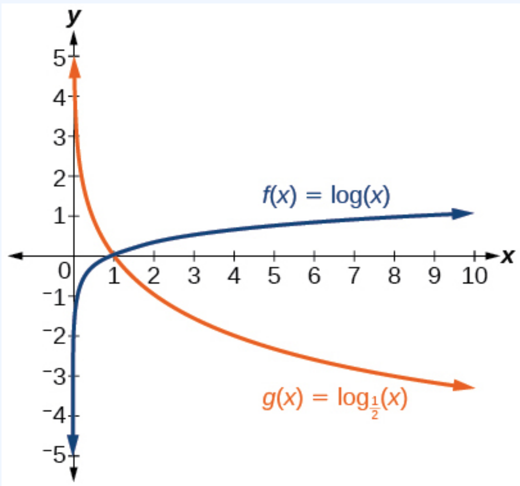
C

For the following exercises, sketch the graphs of each pair of functions on the same axis.

34. $f(x) = \log(x)$ and $g(x) = 10^x$

35. $f(x) = \log(x)$ and $g(x) = \log_{\frac{1}{2}}(x)$

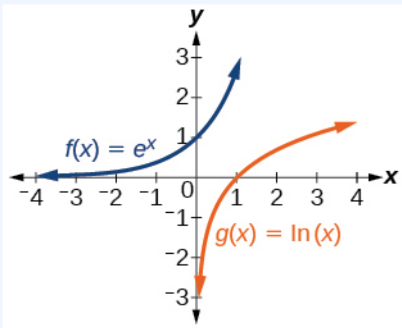
Show Solution



36. $f(x) = \log_4(x)$ and $g(x) = \ln(x)$

37. $f(x) = e^x$ and $g(x) = \ln(x)$

Show Solution



For the following exercises, match each function in (Figure) with the letter corresponding to its graph.

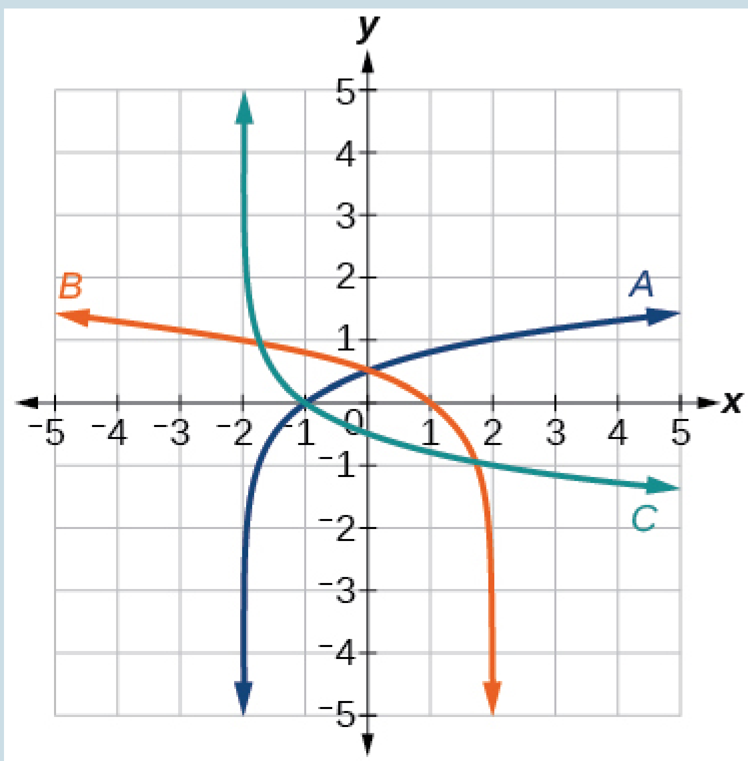


Figure 19.

38. $f(x) = \log_4(-x + 2)$

39. $g\left(x\right)=-\log_{4}\left(x+2\right)$

Show Solution

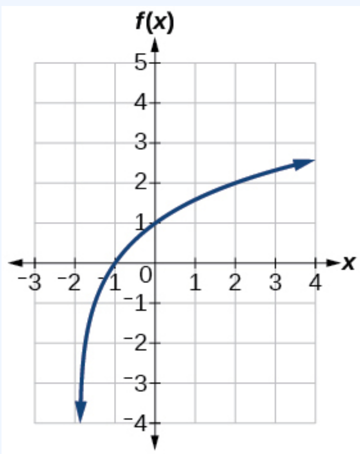
C

40. $h\left(x\right)=\log_{4}\left(x+2\right)$

For the following exercises, sketch the graph of the indicated function.

41. $f\left(x\right)=\log_{2}\left(x+2\right)$

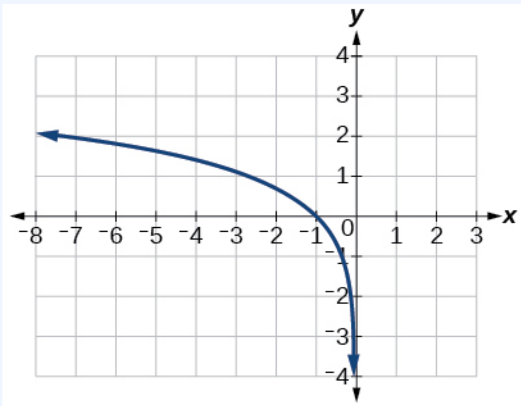
Show Solution



42. $f\left(x\right)=2\log\left(x\right)$

43. $f\left(x\right)=\ln\left(-x\right)$

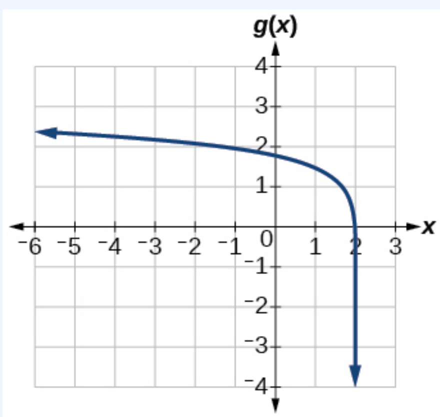
Show Solution



44. $g\left(x\right)=\log\left(4x+16\right)+4$

45. $g\left(x\right)=\log\left(6-3x\right)+1$

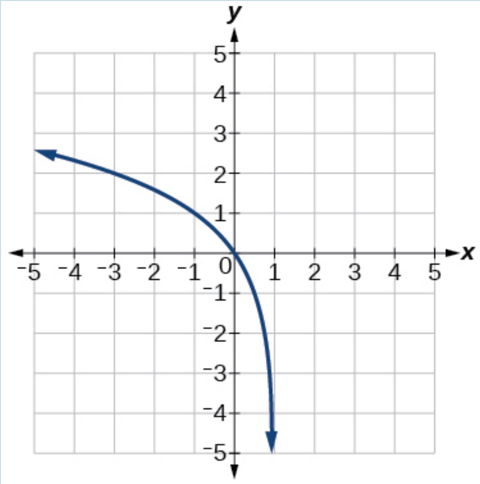
Show Solution



46. $h\left(x\right)=-\frac{1}{2}\ln\left(x+1\right)-3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

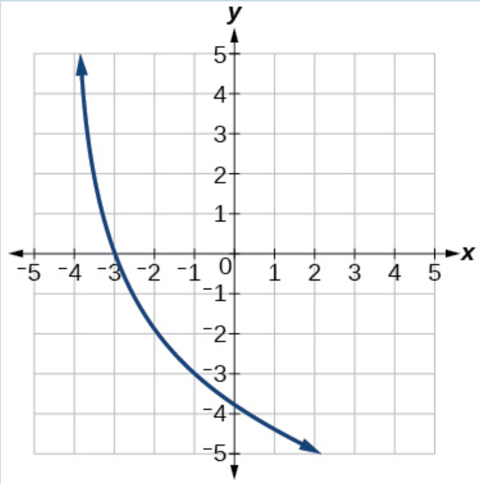
47. Use $y=\log_2\left(x\right)$ as the parent function.



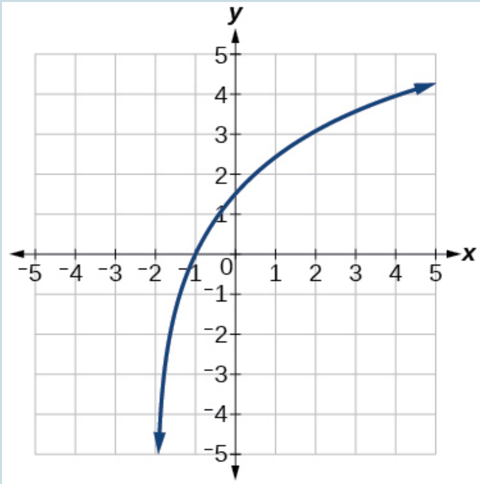
Show Solution

$f\left(x\right)=\log_2\left(-\left(x-1\right)\right)$

48. Use $f\left(x\right)=\log_3\left(x\right)$ as the parent function.



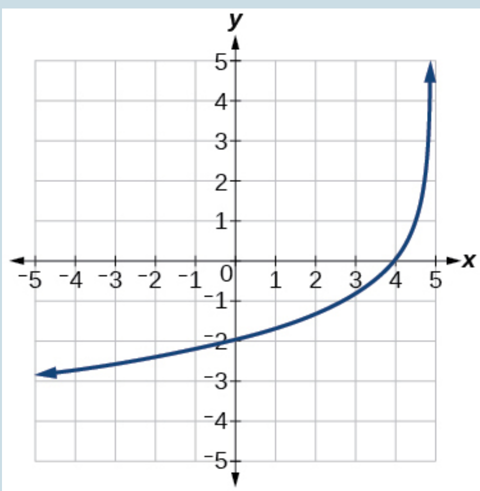
49. Use $f\left(x\right)=\log_4\left(x\right)$ as the parent function.



Show Solution

$f\left(x\right)=3\text{log}_4\left(x+2\right)$

50. Use $f\left(x\right)=\text{log}_5\left(x\right)$ as the parent function.



Technology

For the following exercises, use a graphing calculator to find approximate solutions to each equation.

51. $\text{log}\left(x-1\right)+2=\text{ln}\left(x-1\right)+2$

Show Solution
$x=2$

52. $\log\left(2x-3\right)+2=-\log\left(2x-3\right)+5$

53. $\ln\left(x-2\right)=-\ln\left(x+1\right)$

Show Solution
$x\approx 2.303$

54. $2\ln\left(5x+1\right)=\frac{1}{2}\ln\left(-5x\right)+1$

55. $\frac{1}{3}\log\left(1-x\right)=\log\left(x+1\right)+\frac{1}{3}$

Show Solution
$x\approx -0.472$

Extensions

56. Let b be any positive real number such that $b\neq 1$. What must $\log_b 1$ be equal to? Verify the result.

57. Explore and discuss the graphs of $f\left(x\right)=\log_{\frac{1}{2}}\left(x\right)$ and $g\left(x\right)=-\log_2\left(x\right)$. Make a conjecture based on the result.

Show Solution

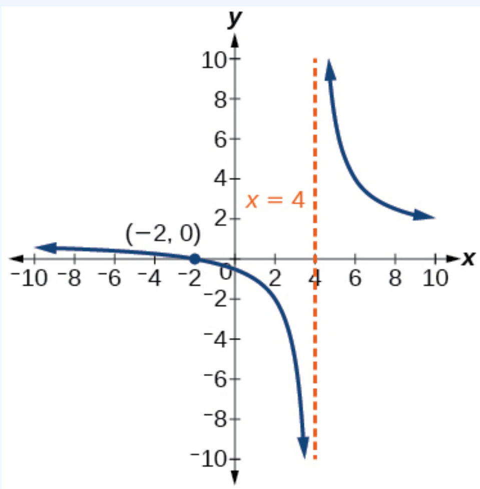
The graphs of $f\left(x\right) = \log_{\frac{1}{2}}\left(x\right)$ and $g\left(x\right) = -\log_2\left(x\right)$ appear to be the same; Conjecture: for any positive base $b \neq 1, \log_b\left(x\right) = -\log_{\frac{1}{b}}\left(x\right).$

58. Prove the conjecture made in the previous exercise.

59. What is the domain of the function $f\left(x\right) = \ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.

Show Solution

Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4} > 0$. From the graph of the function $f\left(x\right) = \frac{x+2}{x-4}$, note that the graph lies above the *x*-axis on the interval $\left(-\infty, -2\right)$ and again to the right of the vertical asymptote, that is $\left(4, \infty\right)$. Therefore, the domain is $\left(-\infty, -2\right) \cup \left(4, \infty\right)$.



60. Use properties of exponents to find the *x*-intercepts of the function $f\left(x\right) = \log\left(x^2 + 4x + 4\right)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

CHAPTER 9.6: LOGARITHMIC PROPERTIES

Learning Objectives

In this section, you will:

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

Figure 1. The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where a is the concentration of hydrogen ion in the solution

$$\begin{aligned}\text{pH} &= -\log\left([H^+]\right)\\ &= \log\left(\frac{1}{[H^+]}\right)\end{aligned}$$

The equivalence of $-\log\left([H^+]\right)$ and $\log\left(\frac{1}{[H^+]}\right)$ is one of the logarithm properties we will examine in this section.

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0$$
$$\log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b\left(b^x\right) = x$$
$$b^{\log_b x} = x, x > 0$$

For example, to evaluate $\log\left(100\right)$, we can rewrite the logarithm as $\log_{10}\left(10^2\right)$, and then apply the inverse property $\log_b\left(b^x\right) = x$ to get $\log_{10}\left(10^2\right) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \text{ if and only if } M = N$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x+5)$ for x. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x :

$$\begin{aligned} 3x &= 2x+5 && \text{Set the arguments equal.} \\ x &= 5 && \text{Subtract } 2x. \end{aligned}$$

But what about the equation $\log_3(3x) + \log_3(2x+5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for } M \text{ and } N. \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents.} \\ &= m+n && \text{Apply the inverse property of logs.} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

How To

Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Using the Product Rule for Logarithms

Expand $\log_3\left(30x\left(3x+4\right)\right)$.

Show Solution

We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3\left(30x\left(3x+4\right)\right)=\log_3\left(2\cdot3\cdot5\cdot x\cdot\left(3x+4\right)\right)$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3\left(30x\left(3x+4\right)\right)=\log_3\left(2\right)+\log_3\left(3\right)+\log_3\left(5\right)+\log_3\left(x\right)+\log_3\left(3x+4\right)$$

Try It

Expand $\log_b\left(8k\right)$.

Show Solution

$\log_b 2+\log_b 2+\log_b 2+\log_b k=3\log_b 2+\log_b k$

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $\frac{x^a}{x^b}=x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, where $b\neq 1$, we will show

$\log_b\left(\frac{M}{N}\right)=\log_b(M)-\log_b(N)$.

Let $m=\log_b M$ and $n=\log_b N$. In exponential form, these equations are $b^m=M$ and $b^n=N$. It follows that

$$\begin{array}{lll}\log_b\left(\frac{M}{N}\right) & =\log_b\left(\frac{b^m}{b^n}\right) & \text{Substitute for } M \text{ and } N.\\ & =\log_b\left(b^{m-n}\right) & \text{Apply the quotient rule for exponents.}\\ & =m-n & \text{Apply the inverse property of logs.}\\ & =\log_b(M)-\log_b(N) & \text{Substitute for } m \text{ and } n.\end{array}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{array}{ll}\log\left(\frac{2x^2+6x}{3x+9}\right)=\log\left(\frac{2x(x+3)}{3(x+3)}\right) & \text{Factor the numerator and denominator.}\\ =\log\left(\frac{2x}{3}\right) & \text{Cancel the common factors.}\end{array}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{array}{l}\log\left(\frac{2x}{3}\right)=\log(2x)-\log(3)\\ =\log(2)+\log(x)-\log(3)\end{array}$$

The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

How To

Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Using the Quotient Rule for Logarithms

Expand $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Show Solution

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2\left(15x\left(x-1\right)\right) - \log_2\left(\left(3x+4\right)\left(2-x\right)\right)$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{array}{l}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) = [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) + \log_2(2-x)] \\ \quad = \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x)\end{array}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2.$ Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2.$ Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Try It

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Show Solution

$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned}\log_b\left(x^2\right) &= \log_b(x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x\end{aligned}$$

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$100 = 10^2 \qquad \sqrt{3} = 3^{\frac{1}{2}} \qquad \frac{1}{e} = e^{-1}$

The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b \left(M^n\right) = n\log_b M$$

How To

Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Expanding a Logarithm with Powers

Expand $\log_2 x^5$.

Show Solution

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2\left(x^5\right) = 5\log_2 x$$

Try It

Expand $\ln x^2$.

Show Solution

$2\ln x$

Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

Show Solution

Expressing the argument as a power, we get $\log_3(25) = \log_3\left(5^2\right)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3\left(5^2\right) = 2\log_3(5)$$

Try It

Expand $\ln\left(\frac{1}{x^2}\right)$.

Show Solution
$-2\ln(x)$

Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Show Solution
Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln\left(x^4\right)$.

Try It

Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Show Solution

$\log_3 16$

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called "laws of logs." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b\left(AC^{-1}\right) \\ &= \log_b(A) + \log_b\left(C^{-1}\right) \\ &= \log_b A + (-1)\log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs.

Show Solution

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4y}{7}\right) = \ln\left(x^4y\right) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln\left(x^4y\right) - \ln(7) = \ln\left(x^4\right) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln\left(x^4\right) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

Try It

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Show Solution

$2\log x + 3\log y - 4\log z$

Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log\left(\sqrt{x}\right)$.

Show Solution

$$\begin{aligned}\log\left(\sqrt{x}\right) &= \log x^{\left(\frac{1}{2}\right)} \\ &= \frac{1}{2}\log x\end{aligned}$$

Try It

Expand $\ln\left(\sqrt[3]{x^2}\right)$.

Show Solution

$\frac{2}{3}\ln x$

Can we expand $\ln\left(x^2+y^2\right)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Show Solution

We can expand by applying the Product and Quotient Rules.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64+\log_6 x^3+\log_6(4x+1)-\log_6(2x-1) && \text{Apply the Quotient Rule.}\\ &= \log_6 2^6+\log_6 x^3+\log_6(4x+1)-\log_6(2x-1) && \text{Simplify by writing 64 as } 2^6.\\ &= 6\log_6 2+3\log_6 x+\log_6(4x+1)-\log_6(2x-1) && \text{Apply the Power Rule.}\end{aligned}$$

Try It

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

Show Solution

$\frac{1}{2}\ln(x-1)+\ln(2x+1)-\ln(x+3)-\ln(x-3)$

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

How To

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a

product.

3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Show Solution

Using the product and quotient rules

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to

$$\log_3(40)-\log_3(2)$$

Then, using the quotient rule

$$\log_3(40)-\log_3(2)=\log_3\left(\frac{40}{2}\right)=\log_3(20)$$

Try It

Condense $\log 3-\log 4+\log 5-\log 6.$

Show Solution

$\log\left(\frac{3\cdot 5}{4\cdot 6}\right)$; can also be written $\log\left(\frac{5}{8}\right)$ by reducing the fraction to lowest terms.

Condensing Complex Logarithmic Expressions

Condense $\log_2\left(x^2\right)+\frac{1}{2}\log_2\left(x-1\right)-3\log_2\left(\left(x+3\right)^2\right)$.

Show Solution

We apply the power rule first:

$$\log_2\left(x^2\right)+\frac{1}{2}\log_2\left(x-1\right)-3\log_2\left(\left(x+3\right)^2\right)=\log_2\left(x^2\right)+\log_2\left(\sqrt{x-1}\right)-\log_2\left(\left(x+3\right)^6\right)$$

Next we apply the product rule to the sum:

$$\log_2\left(x^2\right)+\log_2\left(\sqrt{x-1}\right)-\log_2\left(\left(x+3\right)^6\right)=\log_2\left(x^2\sqrt{x-1}\right)-\log_2\left(\left(x+3\right)^6\right)$$

Finally, we apply the quotient rule to the difference:

$$\log_2\left(x^2\sqrt{x-1}\right)-\log_2\left(\left(x+3\right)^6\right)=\log_2\frac{x^2\sqrt{x-1}}{\left(x+3\right)^6}$$

Rewriting as a Single Logarithm

Rewrite $2\log x-4\log\left(x+5\right)+\frac{1}{x}\log\left(3x+5\right)$ as a single logarithm.

Show Solution

We apply the power rule first:

$$\log\left(x+5\right)+\frac{1}{x}\log\left(3x+5\right)=\log\left(x^2\right)-\log(x+5)^4+\log\left(\left(3x+5\right)^{x^{-1}}\right)$$

Next we rearrange and apply the product rule to the sum:

$$\log\left(x^2\right)-\log(x+5)^4+\log\left(\left(3x+5\right)^{x^{-1}}\right)$$
$$=\log\left(x^2\right)+\log\left(\left(3x+5\right)^{x^{-1}}\right)-\log(x+5)^4$$

$$= \log\left(x^2(3x+5)^{x^{-1}}\right) - \log(x+5)^4$$

Finally, we apply the quotient rule to the difference:

$$= \log\left(x^2(3x+5)^{x^{-1}}\right) - \log\left(x+5\right)^4 = \log\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4}$$

Try It

Rewrite $\log\left(5\right) + 0.5\log\left(x\right) - \log\left(7x-1\right) + 3\log\left(x-1\right)$ as a single logarithm.

Show Solution

$\log\left(\frac{5(x-1)^3\sqrt{x}}{(7x-1)}\right)$

Try It

Condense $4\left(3\log\left(x\right) + \log\left(x+5\right) - \log\left(2x+3\right)\right).$

Show Solution

$\log\frac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log\left(\frac{x^3(x+5)}{(2x+3)}\right)^4.$

Applying of the Laws of Logs

Recall that, in chemistry, $\mathrm{pH} = -\log\left[H^+\right]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Show Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log\left(C\right)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$\mathrm{pH} = -\log\left(2C\right)$$

Using the product rule of logs

$$\mathrm{pH} = -\log\left(2C\right) = -\left(\log\left(2\right) + \log\left(C\right)\right) = -\log\left(2\right) - \log\left(C\right)$$

Since $P = -\log\left(C\right)$, the new pH is

$$\mathrm{pH} = P - \log\left(2\right) \approx P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

Try It

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Show Solution

The pH increases by about 0.301.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the change-of-base formula to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and power rule for logarithms.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$, we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{array}{lll} \log_n (b^y) & = \log_n M & \text{Apply the one-to-one property.} \\ y \log_n b & = \log_n M & \text{Apply the power rule for logarithms.} \\ y & = \frac{\log_n M}{\log_n b} & \text{Isolate } y. \\ \log_b M & = \frac{\log_n M}{\log_n b} & \text{Substitute for } y. \end{array}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{array}{lll} \log_5 36 & = \frac{\log(36)}{\log(5)} & \text{Apply the change of base formula using base 10.} \\ & \approx 2.2266 & \text{Use a calculator to evaluate to 4 decimal places.} \end{array}$$

The Change-of-Base Formula

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b}.$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b}$$

and

$$\log_b M = \frac{\log M}{\log b}$$

How To

Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$, has base e.
2. Rewrite the log as a quotient using the change-of-base formula
    - The numerator of the quotient will be a logarithm with base n and argument M.
    - The denominator of the quotient will be a logarithm with base n and argument b.

Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

Show Solution

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\begin{aligned} \log_b M &= \frac{\ln M}{\ln b} \\ \log_5 3 &= \frac{\ln 3}{\ln 5} \end{aligned}$$

Try It

Change $\log_{0.5}8$ to a quotient of natural logarithms.

Show Solution
$\frac{\ln 8}{\ln 0.5}$

Can we change common logarithms to natural logarithms?

Yes. Remember that $\log 9$ *means* $\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

Show Solution
According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\begin{aligned} \log_2 10 &= \frac{\ln 10}{\ln 2} && \text{Apply the change of base formula using base } e. \\ &\approx 3.3219 && \text{Use a calculator to evaluate to 4 decimal places.} \end{aligned}$$

Try It

Evaluate $\log_5(100)$ using the change-of-base formula.

Show Solution

$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$

Access these online resources for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms
- Expand Logarithmic Expressions
- Evaluate a Natural Logarithmic Expression

Key Equations

The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$
The Quotient Rule for Logarithms	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
The Power Rule for Logarithms	$\log_b(M^n) = n\log_b M$
The Change-of-Base Formula	$\log_b M = \frac{\log_n M}{\log_n b}\ n > 0, n \neq 1, b \neq 1$

Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See (Figure).
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See (Figure).

- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See (Figure), (Figure), and (Figure).
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See (Figure), (Figure), and (Figure).
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See (Figure), (Figure), (Figure), and (Figure).
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See (Figure).
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and e as the quotient of natural or common logs. That way a calculator can be used to evaluate. See (Figure).

Section Exercises

Verbal

1. How does the power rule for logarithms help when solving logarithms with the form $\log_b\left(\sqrt[n]{x}\right)$?

Show Solution

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b\left(x\right).$

2. What does the change-of-base formula do? Why is it useful when using a calculator?

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

3. $\log_b\left(7x \cdot 2y\right)$

Show Solution

$\log_b(2) + \log_b(7) + \log_b(x) + \log_b(y)$

4. $\ln(3ab \cdot 5c)$

5. $\log_b\left(\frac{13}{17}\right)$

Show Solution

$\log_b(13) - \log_b(17)$

6. $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

7. $\ln\left(\frac{1}{4^k}\right)$

Show Solution

$-k\ln(4)$

8. $\log_2(y^x)$

For the following exercises, condense to a single logarithm if possible.

9. $\ln(7) + \ln(x) + \ln(y)$

Show Solution

$\ln(7xy)$

10. $\log_3(2) + \log_3(a) + \log_3(11) + \log_3(b)$

11. $\log_b(28) - \log_b(7)$

Show Solution
$\log_b(4)$

12. $\ln(a) - \ln(d) - \ln(c)$

13. $-\log_b\left(\frac{1}{7}\right)$

Show Solution
$\log_b(7)$

14. $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

15. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

Show Solution
$15\log(x) + 13\log(y) - 19\log(z)$

16. $\ln\left(\frac{a^{-2}}{b^{-4}c^{5}}\right)$

17. $\log\left(\sqrt{x^3y^{-4}}\right)$

Show Solution

$\frac{3}{2}\log(x) - 2\log(y)$

18. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

19. $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

Show Solution

$\frac{8}{3}\log(x) + \frac{14}{3}\log(y)$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

20. $\log(2x^4) + \log(3x^5)$

21. $\ln(6x^9) - \ln(3x^2)$

Show Solution

$\ln(2x^7)$

22. $2\log(x) + 3\log(x+1)$

23. $\log(x) - \frac{1}{2}\log(y) + 3\log(z)$

Show Solution

$\log\left(\frac{xz^3}{\sqrt{y}}\right)$

24. $4\log_7(c) + \frac{\log_7(a)}{3} + \frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

25. $\log_7(15)$ to base e

Show Solution
$\log_7(15) = \frac{\ln(15)}{\ln(7)}$

26. $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6) = a$ and $\log_5(11) = b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

27. $\log_{11}(5)$

Show Solution
$\log_{11}(5) = \frac{\log_5(5)}{\log_5(11)} = \frac{1}{b}$

28. $\log_6(55)$

29. $\log_{11}\left(\frac{6}{11}\right)$

Show Solution
$\log_{11}\left(\frac{6}{11}\right) = \frac{\log_5\left(\frac{6}{11}\right)}{\log_5(11)} = \frac{\log_5(6)-\log_5(11)}{\log_5(11)} = \frac{a-b}{b} = \frac{a}{b} - 1$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

30. $\log_3\left(\frac{1}{9}\right) - 3\log_3(3)$

31. $6\log_8(2) + \frac{\log_8(64)}{3\log_8(4)}$

Show Solution
3

32. $2\log_9(3) - 4\log_9(3) + \log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

33. $\log_3(22)$

Show Solution
2.81359

34. $\log_8(65)$

35. $\log_6(5.38)$

Show Solution
0.93913

36. $\log_4\left(\frac{15}{2}\right)$

37. $\log_{\frac{1}{2}}(4.7)$

Show Solution
-2.23266

Extensions

38. Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2.$ Show the steps for solving.

39. Use the quotient rule for logarithms to find all x values such that $\log_6(x+2)-\log_6(x-3)=1.$ Show the steps for solving.

Show Solution
$x=4;$ By the quotient rule: $\log_6(x+2)-\log_6(x-3)=\log_6\left(\frac{x+2}{x-3}\right)=1.$

Rewriting as an exponential equation and solving for x :

$$\begin{aligned}6^1 &= \frac{x+2}{x-3}\\ 0 &= \frac{x+2}{x-3}-6\\ 0 &= \frac{x+2}{x-3}-\frac{6(x-3)}{(x-3)}\\ 0 &= \frac{x+2-6x+18}{x-3}\\ 0 &= \frac{x-4}{x-3}\\ x &= 4\end{aligned}$$

Checking, we find that $\log_6(4+2)-\log_6(4-3)=\log_6(6)-\log_6(1)$ is defined, so $x=4.$

40. Can the power property of logarithms be derived from the power property of exponents using the equation $b^x=m$? If not, explain why. If so, show the derivation.

41. Prove that $\log_b(n)=\frac{1}{\log_n(b)}$ for any positive integers $b>1$ and $n>1.$

Show Solution

Let b and n be positive integers greater than 1. Then, by the change-of-base formula, $\log_b(n) = \frac{\log_n(n)}{\log_n(b)} = \frac{1}{\log_n(b)}$.

42. Does $\log_{81}(2401) = \log_3(7)$? Verify the claim algebraically.

Glossary

change-of-base formula
: a formula for converting a logarithm with any base to a quotient of logarithms with any other base.

power rule for logarithms
: a rule of logarithms that states that the log of a power is equal to the product of the exponent and the log of its base

product rule for logarithms
: a rule of logarithms that states that the log of a product is equal to a sum of logarithms

quotient rule for logarithms
: a rule of logarithms that states that the log of a quotient is equal to a difference of logarithms

CHAPTER 9.7: EXPONENTIAL AND LOGARITHMIC EQUATIONS

Learning Objectives

In this section, you will:

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

Figure 1. Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the “rabbit plague.” (credit: Richard Taylor, Flickr)

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T, where $b > 0,\ b \neq 1$, $b^S = b^T$ if and only if $S = T$.

In other words, when an exponential equation has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \frac{3^{2x}}{3}$. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x :

$$\begin{array}{lll} 3^{4x-7} & = \frac{3^{2x}}{3} & \\ 3^{4x-7} & = \frac{3^{2x}}{3^1} & \text{Rewrite 3 as } 3^1. \\ 3^{4x-7} & = 3^{2x-1} & \text{Use the division property of exponents.} \\ 4x-7 & = 2x-1 & \text{Apply the one-to-one property of exponents.} \\ 2x & = 6 & \text{Subtract } 2x \text{ and add 7 to both sides.} \\ x & = 3 & \text{Divide by 3.} \end{array}$$

Using the One-to-One Property of Exponential Functions to Solve Exponential Equations

For any algebraic expressions S and T, and any positive real number $b \neq 1$,

$b^S = b^T$ if and only if $S = T$

How To

Given an exponential equation with the form $b^S = b^T$, where S and T are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Show Solution

$2^{x-1} = 2^{2x-4}$	The common base is 2.
$x - 1 = 2x - 4$	By the one-to-one property the exponents must be equal.
$x = 3$	Solve for x.

Try It

Solve $5^{2x} = 5^{3x+2}$.

Show Solution
$x = -2$

Rewriting Equations So All Powers Have the Same Base

Sometimes the common base for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x :

$$\begin{array}{ll} 256 = 4^{x-5} & \\ 2^8 = \left(2^2\right)^{x-5} & \text{Rewrite each side as a power with base 2.} \\ 2^8 = 2^{2x-10} & \text{Use the one-to-one property of exponents.} \\ 8 = 2x - 10 & \text{Apply the one-to-one property of exponents.} \\ 18 = 2x & \text{Add 10 to both sides.} \\ x = 9 & \text{Divide by 2.} \end{array}$$

How To

Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S = T$, for the unknown.

Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

Show Solution

$8^{x+2} = 16^{x+1}$	
$\left(2^3\right)^{x+2} = \left(2^4\right)^{x+1}$	Write 8 and 16 as powers of 2.
$2^{3x+6} = 2^{4x+4}$	To take a power of a power, multiply exponents.
$3x + 6 = 4x + 4$	Use the one-to-one property to set the exponents equal.
$x = 2$	Solve for x.

Try It

Solve $5^{2x} = 25^{3x+2}$.

Show Solution

$x = -1$

Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

Show Solution

$$\begin{aligned} 2^{5x} &= 2^{\frac{1}{2}} && \text{Write the square root of 2 as a power of 2.} \\ 5x &= \tfrac{1}{2} && \text{Use the one-to-one property.} \\ x &= \tfrac{1}{10} && \text{Solve for } x. \end{aligned}$$

Try It

Solve $5^x = \sqrt{5}$.

Show Solution

$x = \frac{1}{2}$

Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?

No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

Solving an Equation with Positive and Negative Powers

Solve $3^{x+1} = -2$.

Show Solution

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Analysis

(Figure) shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

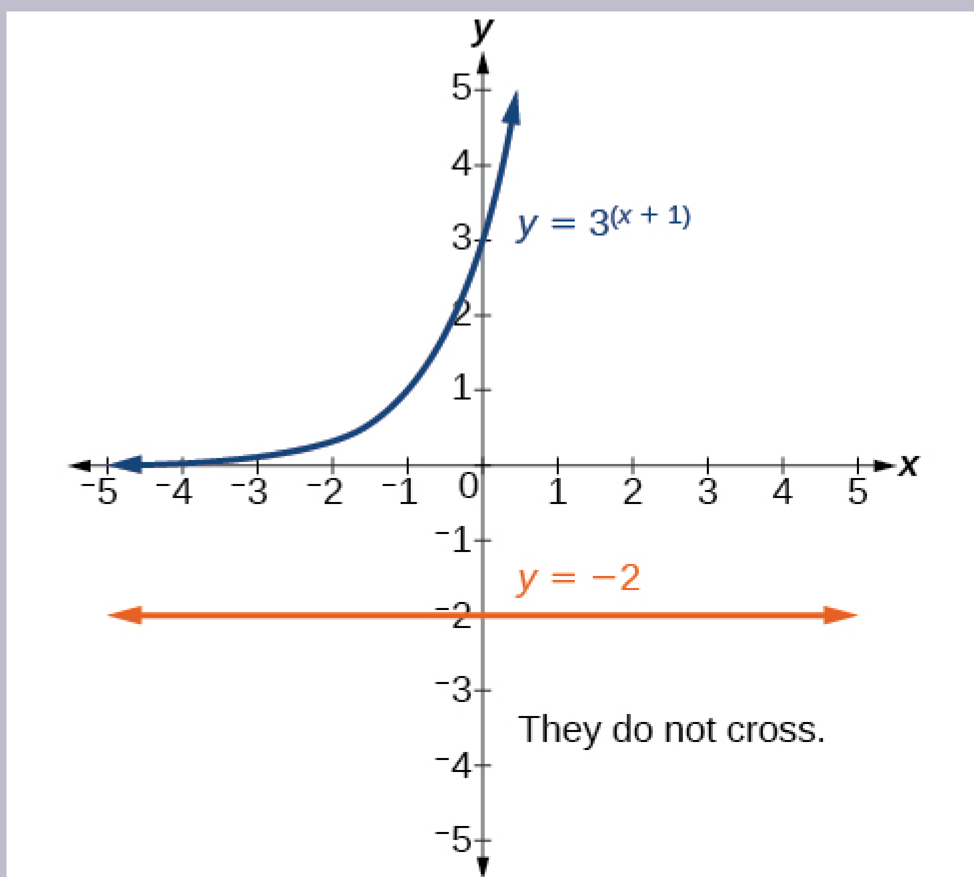


Figure 2.

Try It

Solve $2^x = -100$.

Show Solution
The equation has no solution.

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

How To

Given an exponential equation in which a common base cannot be found, solve for the unknown.

1. Apply the logarithm of both sides of the equation.
    - If one of the terms in the equation has base 10, use the common logarithm.
    - If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^x$.

Show Solution

$5^{x+2} = 4^x$	There is no easy way to get the powers to have the same base.
$\ln 5^{x+2} = \ln 4^x$	Take ln of both sides.
$(x+2)\ln 5 = x\ln 4$	Use laws of logs.
$x\ln 5 + 2\ln 5 = x\ln 4$	Use the distributive law.
$x\ln 5 - x\ln 4 = -2\ln 5$	Get terms containing x on one side, terms without x on the other.
$x(\ln 5 - \ln 4) = -2\ln 5$	On the left hand side, factor out an x.
$x\ln\left(\frac{5}{4}\right) = \ln\left(\frac{1}{25}\right)$	Use the laws of logs.
$x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$	Divide by the coefficient of x.

Try It

Solve $2^x = 3^{x+1}$.

Show Solution

$x = \frac{\ln 3}{\ln\left(\frac{2}{3}\right)}$

Is there any way to solve $2^x = 3^x$?

Yes. The solution is 0.

Equations Containing *e*

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

How To

Given an equation of the form $y = Ae^{kt}$, solve for t.

1. Divide both sides of the equation by A.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by k.

Solve an Equation of the Form *y* = *Ae*^{kt}

Solve $100 = 20e^{2t}$.

Show Solution

$$\begin{array}{lll} 100 & = 20e^{2t} & \\ 5 & = e^{2t} & \text{Divide by the coefficient of the power.} \\ \ln 5 & = 2t & \text{Take ln of both sides. Use the fact that } \ln(x) \text{ and } e^x \text{ are inverse functions.} \\ t & = \frac{\ln 5}{2} & \text{Divide by the coefficient of } t. \end{array}$$

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln\sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

Try It

Solve $3e^{0.5t} = 11$.

Show Solution

$t = 2\ln\left(\frac{11}{3}\right)$ or $\ln\left(\frac{11}{3}\right)^2$

Does every equation of the form $y = Ae^{kt}$ have a solution?

No. There is a solution when $k \neq 0$, and when y and A are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

Solving an Equation That Can Be Simplified to the Form *y* = *Ae*^{kt}

Solve $4e^{2x} + 5 = 12$.

Show Solution

$$\begin{aligned} 4e^{2x} + 5 &= 12 & \\ 4e^{2x} &= 7 & \text{Combine like terms.} \\ e^{2x} &= \frac{7}{4} & \text{Divide by the coefficient of the power.} \\ 2x &= \ln\left(\frac{7}{4}\right) & \text{Take ln of both sides.} \\ x &= \frac{1}{2}\ln\left(\frac{7}{4}\right) & \text{Solve for } x. \end{aligned}$$

Try It

Solve $3 + e^{2t} = 7e^{2t}$.

Show Solution

$t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an extraneous solution, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

Solving Exponential Functions in Quadratic Form

Solve $e^{2x} - e^{x} = 56$.

Show Solution

$e^{2x} - e^{x}$	$= 56$	
$e^{2x} - e^{x} - 56$	$= 0$	Get one side of the equation equal to zero.
$(e^{x} + 7)(e^{x} - 8)$	$= 0$	Factor by the FOIL method.
$e^{x} + 7$	$= 0$ or $e^{x} - 8 = 0$	If a product is zero, then one factor must be zero.
e^{x}	$= -7$ or $e^{x} = 8$	Isolate the exponentials.
e^{x}	$= 8$	Reject the equation in which the power equals a negative number.
x	$= \ln 8$	Solve the equation in which the power equals a positive number.

Analysis

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^x = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

Try It

Solve $e^{2x} = e^x + 2$.

Show Solution
$x = \ln 2$

Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x-5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x :

$$\begin{array}{ll} \log_2(2) + \log_2(3x-5) = 3 & \\ \log_2(2(3x-5)) = 3 & \text{Apply the product rule of logarithms.} \\ \log_2(6x-10) = 3 & \text{Distribute.} \\ 2^3 = 6x-10 & \text{Apply the definition of a logarithm.} \\ 8 = 6x-10 & \text{Calculate } 2^3. \\ 18 = 6x & \text{Add 10 to both sides.} \\ x = 3 & \text{Divide by 6.} \end{array}$$

Using the Definition of a Logarithm to Solve Logarithmic Equations

For any algebraic expression S and real numbers b and c, where $b > 0,\ b \neq 1$,

$\log_b(S) = c$ if and only if $b^c = S$

Using Algebra to Solve a Logarithmic Equation

Solve $2\ln x + 3 = 7$.

Show Solution

$$\begin{array}{ll} 2\ln x + 3 = 7 & \\ 2\ln x = 4 & \text{Subtract 3.} \\ \ln x = 2 & \text{Divide by 2.} \\ x = e^2 & \text{Rewrite in exponential form.} \end{array}$$

Try It

Solve $6 + \ln x = 10$.

Show Solution

$x = e^4$

Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x) = 7$.

Show Solution

$$\begin{array}{ll} 2\ln(6x) = 7 & \\ \ln(6x) = \frac{7}{2} & \text{Divide by 2.} \\ 6x = e^{\left(\frac{7}{2}\right)} & \text{Use the definition of ln.} \\ x = \frac{1}{6}e^{\left(\frac{7}{2}\right)} & \text{Divide by 6.} \end{array}$$

Try It

Solve $2\ln(x + 1) = 10$.

Show Solution

$x = e^5 - 1$

Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x = 3$.

Show Solution

$\ln x = 3$

$x = e^3 \quad \text{Use the definition of the natural logarithm.}$

(Figure) represents the graph of the equation. On the graph, the *x*-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20.$ A calculator gives a better approximation: $e^3 \approx 20.0855.$

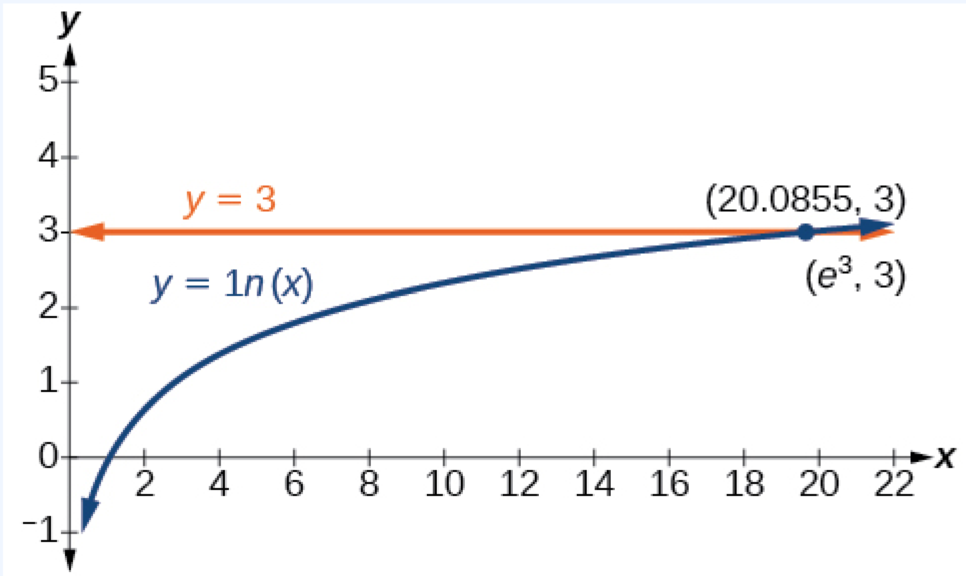


Figure 3. The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3,$ which is approximately (20.0855, 3).

Try It

Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Show Solution

$x \approx 9.97$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8)\text{, then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x, and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:

$$\begin{array}{ll}
\log(3x-2) - \log(2) = \log(x+4) & \\
\log\left(\frac{3x-2}{2}\right) = \log(x+4) & \text{Apply the quotient rule of logarithms.} \\
\frac{3x-2}{2} = x+4 & \text{Apply the one to one property of a logarithm.} \\
3x - 2 = 2x + 8 & \text{Multiply both sides of the equation by 2.} \\
x = 10 & \text{Subtract } 2x \text{ and add 2.}
\end{array}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

$$\begin{array}{ll}
\log(3(10)-2) - \log(2) = \log((10)+4) & \\
\log(28) - \log(2) = \log(14) & \\
\log\left(\frac{28}{2}\right) = \log(14) & \text{The solution checks.}
\end{array}$$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

For any algebraic expressions S and T and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

How To

Given an equation containing logarithms, solve it using the one-to-one property.

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln\left(x^2\right) = \ln\left(2x+3\right)$.

Show Solution

$$\begin{array}{ll} \ln\left(x^2\right) = \ln\left(2x+3\right) & \\ x^2 = 2x+3 & \text{Use the one-to-one property of the logarithm.} \\ x^2-2x-3=0 & \text{Get zero on one side before factoring.} \\ \left(x-3\right)\left(x+1\right)=0 & \text{Factor using FOIL.} \\ x-3=0 \text{ or } x+1=0 & \text{If a product is zero, one of the factors must be zero.} \\ x=3 \text{ or } x=-1 & \text{Solve for } x. \end{array}$$

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

Try It

Solve $\ln\left(x^2\right) = \ln 1$.

Show Solution

$x = 1$ or $x = -1$

Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. (Figure) lists the half-life for several of the more common radioactive substances.

Substance	**Use**	**Half-life**
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A\left(t\right)=A_0e^{\frac{\ln(0.5)}{T}t}$$
$$A\left(t\right)=A_0e^{\ln(0.5)\frac{t}{T}}$$
$$A\left(t\right)=A_0\left(e^{\ln(0.5)}\right)^{\frac{t}{T}}$$
$$A\left(t\right)=A_0\left(\tfrac{1}{2}\right)^{\frac{t}{T}}$$

where

- A_0 is the amount initially present
- T is the half-life of the substance
- t is the time period over which the substance is studied
- y is the amount of the substance present after time t

Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

Show Solution

$$\begin{array}{ll} y=1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} & \\ 900=1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} & \text{After 10\% decays, 900 grams are left.} \\ 0.9=e^{\frac{\ln(0.5)}{703{,}800{,}000}t} & \text{Divide by 1000.} \\ \ln\left(0.9\right)=\ln\left(e^{\frac{\ln(0.5)}{703{,}800{,}000}t}\right) & \text{Take ln of both sides.} \\ \ln\left(0.9\right)=\frac{\ln(0.5)}{703{,}800{,}000}t & \ln\left(e^M\right)=M \\ t=703{,}800{,}000\frac{\ln(0.9)}{\ln(0.5)}\text{years} & \text{Solve for } t. \\ t\approx 106{,}979{,}777\text{ years} & \end{array}$$

Analysis

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

Try It

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

Show Solution

$t = 703,800,000\frac{\ln(0.8)}{\ln(0.5)}$ years $\approx$ $226,572,993$ years.

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

Key Equations

One-to-one property for exponential functions	For any algebraic expressions S and T and any positive real number b, where< $b^S = b^T$ if and only if $S = T$. </td>
Definition of a logarithm	For any algebraic expression S and positive real numbers b and c, where $b \neq 1$, < $\log_b(S) = c$ if and only if $b^c = S$. </td>
One-to-one property for logarithmic functions	For any algebraic expressions S and T and any positive real number b, where $b \neq 1$, < $\log_b S = \log_b T$ if and only if $S = T$. </td>

Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.

- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See (Figure).
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See (Figure), (Figure), and (Figure).
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See (Figure).
- We can solve exponential equations with base e, by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See (Figure) and (Figure).
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See (Figure).
- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See (Figure) and (Figure).
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the *x*-value of the intersecting point. See (Figure).
- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See (Figure).
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See (Figure).

Section Exercises

Verbal

1. How can an exponential equation be solved?

Show Solution

Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

2. When does an extraneous solution occur? How can an extraneous solution be recognized?

3. When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Show Solution

The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

Algebraic

For the following exercises, use like bases to solve the exponential equation.

4. $4^{-3v-2} = 4^{-v}$

5. $64 \cdot 4^{3x} = 16$

Show Solution

$x = -\frac{1}{3}$

6. $3^{2x+1} \cdot 3^{x} = 243$

7. $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$

Show Solution
$n = -1$

8. $625 \cdot 5^{3x+3} = 125$

9. $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$

Show Solution
$b = \frac{6}{5}$

10. $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^6$

For the following exercises, use logarithms to solve.

11. $9^{x-10} = 1$

Show Solution
$x = 10$

12. $2e^{6x} = 13$

13. $e^{r+10} - 10 = -42$

Show Solution
No solution

14. $2 \cdot 10^{9a} = 29$

15. $-8 \cdot 10^{p+7} - 7 = -24$

Show Solution
$p = \log\left(\frac{17}{8}\right) - 7$

16. $7e^{3n-5} + 5 = -89$

17. $e^{-3k} + 6 = 44$

Show Solution
$k = -\frac{\ln(38)}{3}$

18. $-5e^{9x-8} - 8 = -62$

19. $-6e^{9x+8} + 2 = -74$

Show Solution
$x = \frac{\ln\left(\frac{38}{3}\right) - 8}{9}$

20. $2^{x+1} = 5^{2x-1}$

21. $e^{2x} - e^{x} - 132 = 0$

Show Solution
$x = \ln 12$

22. $7e^{8x+8} - 5 = -95$

23. $10e^{8x+3}+2=8$

Show Solution

$x=\frac{\ln\left(\frac{3}{5}\right)-3}{8}$

24. $4e^{3x+3}-7=53$

25. $8e^{-5x-2}-4=-90$

Show Solution

no solution

26. $3^{2x+1}=7^{x-2}$

27. $e^{2x}-e^{x}-6=0$

Show Solution

$x=\ln\left(3\right)$

28. $3e^{3-3x}+6=-31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

29. $\log\left(\frac{1}{100}\right)=-2$

Show Solution
$10^{-2}=\frac{1}{100}$

30. $\log_{324}\left(18\right)=\frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

31. $5\log_{7}n=10$

Show Solution
$n=49$

32. $-8\log_{9}x=16$

33. $4+\log_{2}\left(9k\right)=2$

Show Solution
$k=\frac{1}{36}$

34. $2\log\left(8n+4\right)+6=10$

35. $10-4\ln\left(9-8x\right)=6$

Show Solution
$x=\frac{9-e}{8}$

For the following exercises, use the one-to-one property of logarithms to solve.

36. $\ln(10-3x)=\ln(-4x)$

37. $\log_{13}(5n-2)=\log_{13}(8-5n)$

Show Solution
$n=1$

38. $\log(x+3)-\log(x)=\log(74)$

39. $\ln(-3x)=\ln\left(x^2-6x\right)$

Show Solution
No solution

40. $\log_4(6-m)=\log_4 3m$

41. $\ln(x-2)-\ln(x)=\ln(54)$

Show Solution
No solution

42. $\log_9\left(2n^2-14n\right)=\log_9\left(-45+n^2\right)$

43. $\ln\left(x^2-10\right)+\ln(9)=\ln(10)$

Show Solution
$x=\frac{10}{3}$

For the following exercises, solve each equation for x.

44. $\log(x+12) = \log(x) + \log(12)$

45. $\ln(x) + \ln(x-3) = \ln(7x)$

Show Solution
$x = 10$

46. $\log_2(7x+6) = 3$

47. $\ln(7) + \ln\left(2-4x^2\right) = \ln(14)$

Show Solution
$x = 0$

48. $\log_8(x+6) - \log_8(x) = \log_8(58)$

49. $\ln(3) - \ln(3-3x) = \ln(4)$

Show Solution
$x = \frac{3}{4}$

50. $\log_3(3x) - \log_3(6) = \log_3(77)$

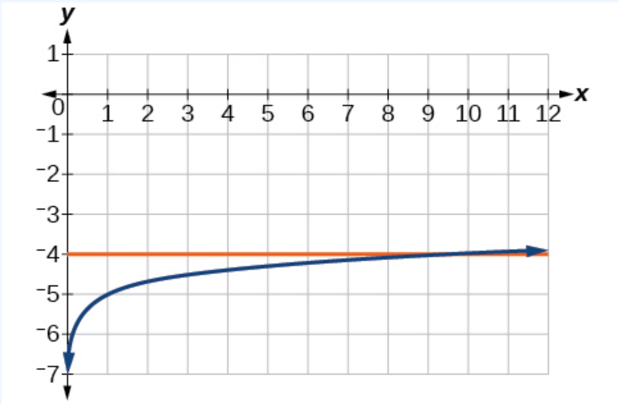
Graphical

For the following exercises, solve the equation for x, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

51. $\log_9(x) - 5 = -4$

Show Solution

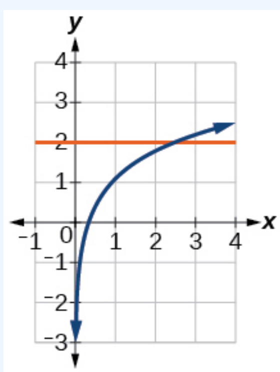
$x = 9$



52. $\log_3(x) + 3 = 2$

53. $\ln(3x) = 2$

Show Solution

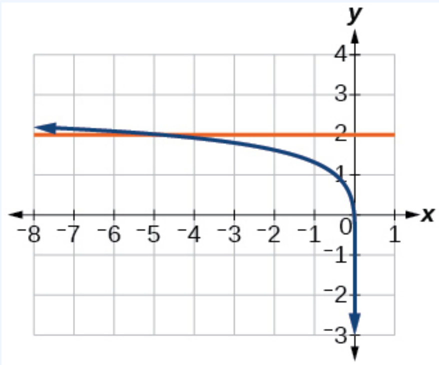
$x = \frac{e^2}{3} \approx 2.5$ 

54. $\ln(x - 5) = 1$

55. $\log(4) + \log(-5x) = 2$

Show Solution

$x = -5$

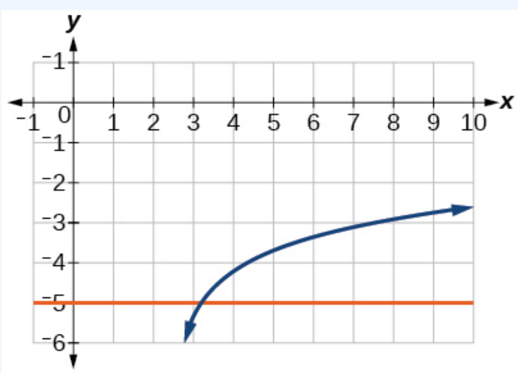


56. $-7 + \log_3(4 - x) = -6$

57. $\ln(4x - 10) - 6 = -5$

Show Solution

$x = \frac{e+10}{4} \approx 3.2$

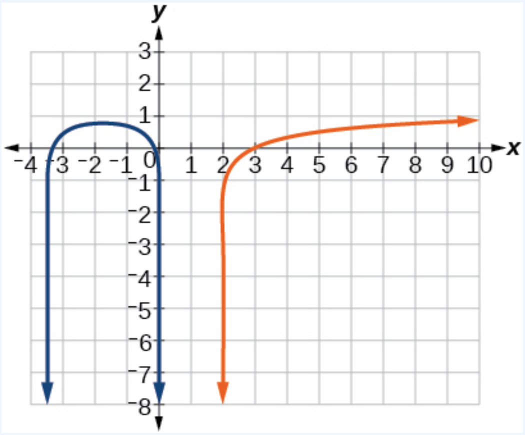


58. $\log(4 - 2x) = \log(-4x)$

59. $\log_{11}\left(-2x^2 - 7x\right) = \log_{11}(x - 2)$

Show Solution

No solution

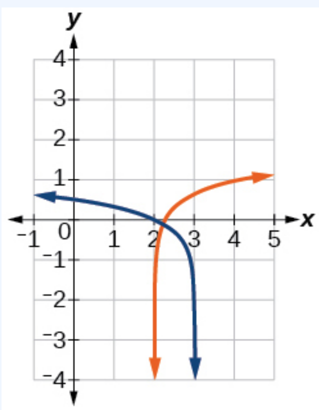


60. $\ln\left(2x+9\right)=\ln\left(-5x\right)$

61. $\log_9\left(3-x\right)=\log_9\left(4x-8\right)$

Show Solution

$x=\frac{11}{5}\approx 2.2$

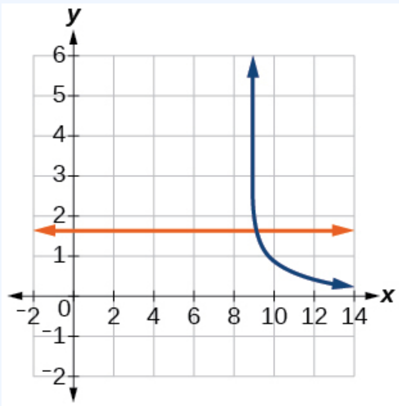


62. $\log\left(x^2+13\right)=\log\left(7x+3\right)$

63. $\frac{3}{\log_2(10)} - \log(x-9) = \log(44)$

Show Solution

$x = \frac{101}{11} \approx 9.2$



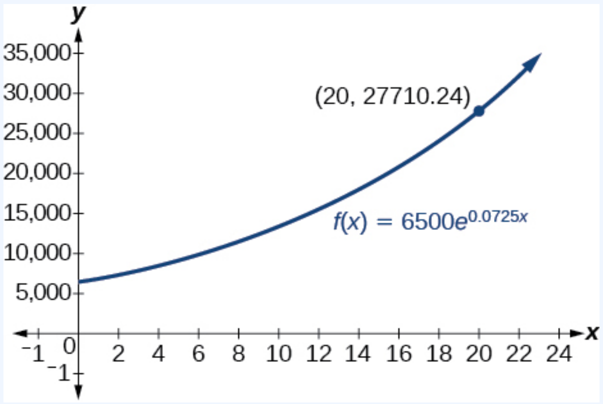
64. $\ln(x) - \ln(x+3) = \ln(6)$

For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

65. An account with an initial deposit of $\$6{,}500$ earns 7.25 annual interest, compounded continuously. How much will the account be worth after 20 years?

Show Solution

about $27{,}710.24$

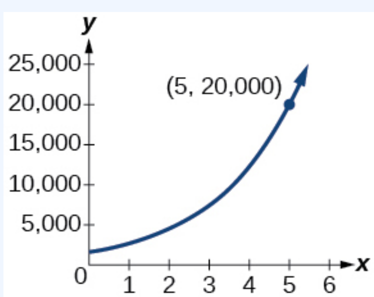


66. The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8.3 \cdot 10^2$ watts per square meter?

67. The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach $20{,}000?$

Show Solution

about 5 years



Technology

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate x to 3 decimal places.

68. $1000(1.03)^t = 5000$ using the common log.

69. $e^{5x} = 17$ using the natural log

Show Solution
$\frac{\ln(17)}{5} \approx 0.567$

70. $3(1.04)^{3t} = 8$ using the common log

71. $3^{4x-5} = 38$ using the common log

Show Solution
$x = \frac{\log(38)+5\log(3)}{4\log(3)} \approx 2.078$

72. $50e^{-0.12t} = 10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

73. $7e^{3x-5} + 7.9 = 47$

Show Solution
$x \approx 2.2401$

74. $\ln(3) + \ln(4.4x + 6.8) = 2$

75. $\log(-0.7x - 9) = 1 + 5\log(5)$

Show Solution
$x \approx -44655.7143$

76. Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (*Hint*: there are 5280 feet in a mile)

77. The magnitude *M* of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules and $E_0 = 10^{4.4}$ is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

Show Solution
about 5.83

Extensions

78. Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x} = x$.

79. Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Show Solution
$t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$

80. Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.

81. Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)\,e^{-kt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Show Solution

$$t = \ln\left(\left(\frac{T - T_s}{T_0 - T_s}\right)^{-\frac{1}{k}}\right)$$

Glossary

extraneous solution
: a solution introduced while solving an equation that does not satisfy the conditions of the original equation

CHAPTER 9.8: EXPONENTIAL AND LOGARITHMIC MODELS

Learning Objectives

In this section, you will:

- Model exponential growth and decay.
- Use Newton's Law of Cooling.
- Use logistic-growth models.
- Choose an appropriate model for data.
- Express an exponential model in base e .

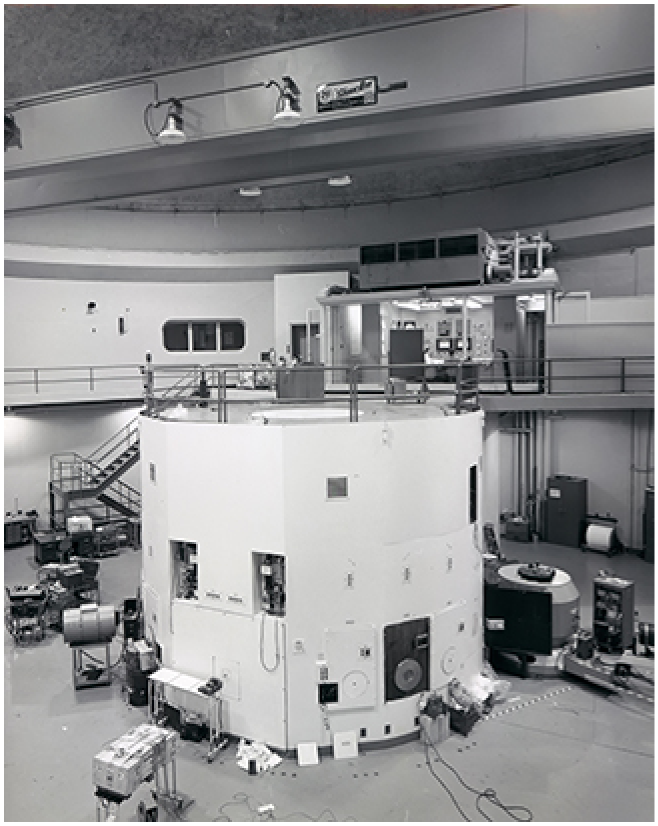


Figure 1. A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton's Law of Cooling.

Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:

$$y = A_0e^{kt}$$

where A_0 is equal to the value at time zero, e is Euler's constant, and k is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time. In some applications, however, as we will see when we discuss the logistic equation, the logistic model sometimes fits the data better than the exponential model.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the exponential decay model. Again, we have the form $y = A_0e^{kt}$ where A_0 is the starting value, and e is Euler's constant. Now k is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating half-life, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in (Figure) and (Figure). It is important to remember that, although parts of each of the two graphs seem to lie on the *x*-axis, they are really a tiny distance above the *x*-axis.

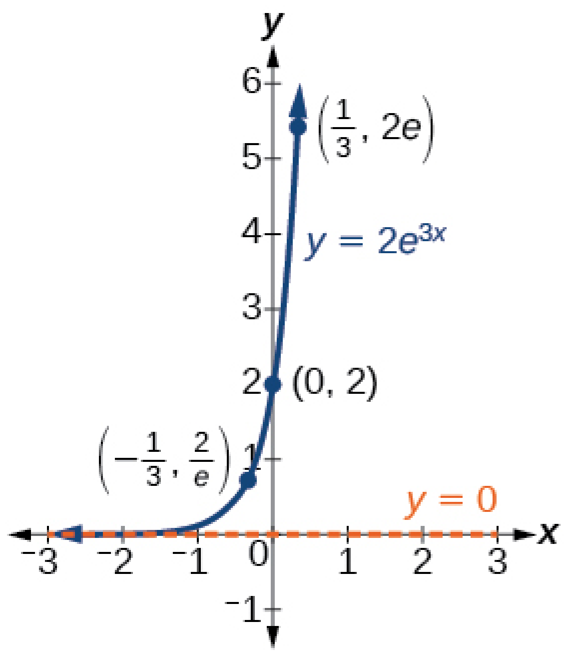


Figure 2. A graph showing exponential growth. The equation is $y = 2e^{3x}$.

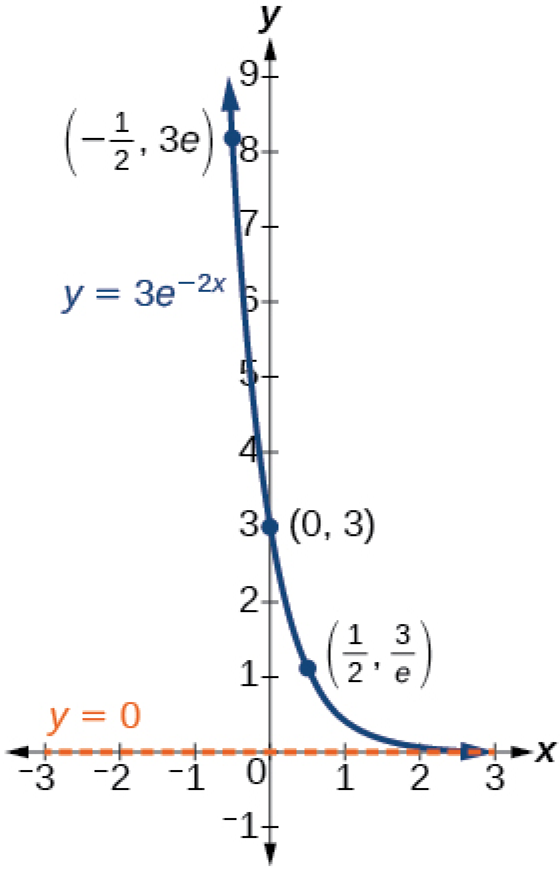


Figure 3. A graph showing exponential decay. The equation is $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude. The order of magnitude is the power of ten, when the number is expressed in scientific notation, with one digit to the left of the decimal. For example, the distance to the nearest star, Proxima Centauri, measured in kilometers, is 40,113,497,200,000 kilometers. Expressed in scientific notation, this is $4.01134972\ 10^{13}$. So, we could describe this number as having order of magnitude 10^{13}.

Characteristics of the Exponential Function, $y = A_0e^{kt}$

An exponential function with the form $y = A_0e^{kt}$ has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$

- x intercept: none
- y-intercept: $(0, A_0)$
- increasing if $k > 0$ (see (Figure))
- decreasing if $k < 0$ (see (Figure))

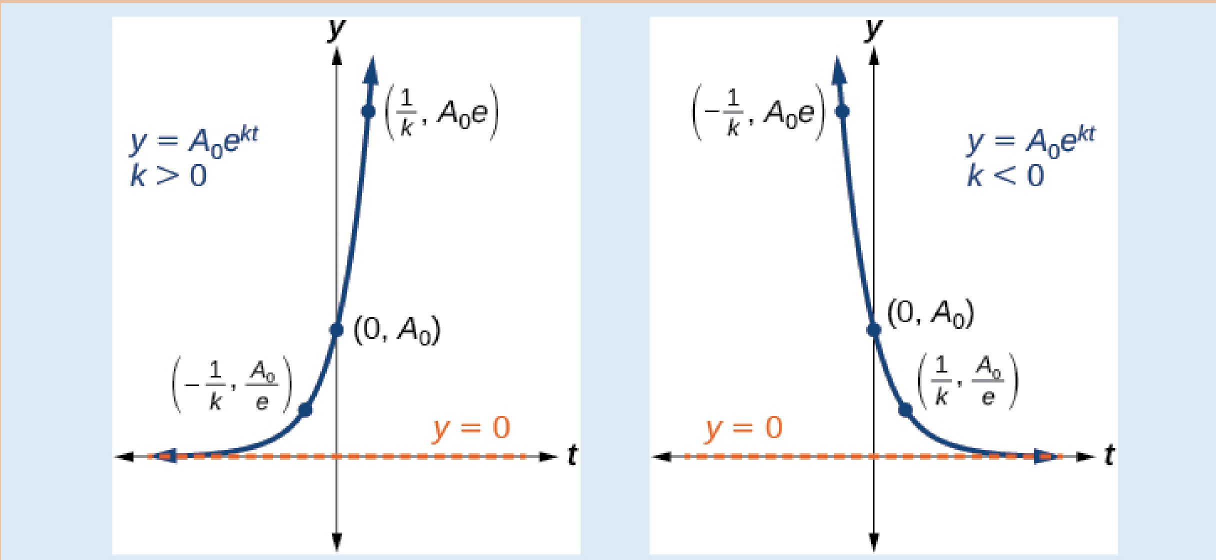


Figure 4. An exponential function models exponential growth when $k > 0$ and exponential decay when $k < 0.$

Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.

Show Solution

When an amount grows at a fixed percent per unit time, the growth is exponential. To find A_0 we use the fact that A_0 is the amount at time zero, so $A_0 = 10.$ To find $k,$ use the fact that after one hour $(t = 1)$ the population doubles from 10 to $20.$ The formula is derived as follows

$$\begin{aligned}20 &= 10e^{k\cdot 1} \\ 2 &= e^{k} && \text{Divide by 10} \\ \ln 2 &= k && \text{Take the natural logarithm}\end{aligned}$$

so $k = \ln(2)$. Thus the equation we want to graph is $y = 10e^{(\ln 2)t} = 10\left(e^{\ln 2}\right)^{t} = 10\Delta 2^{t}$. The graph is shown in (Figure).

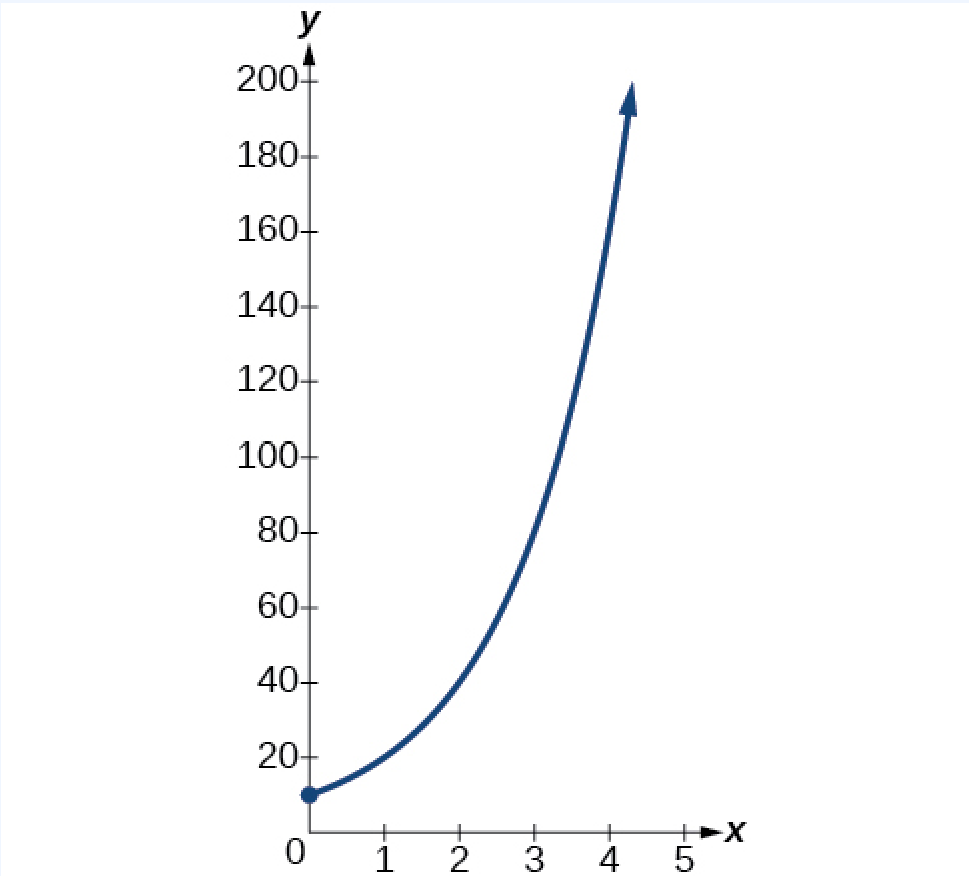


Figure 5. The graph of $y = 10e^{(\ln 2)t}$

Analysis

The population of bacteria after ten hours is 10,240. We could describe this amount is being of the order of magnitude 10^4. The population of bacteria after twenty hours is 10,485,760 which is of the order of magnitude 10^7, so we could say that the population has increased by three orders of magnitude in ten hours.

Half-Life

We now turn to exponential decay. One of the common terms associated with exponential decay, as stated above, is **half-life**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

To find the half-life of a function describing exponential decay, solve the following equation:

$$\frac{1}{2}A_0 = A_o e^{kt}$$

We find that the half-life depends only on the constant k and not on the starting quantity A_0.

The formula is derived as follows

$$\begin{aligned} \frac{1}{2}A_0 &= A_o e^{kt} \\ \frac{1}{2} &= e^{kt} && \text{Divide by } A_0. \\ \ln\left(\frac{1}{2}\right) &= kt && \text{Take the natural log.} \\ -\ln(2) &= kt && \text{Apply laws of logarithms.} \\ -\frac{\ln(2)}{k} &= t && \text{Divide by } k. \end{aligned}$$

Since t, the time, is positive, k must, as expected, be negative. This gives us the half-life formula

$$t = -\frac{\ln(2)}{k}$$

How To

Given the half-life, find the decay rate.

1. Write $A = A_o e^{kt}$.
2. Replace A by $\frac{1}{2}A_0$ and replace t by the given half-life.
3. Solve to find k. Express k as an exact value (do not round).

Note: *It is also possible to find the decay rate using* $k = -\frac{\ln(2)}{t}$.

Finding the Function that Describes Radioactive Decay

The half-life of carbon-14 is 5,730 years. Express the amount of carbon-14 remaining as a function of time, t.

Show Solution

This formula is derived as follows.

$$\begin{array}{ll} A=A_0e^{kt} & \text{The continuous growth formula.} \\ 0.5A_0=A_0e^{k\cdot 5730} & \text{Substitute the half-life for } t \text{ and } 0.5A_0 \text{ for } f\left(t\right). \\ 0.5=e^{5730k} & \text{Divide by } A_0. \\ \ln\left(0.5\right)=5730k & \text{Take the natural log of both sides.} \\ k=\frac{\ln(0.5)}{5730} & \text{Divide by the coefficient of } k. \\ A=A_0e^{\left(\frac{\ln(0.5)}{5730}\right)t} & \text{Substitute for } r \text{ in the continuous growth formula.} \end{array}$$

The function that describes this continuous decay is $f\left(t\right)=A_0e^{\left(\frac{\ln(0.5)}{5730}\right)t}$. We observe that the coefficient of t, $\frac{\ln(0.5)}{5730}\approx-1.209710^{-4}$ is negative, as expected in the case of exponential decay.

Try It

The half-life of plutonium-244 is 80,000,000 years. Find function gives the amount of carbon-14 remaining as a function of time, measured in years.

Show Solution

$f\left(t\right)=A_0e^{-0.0000000087t}$

Radiocarbon Dating

The formula for radioactive decay is important in radiocarbon dating, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about 1% error for plants or animals that died within the last 60,000 years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of 5,730 years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of 12 and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last 60,000 years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is 5,730 years, the formula for the amount of carbon-14 remaining after t years is

$$A \approx A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$$

where

- A is the amount of carbon-14 remaining
- A_0 is the amount of carbon-14 when the plant or animal began decaying.

This formula is derived as follows:

$$\begin{array}{ll} A = A_0 e^{kt} & \text{The continuous growth formula.} \\ 0.5A_0 = A_0 e^{k \cdot 5730} & \text{Substitute the half-life for } t \text{ and } 0.5A_0 \text{ for } f\left(t\right). \\ 0.5 = e^{5730k} & \text{Divide by } A_0. \\ \ln\left(0.5\right) = 5730k & \text{Take the natural log of both sides.} \\ k = \frac{\ln(0.5)}{5730} & \text{Divide by the coefficient of } k. \\ A = A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} & \text{Substitute for } r \text{ in the continuous growth formula.} \end{array}$$

To find the age of an object, we solve this equation for t :

$$t = \frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$$

Out of necessity, we neglect here the many details that a scientist takes into consideration when doing carbon-14 dating, and we only look at the basic formula. The ratio of carbon-14 to carbon-12 in the

atmosphere is approximately 0.0000000001%. Let r be the ratio of carbon-14 to carbon-12 in the organic artifact or fossil to be dated, determined by a method called liquid scintillation. From the equation $A \approx A_0 e^{-0.000121t}$ we know the ratio of the percentage of carbon-14 in the object we are dating to the percentage of carbon-14 in the atmosphere is $r = \frac{A}{A_0} \approx e^{-0.000121t}$. We solve this equation for t, to get
$t = \frac{\ln(r)}{-0.000121}$

Given the percentage of carbon-14 in an object, determine its age.

1. Express the given percentage of carbon-14 as an equivalent decimal, k.
2. Substitute for *k* in the equation $t = \frac{\ln(r)}{-0.000121}$ and solve for the age, t.

Finding the Age of a Bone

A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Show Solution

We substitute 20 for k in the equation and solve for t :

$$\begin{aligned} t &= \frac{\ln(r)}{-0.000121} && \text{Use the general form of the equation.} \\ &= \frac{\ln(0.20)}{-0.000121} && \text{Substitute for } r. \\ &\approx 13301 && \text{Round to the nearest year.} \end{aligned}$$

The bone fragment is about 13,301 years old.

Analysis

The instruments that measure the percentage of carbon-14 are extremely sensitive and, as we mention above, a scientist will need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about 1%, so this age should be given as 13,301 years $\pm$ 1% or 13,301 years $\pm$ 133 years.

Try It

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Show Solution
less than 230 years, 229.3157 to be exact

Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the doubling time.

Given the basic exponential growth equation $A = A_0e^{kt}$, doubling time can be found by solving for when the original quantity has doubled, that is, by solving $2A_0 = A_0e^{kt}$.

The formula is derived as follows:

$$\begin{aligned} 2A_0 &= A_0e^{kt} & \\ 2 &= e^{kt} & \text{Divide by } A_0. \\ \ln 2 &= kt & \text{Take the natural logarithm.} \\ t &= \frac{\ln 2}{k} & \text{Divide by the coefficient of } t. \end{aligned}$$

Thus the doubling time is

$$t = \frac{\ln 2}{k}$$

Finding a Function That Describes Exponential Growth

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

Show Solution

The formula is derived as follows:

$t=\frac{\ln 2}{k}$	The doubling time formula.
$2=\frac{\ln 2}{k}$	Use a doubling time of two years.
$k=\frac{\ln 2}{2}$	Multiply by k and divide by 2.
$A\ =A_0e^{\frac{\ln 2}{2}t}$	Substitute k into the continuous growth formula.

The function is $A=A_0e^{\frac{\ln 2}{2}t}$.

Try It

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

Show Solution

$f\left(t\right)=A_0e^{\frac{\ln 2}{3}t}$

Using Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a vertical shift of the generic exponential decay function. This translation leads to Newton's Law of Cooling, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

$$T\left(t\right)=a{e}^{kt}+{T}_{s}$$

This formula is derived as follows:

$$\begin{array}{ll}T\left(t\right)=A{b}^{ct}+{T}_{s} & \\ T\left(t\right)=A{e}^{\mathrm{ln}\left({b}^{ct}\right)}+{T}_{s} & \text{Laws of logarithms.}\\ T\left(t\right)=A{e}^{ct\mathrm{ln}b}+{T}_{s} & \text{Laws of logarithms.}\\ T\left(t\right)=A{e}^{kt}+{T}_{s} & \text{Rename the constant }c\mathrm{ln}b,\text{ calling it }k.\end{array}$$

Newton's Law of Cooling

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T\left(t\right)=A{e}^{kt}+{T}_{s}$$

where

- t is time
- A is the difference between the initial temperature of the object and the surroundings
- k is a constant, the continuous rate of cooling of the object

How To

Given a set of conditions, apply Newton's Law of Cooling.

1. Set T_s equal to the *y*-coordinate of the horizontal asymptote (usually the ambient temperature).
2. Substitute the given values into the continuous growth formula $T\left(t\right)=A{e}^{kt}+{T}_{s}$ to find the parameters A and k.
3. Substitute in the desired time to find the temperature or the desired temperature to find the time.

Using Newton's Law of Cooling

A cheesecake is taken out of the oven with an ideal internal temperature of $165\mathrm{F}$, and is placed into a $35F$ refrigerator. After 10 minutes, the cheesecake has cooled to $150\mathrm{F}$. If we must wait until the cheesecake has cooled to $70\mathrm{F}$ before we eat it, how long will we have to wait?

Show Solution

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation

$$T\left(t\right)=Ae^{kt}+35$$

We know the initial temperature was 165, so $T\left(0\right)=165.$

$$\begin{array}{ll} 165=Ae^{k0}+35 & \text{Substitute } \left(0,165\right). \\ A=130 & \text{Solve for } A. \end{array}$$

We were given another data point, $T\left(10\right)=150$, which we can use to solve for k.

$$\begin{array}{ll} 150=130e^{k10}+35 & \text{Substitute } (10,\ 150). \\ 115=130e^{k10} & \text{Subtract } 35. \\ \frac{115}{130}=e^{10k} & \text{Divide by } 130. \\ \ln\left(\frac{115}{130}\right)=10k & \text{Take the natural log of both sides.} \\ k=\frac{\ln\left(\frac{115}{130}\right)}{10}=-0.0123 & \text{Divide by the coefficient of } k. \end{array}$$

This gives us the equation for the cooling of the cheesecake: $T\left(t\right)=130e^{-0.0123t}+35.$

Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$\begin{array}{rl} 70=130e^{-0.0123t}+35 & \text{Substitute in 70 for } T\left(t\right). \\ 35=130e^{-0.0123t} & \text{Subtract } 35. \\ \frac{35}{130}=e^{-0.0123t} & \text{Divide by } 130. \\ \ln\left(\frac{35}{130}\right)=-0.0123t & \text{Take the natural log of both sides} \\ t=\frac{\ln\left(\frac{35}{130}\right)}{-0.0123}\approx 106.68 & \text{Divide by the coefficient of } t. \end{array}$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to $70\mathrm{F}$.

Try It

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Show Solution
6.026 hours

Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the exponential growth model is still useful over a short term, before approaching the limiting value.

The logistic growth model is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the carrying capacity. For constants a, b, and c, the logistic growth of a population over time x is represented by the model

$$f\left(x\right)=\frac{c}{1+ae^{-bx}}$$

The graph in (Figure) shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

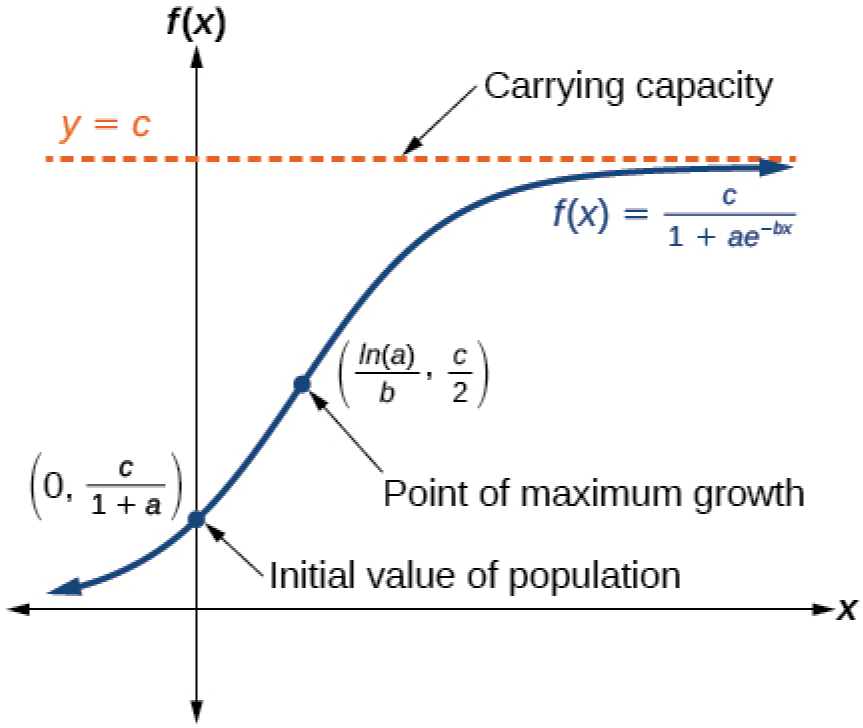


Figure 6.

Logistic Growth

The logistic growth model is

$f(x) = \frac{c}{1+ae^{-bx}}$

where

- $\frac{c}{1+a}$ is the initial value
- c is the *carrying capacity*, or *limiting value*
- b is a constant determined by the rate of growth.

Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors:

The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030.$ Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

Show Solution

We substitute the given data into the logistic growth model

$f\left(x\right) = \frac{c}{1+ae^{-bx}}$

Because at most 1,000 people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000.$ To find $a,$ we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1,$ from which it follows that $a = 999.$ This model predicts that, after ten days, the number of people who have had the flu is $f\left(x\right) = \frac{1000}{1+999e^{-0.6030x}} \approx 293.8.$ Because the actual number must be a whole number (a person has either had the flu or not) we round to 294. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000.$

Analysis

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in (Figure) gives a good picture of how this model fits the data.

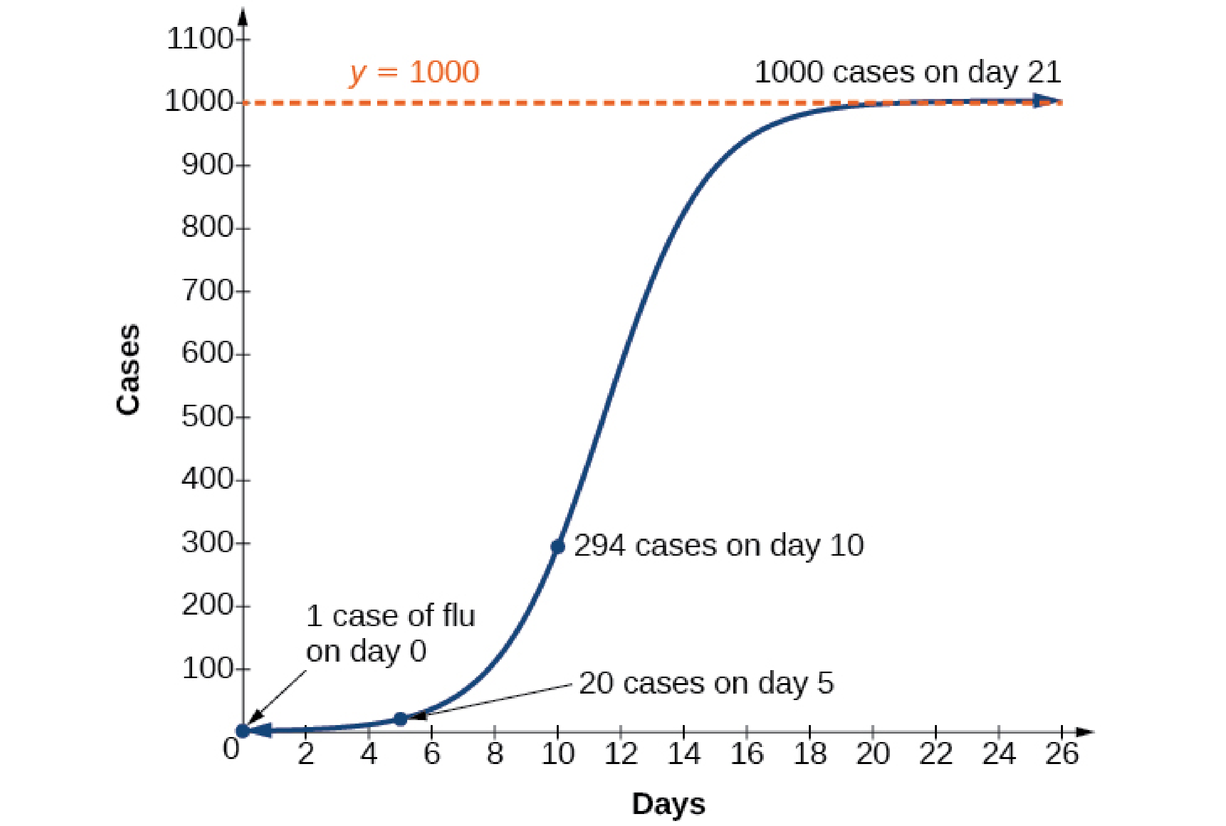


Figure 7. The graph of $f\left(x\right)=\frac{1000}{1+999e^{-0.6030x}}$

Try It

Using the model in (Figure), estimate the number of cases of flu on day 15.

Show Solution

895 cases on day 15

Choosing an Appropriate Model for Data

Now that we have discussed various mathematical models, we need to learn how to choose the appropriate model for the raw data we have. Many factors influence the choice of a mathematical model, among which are experience, scientific laws, and patterns in the data itself. Not all data can be described by elementary functions. Sometimes, a function is chosen that approximates the data over a given interval. For instance, suppose data were gathered on the number of homes bought in the United States from the years 1960 to 2013. After plotting these data in a scatter plot, we notice that the shape of the data from the years 2000 to 2013 follow a logarithmic curve. We could restrict the interval from 2000 to 2010, apply regression analysis using a logarithmic model, and use it to predict the number of home buyers for the year 2015.

Three kinds of functions that are often useful in mathematical models are linear functions, exponential functions, and logarithmic functions. If the data lies on a straight line, or seems to lie approximately along a straight line, a linear model may be best. If the data is non-linear, we often consider an exponential or logarithmic model, though other models, such as quadratic models, may also be considered.

In choosing between an exponential model and a logarithmic model, we look at the way the data curves. This is called the concavity. If we draw a line between two data points, and all (or most) of the data between those two points lies above that line, we say the curve is concave down. We can think of it as a bowl that bends downward and therefore cannot hold water. If all (or most) of the data between those two points lies below the line, we say the curve is concave up. In this case, we can think of a bowl that bends upward and can therefore hold water. An exponential curve, whether rising or falling, whether representing growth or decay, is always concave up away from its horizontal asymptote. A logarithmic curve is always concave away from its vertical asymptote. In the case of positive data, which is the most common case, an exponential curve is always concave up, and a logarithmic curve always concave down.

A logistic curve changes concavity. It starts out concave up and then changes to concave down beyond a certain point, called a point of inflection.

After using the graph to help us choose a type of function to use as a model, we substitute points, and solve to find the parameters. We reduce round-off error by choosing points as far apart as possible.

Choosing a Mathematical Model

Does a linear, exponential, logarithmic, or logistic model best fit the values listed in (Figure)? Find the model, and use a graph to check your choice.

x	1	2	3	4	5	6	7	8	9
y	0	1.386	2.197	2.773	3.219	3.584	3.892	4.159	4.394

Show Solution

First, plot the data on a graph as in (Figure). For the purpose of graphing, round the data to two significant digits.

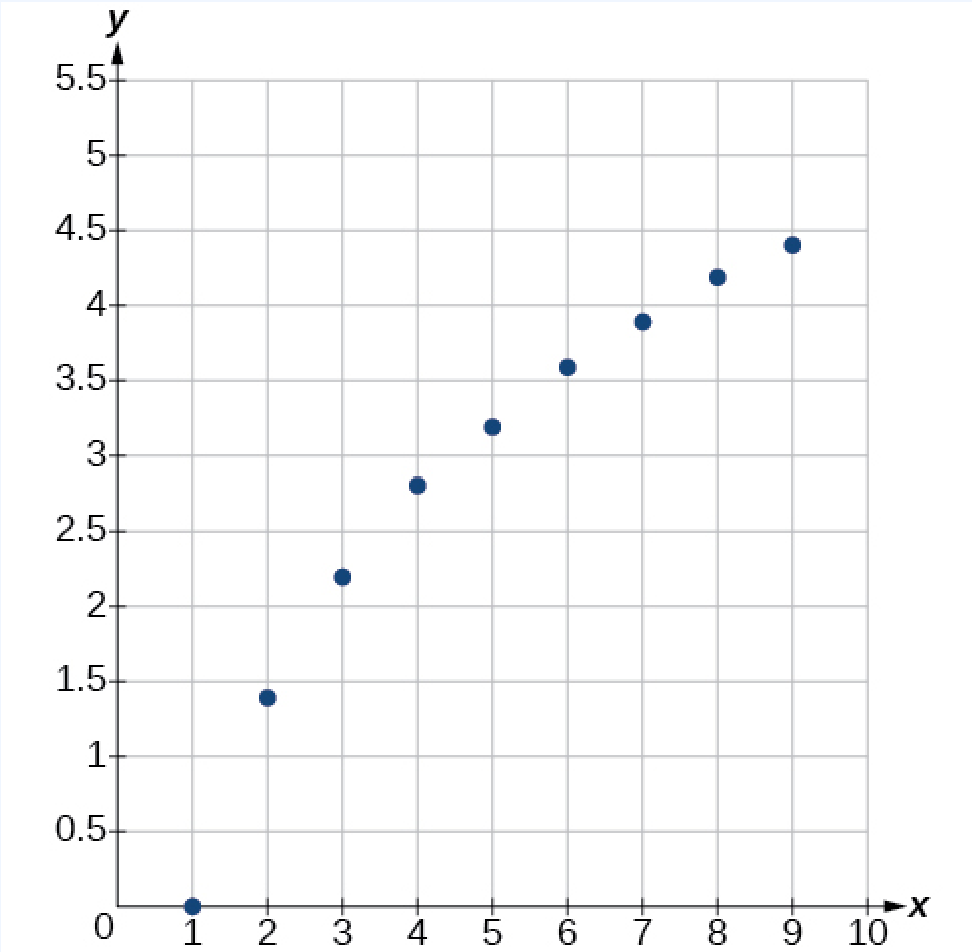


Figure 8.

Clearly, the points do not lie on a straight line, so we reject a linear model. If we draw a line between any two of the points, most or all of the points between those two points lie above the line, so the graph is concave down, suggesting a logarithmic model. We can try $y = a\ln\left(bx\right)$.

Plugging in the first point, $(1{,}0)$, gives $0 = a\ln b$. We reject the case that $a = 0$ (if it were, all outputs would be 0), so we know $\ln(b) = 0$. Thus $b = 1$ and $y = a\ln(\mathrm{x})$. Next we can use the point $(9{,}4.394)$ to solve for a :

$$\begin{aligned} y &= a\ln(x) \\ 4.394 &= a\ln(9) \\ a &= \frac{4.394}{\ln(9)} \end{aligned}$$

Because $a = \frac{4.394}{\ln(9)} \approx 2,$ an appropriate model for the data is $y = 2\ln(x)$.

To check the accuracy of the model, we graph the function together with the given points as in (Figure).

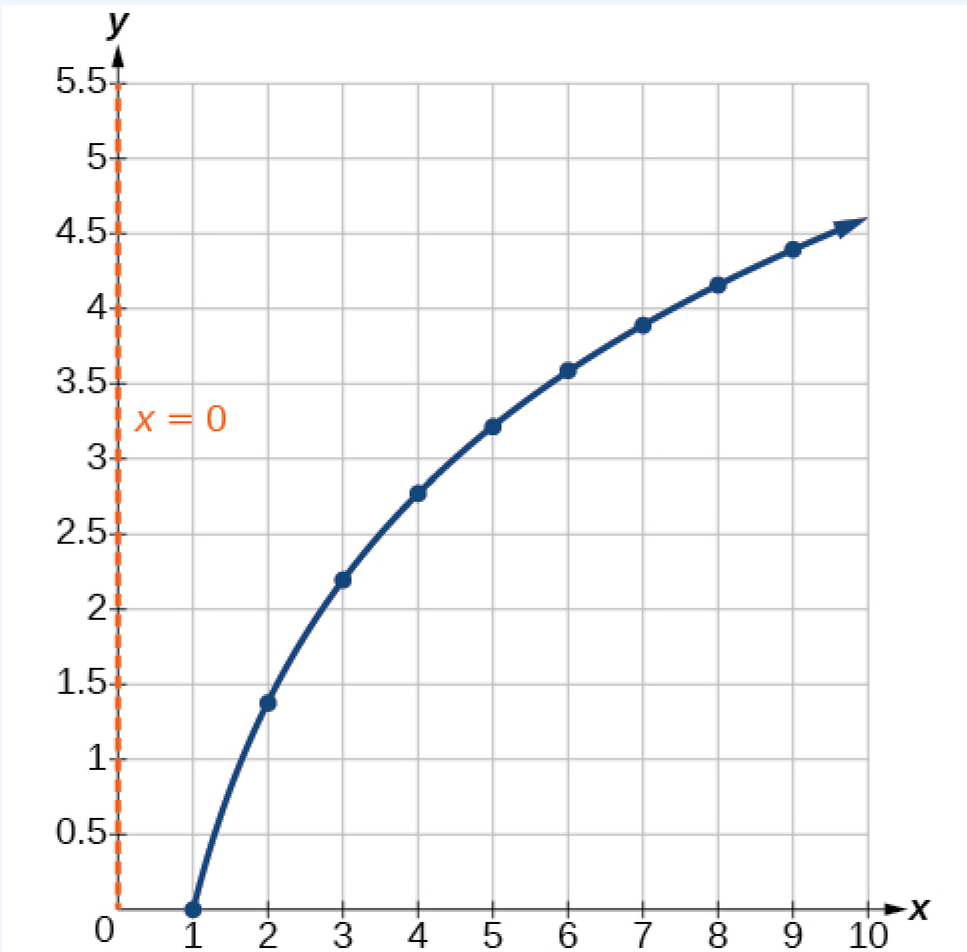


Figure 9. The graph of $y = 2\ln x$.

We can conclude that the model is a good fit to the data.

Compare (Figure) to the graph of $y = \ln\left(x^2\right)$ shown in (Figure).

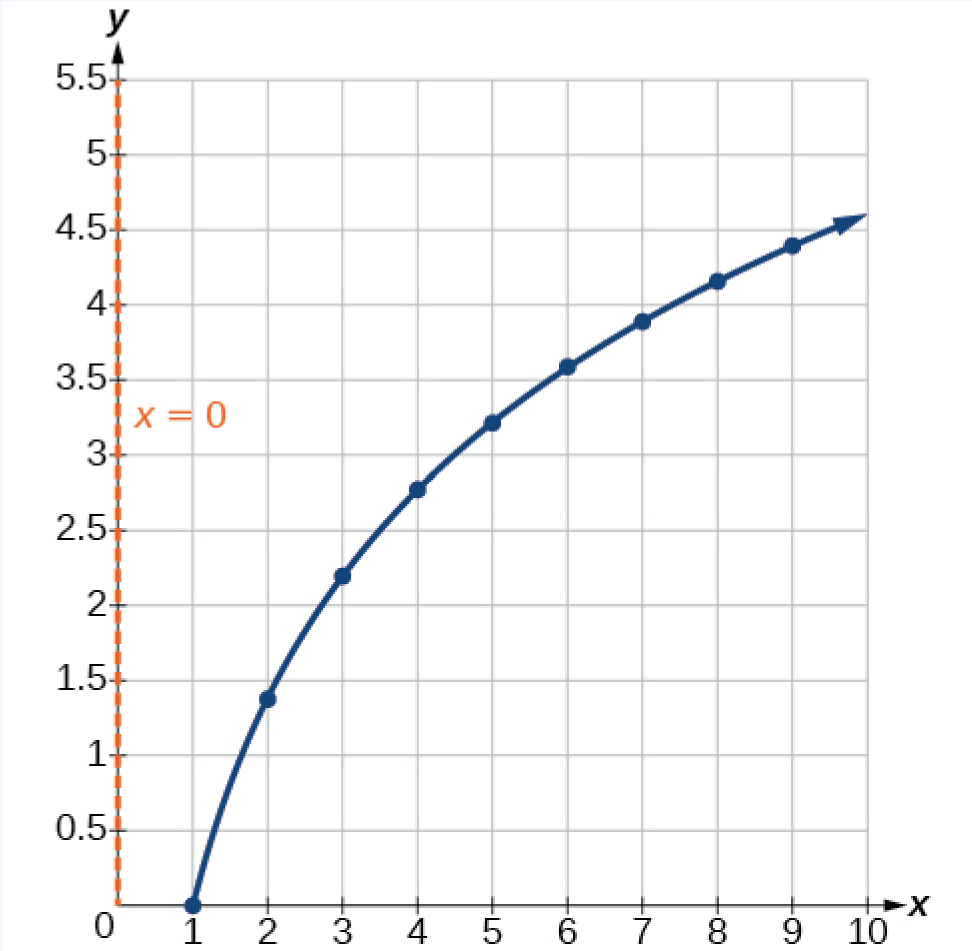


Figure 10. The graph of $y = \ln\left(x^2\right)$

The graphs appear to be identical when $x > 0.$ A quick check confirms this conclusion: $y = \ln\left(x^2\right) = 2\ln\left(x\right)$ for $x > 0.$

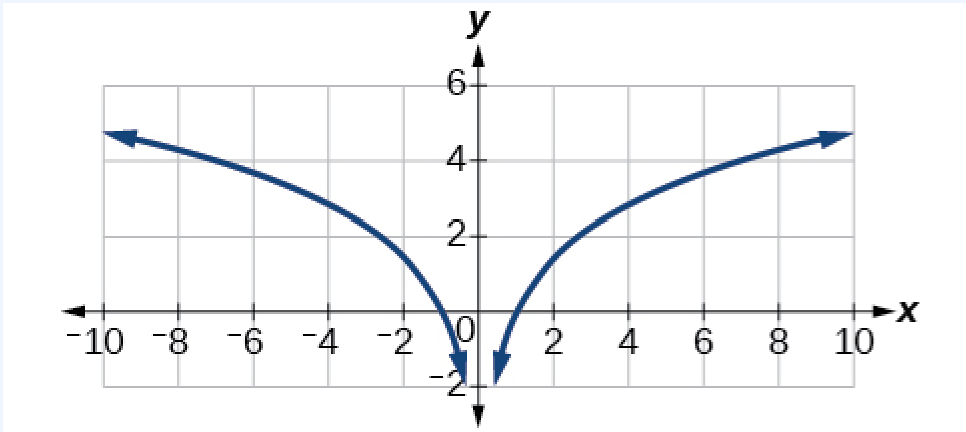


Figure 11.

However, if $x < 0$, the graph of $y = \ln\left(x^2\right)$ includes a "extra" branch, as shown in (Figure). This occurs because, while $y = 2\ln\left(x\right)$ cannot have negative values in the domain (as such values would force the argument to be negative), the function $y = \ln\left(x^2\right)$ can have negative domain values.

Try It

Does a linear, exponential, or logarithmic model best fit the data in (Figure)? Find the model.

x	1	2	3	4	5	6	7	8	9
y	3.297	5.437	8.963	14.778	24.365	40.172	66.231	109.196	180.034

Show Solution

Exponential. $y = 2e^{0.5x}$.

Expressing an Exponential Model in Base *e*

While powers and logarithms of any base can be used in modeling, the two most common bases are 10 and e. In science and mathematics, the base e is often preferred. We can use laws of exponents and laws of logarithms to change any base to base e.

How To

Given a model with the form $y = ab^x$, change it to the form $y = A_0e^{kx}$.

1. Rewrite $y = ab^x$ as $y = ae^{\ln(b^x)}$.
2. Use the power rule of logarithms to rewrite y as $y = ae^{x\ln(b)} = ae^{\ln(b)x}$.
3. Note that $a = A_0$ and $k = \ln(b)$ in the equation $y = A_0e^{kx}$.

Changing to base *e*

Change the function $y = 2.5(3.1)^x$ so that this same function is written in the form $y = A_0e^{kx}$.

Show Solution

The formula is derived as follows

$$
\begin{aligned}
y &= 2.5(3.1)^x \\
&= 2.5e^{\ln(3.1^x)} && \text{Insert exponential and its inverse.} \\
&= 2.5e^{x\ln 3.1} && \text{Laws of logs.} \\
&= 2.5e^{(\ln 3.1)x} && \text{Commutative law of multiplication}
\end{aligned}
$$

Try It

Change the function $y = 3(0.5)^x$ to one having e as the base.

Show Solution

$y = 3e^{(\ln 0.5)x}$

Access these online resources for additional instruction and practice with exponential and logarithmic models.

- Logarithm Application – pH
- Exponential Model – Age Using Half-Life
- Newton's Law of Cooling
- Exponential Growth Given Doubling Time
- Exponential Growth – Find Initial Amount Given Doubling Time

Key Equations

Half-life formula	If $A = A_0 e^{kt}$, $k < 0$, the half-life is $t = -\frac{\ln(2)}{k}$.
Carbon-14 dating	$t = \frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$.
Doubling time formula	If $A = A_0 e^{kt}$, $k > 0$, the doubling time is $t = \frac{\ln 2}{k}$
Newton's Law of Cooling	$T(t) = Ae^{kt} + T_s$, where T_s is the ambient temperature, $A = T(0) - T_s$, and k is the continuous rate of cooling.

Key Concepts

- The basic exponential function is $f(x) = ab^x$. If $b > 1$, we have exponential growth; if $0 < b < 1$, we have exponential decay.
- We can also write this formula in terms of continuous growth as $A = A_0 e^{kx}$, where A_0 is the starting value. If A_0 is positive, then we have exponential growth when $k > 0$ and exponential decay when $k < 0$. See (Figure).
- In general, we solve problems involving exponential growth or decay in two steps. First, we set up a model and use the model to find the parameters. Then we use the formula with these parameters to predict growth and decay. See (Figure).
- We can find the age, t, of an organic artifact by measuring the amount, k, of carbon-14 remaining in the artifact and using the formula $t = \frac{\ln(k)}{-0.000121}$ to solve for t. See (Figure).
- Given a substance's doubling time or half-time, we can find a function that represents its exponential growth or decay. See (Figure).
- We can use Newton's Law of Cooling to find how long it will take for a cooling object to reach a desired temperature, or to find what temperature an object will be after a given time. See (Figure).
- We can use logistic growth functions to model real-world situations where the rate of growth changes over time, such as population growth, spread of disease, and spread of rumors. See (Figure).
- We can use real-world data gathered over time to observe trends. Knowledge of linear, exponential, logarithmic, and logistic graphs help us to develop models that best fit our data. See (Figure).
- Any exponential function with the form $y = ab^x$ can be rewritten as an equivalent

exponential function with the form $y = A_0e^{kx}$ where $k = \ln b$. See (Figure).

Section Exercises

Verbal

1. With what kind of exponential model would *half-life* be associated? What role does half-life play in these models?

Show Solution
Half-life is a measure of decay and is thus associated with exponential decay models. The half-life of a substance or quantity is the amount of time it takes for half of the initial amount of that substance or quantity to decay.

2. What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.

3. With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?

Show Solution
Doubling time is a measure of growth and is thus associated with exponential growth models. The doubling time of a substance or quantity is the amount of time it takes for the initial amount of that substance or quantity to double in size.

4. Define Newton's Law of Cooling. Then name at least three real-world situations where Newton's Law of Cooling would be applied.

5. What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

Show Solution

An order of magnitude is the nearest power of ten by which a quantity exponentially grows. It is also an approximate position on a logarithmic scale; Sample response: Orders of magnitude are useful when making comparisons between numbers that differ by a great amount. For example, the mass of Saturn is 95 times greater than the mass of Earth. This is the same as saying that the mass of Saturn is about 10^2 times, or *2 orders of magnitude* greater, than the mass of Earth.

Numeric

6. The temperature of an object in degrees Fahrenheit after *t* minutes is represented by the equation $T\left(t\right)=68{e}^{-0.0174t}+72.$ To the nearest degree, what is the temperature of the object after one and a half hours?

For the following exercises, use the logistic growth model $f\left(x\right)=\frac{150}{1+8{e}^{-2x}}.$

7. Find and interpret $f\left(0\right)$. Round to the nearest tenth.

Show Solution

$f\left(0\right)\approx 16.7;$ The amount initially present is about 16.7 units.

8. Find and interpret $f\left(4\right)$. Round to the nearest tenth.

9. Find the carrying capacity.

Show Solution

150

10. Graph the model.

11. Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

x	$f(x)$
–2	0.694
–1	0.833
0	1
1	1.2
2	1.44
3	1.728
4	2.074
5	2.488

Show Solution
exponential; $f(x) = 1.2^x$

12. Rewrite $f(x) = 1.68(0.65)^x$ as an exponential equation with base e to five significant digits.

Technology

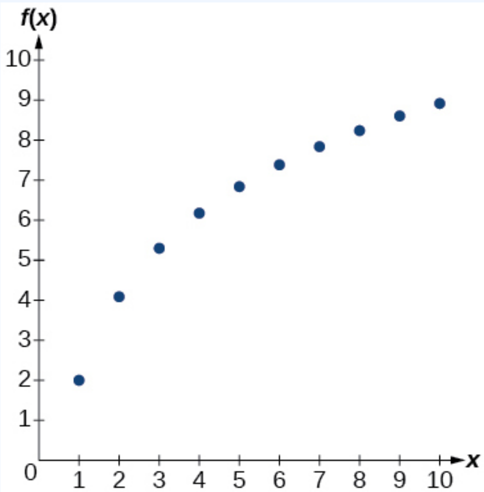
For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

13.

x	$f(x)$
1	2
2	4.079
3	5.296
4	6.159
5	6.828
6	7.375
7	7.838
8	8.238
9	8.592
10	8.908

Show Solution

logarithmic



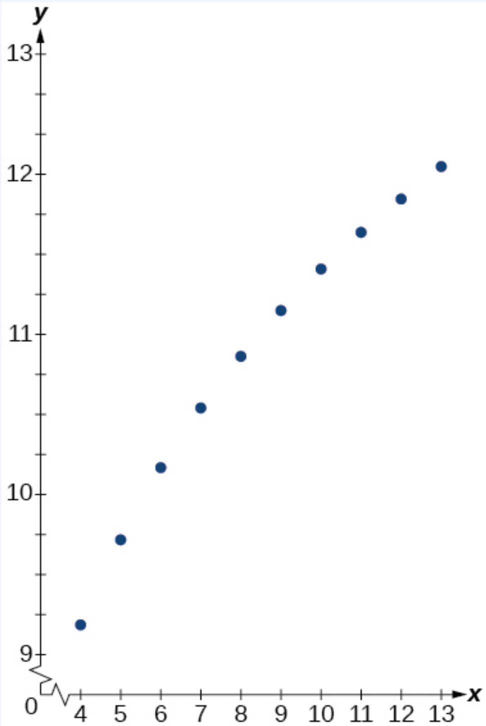
14.

x	$f(x)$
1	2.4
2	2.88
3	3.456
4	4.147
5	4.977
6	5.972
7	7.166
8	8.6
9	10.32
10	12.383

15.

x	$f(x)$
4	9.429
5	9.972
6	10.415
7	10.79
8	11.115
9	11.401
10	11.657
11	11.889
12	12.101
13	12.295

Show Solution
logarithmic



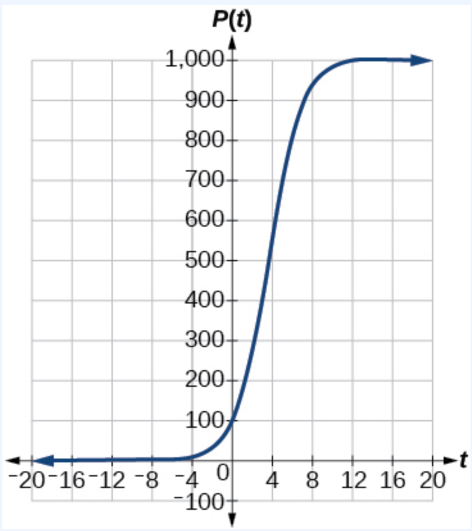
16.

x	$f(x)$
1.25	5.75
2.25	8.75
3.56	12.68
4.2	14.6
5.65	18.95
6.75	22.25
7.25	23.75
8.6	27.8
9.25	29.75
10.5	33.5

For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in t years is modeled by the equation $P\left(t\right)=\frac{1000}{1+9e^{-0.6t}}$.

17. Graph the function.

Show Solution



18. What is the initial population of fish?

19. To the nearest tenth, what is the doubling time for the fish population?

Show Solution
about 1.4 years

20. To the nearest whole number, what will the fish population be after 2 years?

21. To the nearest tenth, how long will it take for the population to reach 900?

Show Solution

about 7.3 years

22. What is the carrying capacity for the fish population? Justify your answer using the graph of P.

Extensions

23. A substance has a half-life of 2.045 minutes. If the initial amount of the substance was 132.8 grams, how many half-lives will have passed before the substance decays to 8.3 grams? What is the total time of decay?

Show Solution
4 half-lives; 8.18 minutes

24. The formula for an increasing population is given by $P\left(t\right)={P}_{0}{e}^{rt}$ where ${P}_{0}$ is the initial population and $r>0.$ Derive a general formula for the time *t* it takes for the population to increase by a factor of *M*.

25. Recall the formula for calculating the magnitude of an earthquake, $M=\frac{2}{3}\log\left(\frac{S}{{S}_{0}}\right)$. Show each step for solving this equation algebraically for the seismic moment $S.$

Show Solution

$$M=\frac{2}{3}\log\left(\frac{S}{{S}_{0}}\right)$$
$$\log\left(\frac{S}{{S}_{0}}\right)=\frac{3}{2}M$$
$$\frac{S}{{S}_{0}}={10}^{\frac{3M}{2}}$$
$$S={S}_{0}{10}^{\frac{3M}{2}}$$

26. What is the *y*-intercept of the logistic growth model $y=\frac{c}{1+a{e}^{-rx}}$? Show the steps for calculation. What does this point tell us about the population?

27. Prove that $b^x = e^{x \ln(b)}$ for positive $b \neq 1.$

Show Solution

Let $y = b^x$ for some non-negative real number b such that $b \neq 1.$ Then,

$\ln(y) = \ln(b^x)$
$\ln(y) = x \ln(b)$
$e^{\ln(y)} = e^{x \ln(b)}$
$y = e^{x \ln(b)}$

Real-World Applications

For the following exercises, use this scenario: A doctor prescribes 125 milligrams of a therapeutic drug that decays by about 30% each hour.

28. To the nearest hour, what is the half-life of the drug?

29. Write an exponential model representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 3 hours. Round to the nearest milligram.

Show Solution

$A = 125e^{(-0.3567t)}; A \approx 43$ mg

30. Using the model found in the previous exercise, find $f(10)$ and interpret the result. Round to the nearest hundredth.

For the following exercises, use this scenario: A tumor is injected with 0.5 grams of Iodine-125, which has a decay rate of 1.15 per day.

31. To the nearest day, how long will it take for half of the Iodine-125 to decay?

Show Solution
about 60 days

32. Write an exponential model representing the amount of Iodine-125 remaining in the tumor after t days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after 60 days. Round to the nearest tenth of a gram.

33. A scientist begins with 250 grams of a radioactive substance. After 250 minutes, the sample has decayed to 32 grams. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?

Show Solution
$f\left(t\right)=250e^{\left(-0.00914t\right)};$ half-life: about 76 minutes

34. The half-life of Radium-226 is 1590 years. What is the annual decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

35. The half-life of Erbium-165 is 10.4 hours. What is the hourly decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

Show Solution
$r\approx-0.0667,$ So the hourly decay rate is about 6.67

36. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is 5730 years.)

37. A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five significant digits, write

an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

Show Solution

$f\left(t\right)=1350{e}^{\left(0.03466t\right)}$; after 3 hours: $P\left(180\right)\approx 691,200$

For the following exercises, use this scenario: A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.

38. To the nearest whole number, what was the initial population in the culture?

39. Rounding to six significant digits, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

Show Solution

$f\left(t\right)=256{e}^{\left(0.068110t\right)}$; doubling time: about 10 minutes

For the following exercises, use this scenario: A pot of boiling soup with an internal temperature of 100 Fahrenheit was taken off the stove to cool in a 69 F room. After fifteen minutes, the internal temperature of the soup was $95\text{ F}.$

40. Use Newton's Law of Cooling to write a formula that models this situation.

41. To the nearest minute, how long will it take the soup to cool to $80\text{ F}?$

Show Solution

about 88 minutes

42. To the nearest degree, what will the temperature be after 2 and a half hours?

For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of 165F and is allowed to cool in a 75F room. After half an hour, the internal temperature of the turkey is 145F.

43. Write a formula that models this situation.

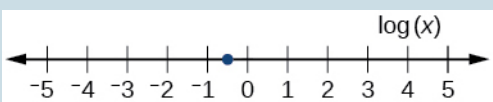
Show Solution
$T\left(t\right)=90e^{\left(-0.008377t\right)}+75,$ where t is in minutes.

44. To the nearest degree, what will the temperature be after 50 minutes?

45. To the nearest minute, how long will it take the turkey to cool to $110\ \text{F}$?

Show Solution
about 113 minutes

For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.

46. 

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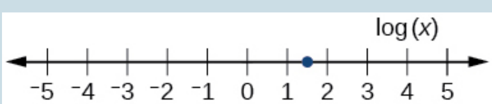
47. 

Show Solution
$\log\left(x\right)=1.5;\ x\approx 31.623$

48. Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\frac{W}{m^2}$, Vacuum: $10^{-4}\frac{W}{m^2}$, Jet: $10^{2}\frac{W}{m^2}$

49. Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

Show Solution
MMS magnitude: 5.82

For the following exercises, use this scenario: The equation $N\left(t\right) = \frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after t days.

50. How many people started the rumor?

51. To the nearest whole number, how many people will have heard the rumor after 3 days?

Show Solution
$N\left(3\right) \approx 71$

52. As t increases without bound, what value does $N\left(t\right)$ approach? Interpret your answer.

For the following exercise, choose the correct answer choice.

53. A doctor and injects a patient with 13 milligrams of radioactive dye that decays exponentially. After 12 minutes, there are 4.75 milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?

A. $f\left(t\right) = 13(0.0805)^t$
B. $f\left(t\right) = 13e^{0.9195t}$
C. $f\left(t\right) = 13e^{(-0.0839t)}$

D. $f\left(t\right)=\frac{4.75}{1+13e^{-0.83925t}}$

Show Solution

C

Glossary

carrying capacity
: in a logistic model, the limiting value of the output

doubling time
: the time it takes for a quantity to double

half-life
: the length of time it takes for a substance to exponentially decay to half of its original quantity

logistic growth model
: a function of the form $f\left(x\right)=\frac{c}{1+ae^{-bx}}$ where $\frac{c}{1+a}$ is the initial value, c is the carrying capacity, or limiting value, and b is a constant determined by the rate of growth

Newton's Law of Cooling
: the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

order of magnitude
: the power of ten, when a number is expressed in scientific notation, with one non-zero digit to the left of the decimal

CHAPTER 9.9: FITTING EXPONENTIAL MODELS TO DATA

Learning Objectives

In this section, you will:

- Build an exponential model from data.
- Build a logarithmic model from data.
- Build a logistic model from data.

In previous sections of this chapter, we were either given a function explicitly to graph or evaluate, or we were given a set of points that were guaranteed to lie on the curve. Then we used algebra to find the equation that fit the points exactly. In this section, we use a modeling technique called *regression analysis* to find a curve that models data collected from real-world observations. With regression analysis, we don't expect all the points to lie perfectly on the curve. The idea is to find a model that best fits the data. Then we use the model to make predictions about future events.

Do not be confused by the word *model*. In mathematics, we often use the terms *function*, *equation*, and *model* interchangeably, even though they each have their own formal definition. The term *model* is typically used to indicate that the equation or function approximates a real-world situation.

We will concentrate on three types of regression models in this section: exponential, logarithmic, and logistic. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review. Take a moment to rethink each of these functions, reflect on the work we've done so far, and then explore the ways regression is used to model real-world phenomena.

Building an Exponential Model from Data

As we've learned, there are a multitude of situations that can be modeled by exponential functions, such as investment growth, radioactive decay, atmospheric pressure changes, and temperatures of a cooling object.

What do these phenomena have in common? For one thing, all the models either increase or decrease as time moves forward. But that's not the whole story. It's the *way* data increase or decrease that helps us determine whether it is best modeled by an exponential equation. Knowing the behavior of exponential functions in general allows us to recognize when to use exponential regression, so let's review exponential growth and decay.

Recall that exponential functions have the form $y = ab^x$ or $y = A_0e^{kx}$. When performing regression analysis, we use the form most commonly used on graphing utilities, $y = ab^x$. Take a moment to reflect on the characteristics we've already learned about the exponential function $y = ab^x$ (assume $a > 0$):

- b must be greater than zero and not equal to one.
- The initial value of the model is $y = a$.
    - If $b > 1$, the function models exponential growth. As x increases, the outputs of the model increase slowly at first, but then increase more and more rapidly, without bound.
    - If $0 < b < 1$, the function models exponential decay. As x increases, the outputs for the model decrease rapidly at first and then level off to become asymptotic to the *x*-axis. In other words, the outputs never become equal to or less than zero.

As part of the results, your calculator will display a number known as the *correlation coefficient*, labeled by the variable r, or r^2. (You may have to change the calculator's settings for these to be shown.) The values are an indication of the "goodness of fit" of the regression equation to the data. We more commonly use the value of r^2 instead of r, but the closer either value is to 1, the better the regression equation approximates the data.

Exponential Regression

Exponential regression is used to model situations in which growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero. We use the command "ExpReg" on a graphing utility to fit an exponential function to a set of data points. This returns an equation of the form, $y = ab^x$

Note that:

- b must be non-negative.
- when $b > 1$, we have an exponential growth model.
- when $0 < b < 1$, we have an exponential decay model.

How To

Given a set of data, perform exponential regression using a graphing utility.

1. Use the STAT then EDIT menu to enter given data.
   a. Clear any existing data from the lists.
   b. List the input values in the L1 column.
   c. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   a. Use ZOOM [9] to adjust axes to fit the data.
   b. Verify the data follow an exponential pattern.
3. Find the equation that models the data.
   a. Select "ExpReg" from the STAT then CALC menu.
   b. Use the values returned for *a* and *b* to record the model, $y = ab^x$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

Using Exponential Regression to Fit a Model to Data

In 2007, a university study was published investigating the crash risk of alcohol impaired driving. Data from 2,871 crashes were used to measure the association of a person's blood alcohol level (BAC) with the risk of being in an accident. (Figure) shows results from the study[1] . The *relative risk* is a measure of how many times more likely a person is to crash. So, for example, a person with a BAC of 0.09 is 3.54 times as likely to crash as a person who has not been drinking alcohol.

1. •Source: Indiana University Center for Studies of Law in Action, 2007

BAC	0	0.01	0.03	0.05	0.07	0.09
Relative Risk of Crashing	1	1.03	1.06	1.38	2.09	3.54
BAC	0.11	0.13	0.15	0.17	0.19	0.21
Relative Risk of Crashing	6.41	12.6	22.1	39.05	65.32	99.78

a. Let x represent the BAC level, and let y represent the corresponding relative risk. Use exponential regression to fit a model to these data.
b. After 6 drinks, a person weighing 160 pounds will have a BAC of about $0.16.$ How many times more likely is a person with this weight to crash if they drive after having a 6-pack of beer? Round to the nearest hundredth.

Show Solution

a. Using the STAT then EDIT menu on a graphing utility, list the BAC values in L1 and the relative risk values in L2. Then use the STATPLOT feature to verify that the scatterplot follows the exponential pattern shown in (Figure):

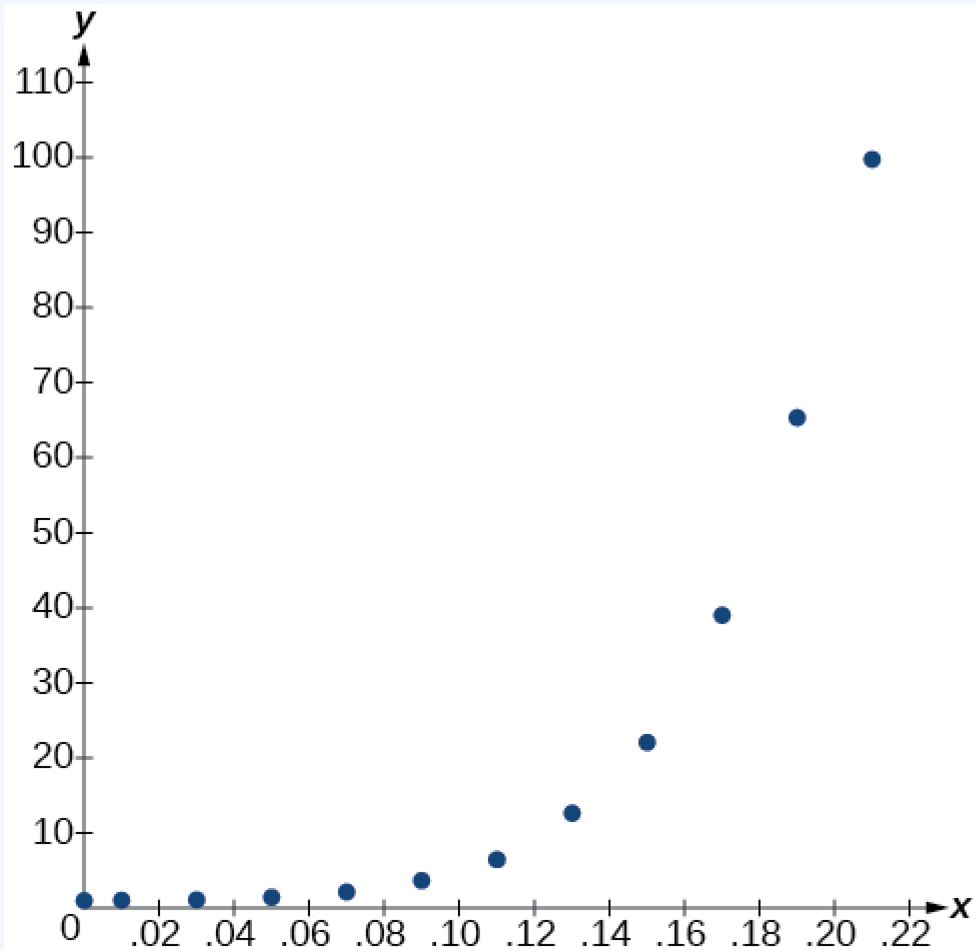


Figure 1.

Use the “ExpReg” command from the STAT then CALC menu to obtain the exponential model,

$$y = 0.58304829(2.20720213\text{E}10)^x$$

Converting from scientific notation, we have:

$$y = 0.58304829(22{,}072{,}021{,}300)^x$$

Notice that $r^2 \approx 0.97$ which indicates the model is a good fit to the data. To see this, graph the model in the same window as the scatterplot to verify it is a good fit as shown in (Figure):

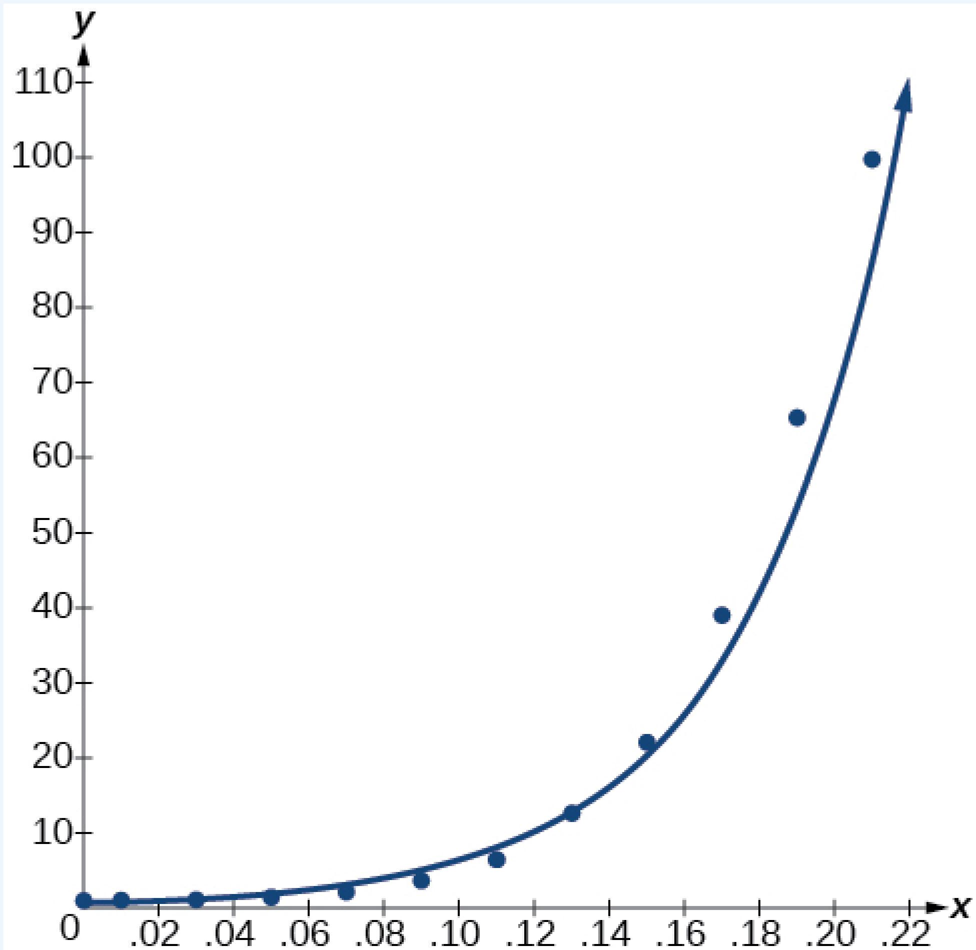


Figure 2.

b. Use the model to estimate the risk associated with a BAC of 0.16. Substitute 0.16 for x in the model and solve for y.

$$\begin{aligned} y &= 0.58304829(22{,}072{,}021{,}300)^x && \text{Use the regression model found in part (a).} \\ &= 0.58304829(22{,}072{,}021{,}300)^{0.16} && \text{Substitute 0.16 for } x. \\ &\approx 26.35 && \text{Round to the nearest hundredth.} \end{aligned}$$

If a 160-pound person drives after having 6 drinks, he or she is about 26.35 times more likely to crash than if driving while sober.

Try It

(Figure) shows a recent graduate's credit card balance each month after graduation.

Month	1	2	3	4	5	6	7	8
Debt ($)	620.00	761.88	899.80	1039.93	1270.63	1589.04	1851.31	2154.92

a. Use exponential regression to fit a model to these data.
b. If spending continues at this rate, what will the graduate's credit card debt be one year after graduating?

Show Solution

a. The exponential regression model that fits these data is $y = 522.88585984(1.19645256)^x$.
b. If spending continues at this rate, the graduate's credit card debt will be $4,499.38 after one year.

Is it reasonable to assume that an exponential regression model will represent a situation indefinitely?

No. Remember that models are formed by real-world data gathered for regression. It is usually reasonable to make estimates within the interval of original observation (interpolation). However, when a model is used to make predictions, it is important to use reasoning skills to determine whether the model makes sense for inputs far beyond the original observation interval (extrapolation).

Building a Logarithmic Model from Data

Just as with exponential functions, there are many real-world applications for logarithmic functions: intensity of sound, pH levels of solutions, yields of chemical reactions, production of goods, and growth of infants. As with exponential models, data modeled by logarithmic functions are either always increasing or always decreasing as time moves forward. Again, it is the *way* they increase or decrease that helps us determine whether a logarithmic model is best.

Recall that logarithmic functions increase or decrease rapidly at first, but then steadily slow as time moves on. By reflecting on the characteristics we've already learned about this function, we can better analyze real world situations that reflect this type of growth or decay. When performing logarithmic regression analysis, we use the form of the logarithmic function most commonly used on graphing utilities, $y = a + b\ln(x)$. For this function

- All input values, x, must be greater than zero.
- The point $(1, a)$ is on the graph of the model.
- If $b > 0$, the model is increasing. Growth increases rapidly at first and then steadily slows over time.
- If $b < 0$, the model is decreasing. Decay occurs rapidly at first and then steadily slows over time.

Logarithmic Regression

Logarithmic regression is used to model situations where growth or decay accelerates rapidly at first and then slows over time. We use the command “LnReg” on a graphing utility to fit a logarithmic function to a set of data points. This returns an equation of the form,

$y = a + b\ln(x)$
Note that

- all input values, x, must be non-negative.
- when $b > 0$, the model is increasing.
- when $b < 0$, the model is decreasing.

How To

Given a set of data, perform logarithmic regression using a graphing utility.

1. Use the STAT then EDIT menu to enter given data.
   a. Clear any existing data from the lists.
   b. List the input values in the L1 column.
   c. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   a. Use ZOOM [9] to adjust axes to fit the data.
   b. Verify the data follow a logarithmic pattern.
3. Find the equation that models the data.
   a. Select "LnReg" from the STAT then CALC menu.
   b. Use the values returned for *a* and *b* to record the model, $y = a + b\ln\left(x\right)$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

Using Logarithmic Regression to Fit a Model to Data

Due to advances in medicine and higher standards of living, life expectancy has been increasing in most developed countries since the beginning of the 20th century.

(Figure) shows the average life expectancies, in years, of Americans from 1900–2010[2].

2. •Source: Center for Disease Control and Prevention, 2013

Year	1900	1910	1920	1930	1940	1950
Life Expectancy(Years)	47.3	50.0	54.1	59.7	62.9	68.2
Year	1960	1970	1980	1990	2000	2010
Life Expectancy(Years)	69.7	70.8	73.7	75.4	76.8	78.7

a. Let x represent time in decades starting with $x = 1$ for the year 1900, $x = 2$ for the year 1910, and so on. Let y represent the corresponding life expectancy. Use logarithmic regression to fit a model to these data.
b. Use the model to predict the average American life expectancy for the year 2030.

Show Solution

a. Using the STAT then EDIT menu on a graphing utility, list the years using values 1–12 in L1 and the corresponding life expectancy in L2. Then use the STATPLOT feature to verify that the scatterplot follows a logarithmic pattern as shown in (Figure):

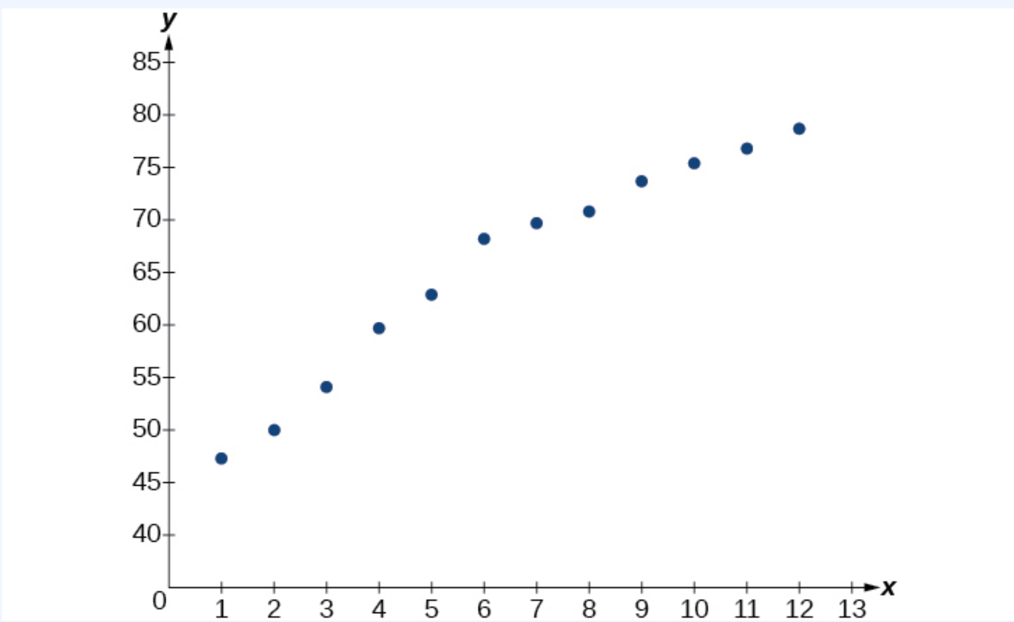


Figure 3.

Use the "LnReg" command from the STAT then CALC menu to obtain the logarithmic model,

$y = 42.52722583 + 13.85752327 \ln(x)$

Next, graph the model in the same window as the scatterplot to verify it is a good fit as shown in (Figure):

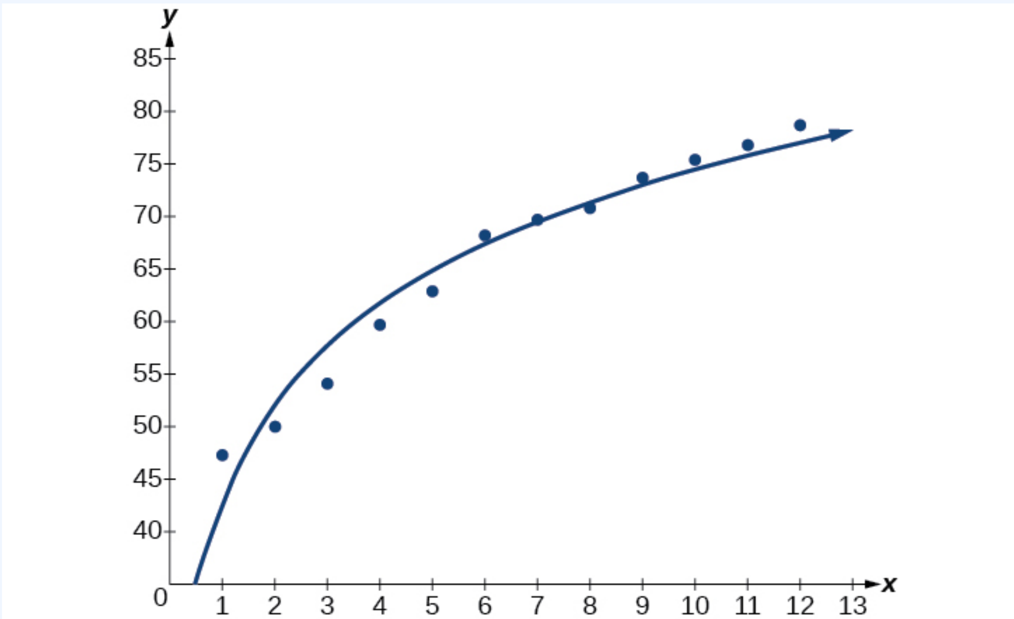


Figure 4.

b. To predict the life expectancy of an American in the year 2030, substitute $x = 14$ for the in the model and solve for y :

$$\begin{aligned} y &= 42.52722583 + 13.85752327\ln(x) && \text{Use the regression model found in part (a).} \\ &= 42.52722583 + 13.85752327\ln(14) && \text{Substitute 14 for } x. \\ &\approx 79.1 && \text{Round to the nearest tenth.} \end{aligned}$$

If life expectancy continues to increase at this pace, the average life expectancy of an American will be 79.1 by the year 2030.

Try It

Sales of a video game released in the year 2000 took off at first, but then steadily slowed as time moved on. (Figure) shows the number of games sold, in thousands, from the years 2000–2010.

Year	2000	2001	2002	2003	2004	2005
Number Sold (thousands)	142	149	154	155	159	161
Year	2006	2007	2008	2009	2010	–
Number Sold (thousands)	163	164	164	166	167	–

a. Let x represent time in years starting with $x = 1$ for the year 2000. Let y represent the number of games sold in thousands. Use logarithmic regression to fit a model to these data.
b. If games continue to sell at this rate, how many games will sell in 2015? Round to the nearest thousand.

Show Solution

a. The logarithmic regression model that fits these data is $y = 141.91242949 + 10.45366573 \ln(x)$
b. If sales continue at this rate, about 171,000 games will be sold in the year 2015.

Building a Logistic Model from Data

Like exponential and logarithmic growth, logistic growth increases over time. One of the most notable differences with logistic growth models is that, at a certain point, growth steadily slows and the function approaches an upper bound, or *limiting value*. Because of this, logistic regression is best for modeling phenomena where there are limits in expansion, such as availability of living space or nutrients.

It is worth pointing out that logistic functions actually model resource-limited exponential growth. There are many examples of this type of growth in real-world situations, including population growth and spread of disease, rumors, and even stains in fabric. When performing logistic regression analysis, we use the form most commonly used on graphing utilities:

$$y = \frac{c}{1 + ae^{-bx}}$$

Recall that:

- $\frac{c}{1+a}$ is the initial value of the model.
- when $b > 0$, the model increases rapidly at first until it reaches its point of maximum growth rate, $\left(\frac{\ln(a)}{b}, \frac{c}{2}\right)$. At that point, growth steadily slows and the function becomes asymptotic to the upper bound $y = c$.
- c is the limiting value, sometimes called the *carrying capacity*, of the model.

Logistic Regression

Logistic regression is used to model situations where growth accelerates rapidly at first and then steadily slows to an upper limit. We use the command "Logistic" on a graphing utility to fit a logistic function to a set of data points. This returns an equation of the form

$y = \frac{c}{1+ae^{-bx}}$

Note that

- The initial value of the model is $\frac{c}{1+a}$.
- Output values for the model grow closer and closer to $y = c$ as time increases.

How To

Given a set of data, perform logistic regression using a graphing utility.

1. Use the STAT then EDIT menu to enter given data.
   a. Clear any existing data from the lists.
   b. List the input values in the L1 column.
   c. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.

a. Use ZOOM [9] to adjust axes to fit the data.
b. Verify the data follow a logistic pattern.

3. Find the equation that models the data.

a. Select “Logistic” from the STAT then CALC menu.
b. Use the values returned for a, b, and c to record the model, $y=\frac{c}{1+ae^{-bx}}$.

4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

Using Logistic Regression to Fit a Model to Data

Mobile telephone service has increased rapidly in America since the mid 1990s. Today, almost all residents have cellular service. (Figure) shows the percentage of Americans with cellular service between the years 1995 and 2012[3] .

Year	Americans with Cellular Service (%)	Year	Americans with Cellular Service (%)
1995	12.69	2004	62.852
1996	16.35	2005	68.63
1997	20.29	2006	76.64
1998	25.08	2007	82.47
1999	30.81	2008	85.68
2000	38.75	2009	89.14
2001	45.00	2010	91.86
2002	49.16	2011	95.28
2003	55.15	2012	98.17

3. •Source: The World Bank, 2013

a. Let x represent time in years starting with $x = 0$ for the year 1995. Let y represent the corresponding percentage of residents with cellular service. Use logistic regression to fit a model to these data.
b. Use the model to calculate the percentage of Americans with cell service in the year 2013. Round to the nearest tenth of a percent.
c. Discuss the value returned for the upper limit, c. What does this tell you about the model? What would the limiting value be if the model were exact?

Show Solution

a. Using the STAT then EDIT menu on a graphing utility, list the years using values 0–15 in L1 and the corresponding percentage in L2. Then use the STATPLOT feature to verify that the scatterplot follows a logistic pattern as shown in (Figure):

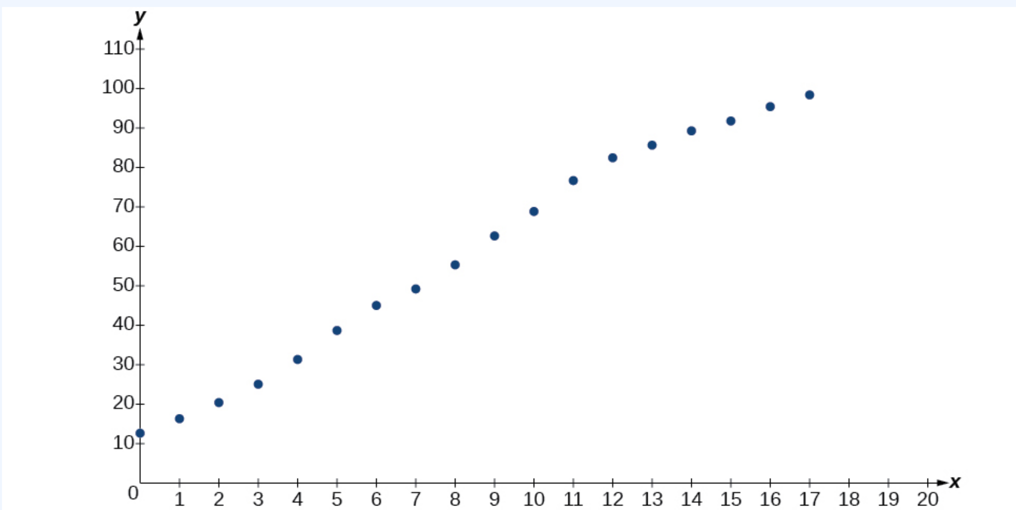


Figure 5.

Use the "Logistic" command from the STAT then CALC menu to obtain the logistic model,

$$y = \frac{105.7379526}{1+6.88328979e^{-0.2595440013x}}$$

Next, graph the model in the same window as shown in (Figure) the scatterplot to verify it is a good fit:

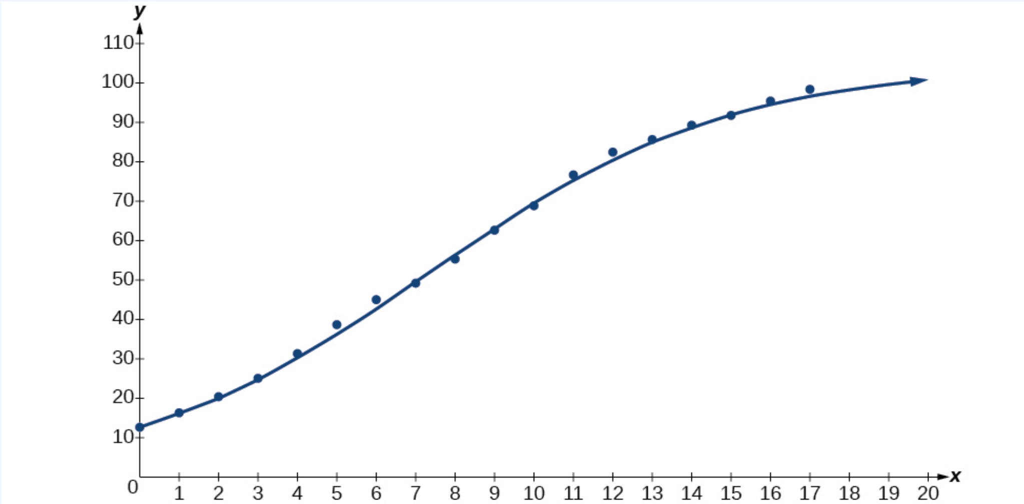


Figure 6.

b. To approximate the percentage of Americans with cellular service in the year 2013, substitute $x = 18$ for the in the model and solve for y :

$$\begin{aligned} y &= \frac{105.7379526}{1+6.88328979e^{-0.2595440013x}} && \text{Use the regression model found in part (a).} \\ &= \frac{105.7379526}{1+6.88328979e^{-0.2595440013(18)}} && \text{Substitute 18 for } x. \\ &\approx 99.3 && \text{Round to the nearest tenth} \end{aligned}$$

According to the model, about 98.8% of Americans had cellular service in 2013.

c. The model gives a limiting value of about 105. This means that the maximum possible percentage of Americans with cellular service would be 105%, which is impossible. (How could over 100% of a population have cellular service?) If the model were exact, the limiting value would be $c = 100$ and the model's outputs would get very close to, but never actually reach 100%. After all, there will always be someone out there without cellular service!

Try It

(Figure) shows the population, in thousands, of harbor seals in the Wadden Sea over the years 1997 to 2012.

Year	Seal Population (Thousands)	Year	Seal Population (Thousands)
1997	3.493	2005	19.590
1998	5.282	2006	21.955
1999	6.357	2007	22.862
2000	9.201	2008	23.869
2001	11.224	2009	24.243
2002	12.964	2010	24.344
2003	16.226	2011	24.919
2004	18.137	2012	25.108

a. Let x represent time in years starting with $x=0$ for the year 1997. Let y represent the number of seals in thousands. Use logistic regression to fit a model to these data.
b. Use the model to predict the seal population for the year 2020.
c. To the nearest whole number, what is the limiting value of this model?

Show Solution

a. The logistic regression model that fits these data is $y=\frac{25.65665979}{1+6.113686306e^{-0.3852149008x}}$.
b. If the population continues to grow at this rate, there will be about $25{,}634$ seals in 2020.
c. To the nearest whole number, the carrying capacity is 25,657.

Access this online resource for additional instruction and practice with exponential function models.

- Exponential Regression on a Calculator

Visit this website for additional practice questions from Learningpod.

Key Concepts

- Exponential regression is used to model situations where growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero.
- We use the command “ExpReg” on a graphing utility to fit function of the form $y = ab^x$ to a set of data points. See (Figure).
- Logarithmic regression is used to model situations where growth or decay accelerates rapidly at first and then slows over time.
- We use the command “LnReg” on a graphing utility to fit a function of the form $y = a + b\ln\left(x\right)$ to a set of data points. See (Figure).
- Logistic regression is used to model situations where growth accelerates rapidly at first and then steadily slows as the function approaches an upper limit.
- We use the command “Logistic” on a graphing utility to fit a function of the form $y = \frac{c}{1+ae^{-bx}}$ to a set of data points. See (Figure).

Section Exercises

Verbal

1. What situations are best modeled by a logistic equation? Give an example, and state a case for why the example is a good fit.

Show Solution
Logistic models are best used for situations that have limited values. For example, populations cannot grow indefinitely since resources such as food, water, and space are limited, so a logistic model best describes populations.

2. What is a carrying capacity? What kind of model has a carrying capacity built into its formula? Why does this make sense?

3. What is regression analysis? Describe the process of performing regression analysis on a graphing utility.

Show Solution
Regression analysis is the process of finding an equation that best fits a given set of data points. To perform a regression analysis on a graphing utility, first list the given points using the STAT then EDIT menu. Next graph the scatter plot using the STAT PLOT feature. The shape of the data points on the scatter graph can help determine which regression feature to use. Once this is determined, select the appropriate regression analysis command from the STAT then CALC menu.

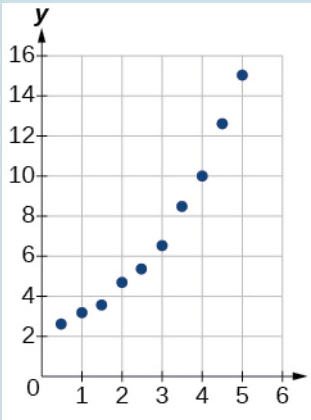
4. What might a scatterplot of data points look like if it were best described by a logarithmic model?

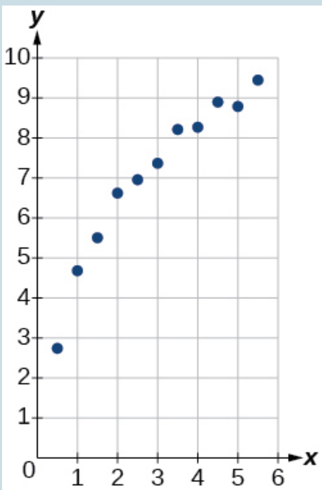
5. What does the y-intercept on the graph of a logistic equation correspond to for a population modeled by that equation?

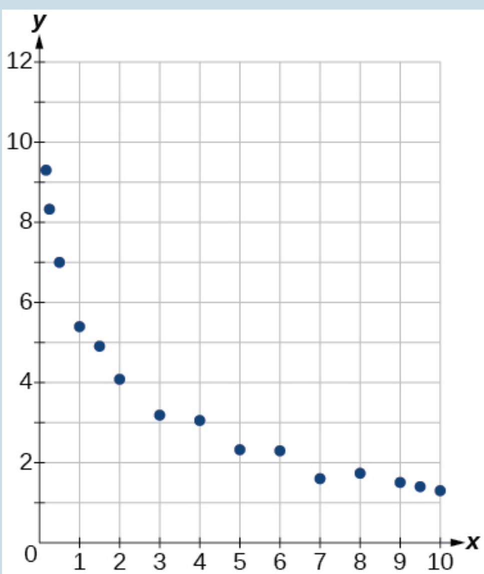
Show Solution
The y-intercept on the graph of a logistic equation corresponds to the initial population for the population model.

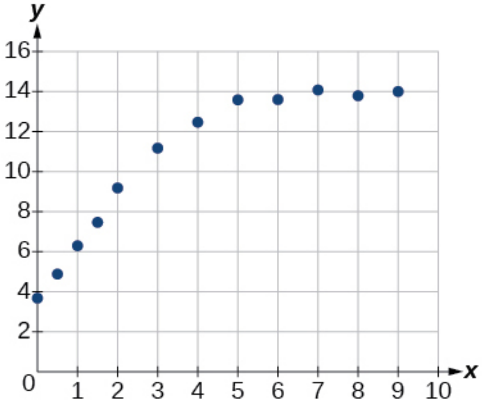
Graphical

For the following exercises, match the given function of best fit with the appropriate scatterplot in (Figure) through (Figure)**.** Answer using the letter beneath the matching graph.

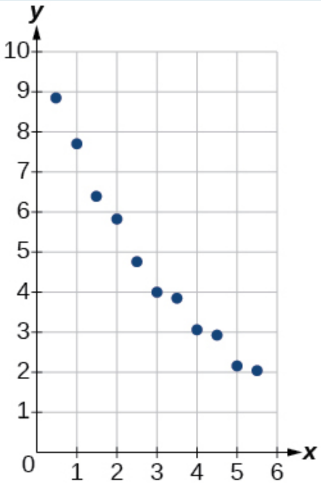

y
16
14
12
10
8
6
4
2
0
1 2 3 4 5 6
(a)


y
10
9
8
7
6
5
4
3
2
1
0
1 2 3 4 5 6
x
(b)


y
12
10
8
6
4
2
0
1 2 3 4 5 6 7 8 9 10
x
(c)



(d)



(e)

6. $y = 10.209e^{-0.294x}$

7. $y = 5.598 - 1.912\ln(x)$

Show Solution

C

8. $y = 2.104(1.479)^x$

9. $y = 4.607 + 2.733\ln(x)$

Show Solution

B

10. $y = \frac{14.005}{1+2.79e^{-0.812x}}$

Numeric

11. To the nearest whole number, what is the initial value of a population modeled by the logistic equation $P\left(t\right) = \frac{175}{1+6.995e^{-0.68t}}$? What is the carrying capacity?

Show Solution

$P\left(0\right) = 22$; 175

12. Rewrite the exponential model $A\left(t\right) = 1550(1.085)^{x}$ as an equivalent model with base e. Express the exponent to four significant digits.

13. A logarithmic model is given by the equation $h\left(p\right) = 67.682 - 5.792\ln\left(p\right)$. To the nearest hundredth, for what value of p does $h\left(p\right) = 62$?

Show Solution

$p \approx 2.67$

14. A logistic model is given by the equation $P\left(t\right) = \frac{90}{1+5e^{-0.42t}}$. To the nearest hundredth, for what value of t does $P\left(t\right) = 45$?

15. What is the y-intercept on the graph of the logistic model given in the previous exercise?

Show Solution
y-intercept: $(0, 15)$

Technology

For the following exercises, use this scenario: The population P of a koi pond over x months is modeled by the function $P\left(x\right)=\frac{68}{1+16e^{-0.28x}}$.

16. Graph the population model to show the population over a span of 3 years.

17. What was the initial population of koi?

Show Solution
4 koi

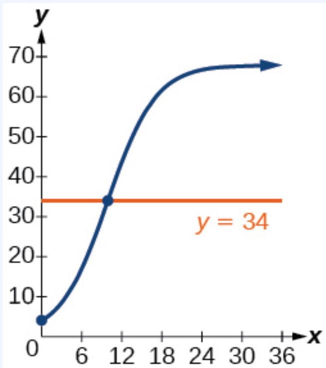
18. How many koi will the pond have after one and a half years?

19. How many months will it take before there are 20 koi in the pond?

Show Solution
about 6.8 months.

Use the intersect feature to approximate the number of months it will take before the population of the pond reaches half its carrying capacity.

Show Solution



For the following exercises, use this scenario: The population P of an endangered species habitat for wolves is modeled by the function $P\left(x\right)=\frac{558}{1+54.8e^{-0.462x}}$, where x is given in years.

20. Graph the population model to show the population over a span of 10 years.

21. What was the initial population of wolves transported to the habitat?

Show Solution
10 wolves

22. How many wolves will the habitat have after 3 years?

23. How many years will it take before there are 100 wolves in the habitat?

Show Solution
about 5.4 years.

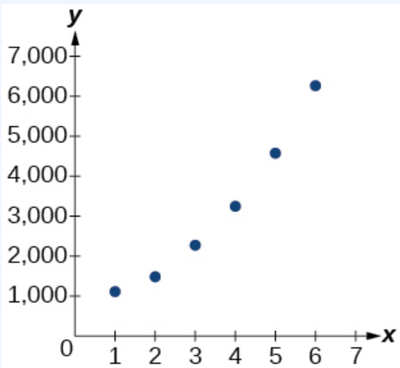
24. Use the intersect feature to approximate the number of years it will take before the population of the habitat reaches half its carrying capacity.

For the following exercises, refer to (Figure).

x	$f(x)$
1	1125
2	1495
3	2310
4	3294
5	4650
6	6361

25. Use a graphing calculator to create a scatter diagram of the data.

Show Solution



26. Use the regression feature to find an exponential function that best fits the data in the table.

27. Write the exponential function as an exponential equation with base e.

Show Solution

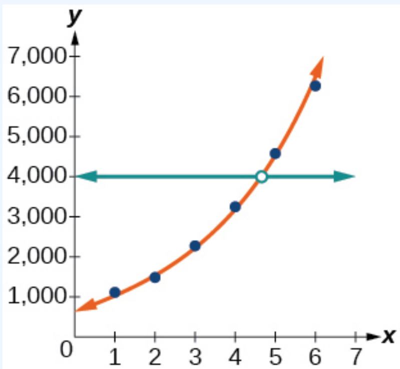
$f\left(x\right)=776.682{e}^{0.3549x}$

28. Graph the exponential equation on the scatter diagram.

29. Use the intersect feature to find the value of x for which $f\left(x\right)=4000.$

Show Solution

When $f\left(x\right)=4000, x\approx 4.6.$



For the following exercises, refer to (Figure).

x	*f(x)*
1	555
2	383
3	307
4	210
5	158
6	122

30. Use a graphing calculator to create a scatter diagram of the data.

31. Use the regression feature to find an exponential function that best fits the data in the table.

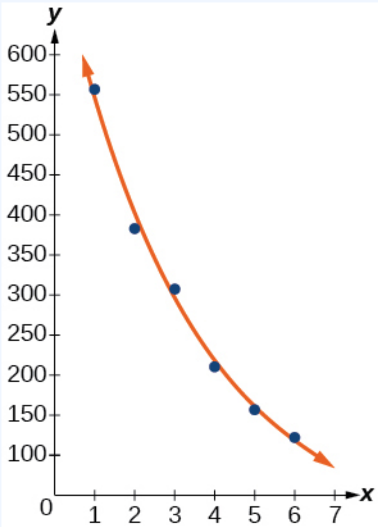
Show Solution

$f\left(x\right)=731.92{\left(0.738\right)}^{x}$

32. Write the exponential function as an exponential equation with base e.

33. Graph the exponential equation on the scatter diagram.

Show Solution



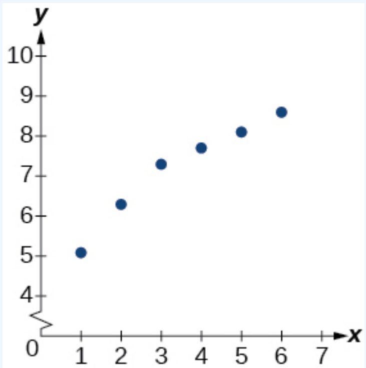
34. Use the intersect feature to find the value of x for which $f(x) = 250.$

For the following exercises, refer to (Figure).

x	*f(x)*
1	5.1
2	6.3
3	7.3
4	7.7
5	8.1
6	8.6

35. Use a graphing calculator to create a scatter diagram of the data.

Show Solution



36. Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

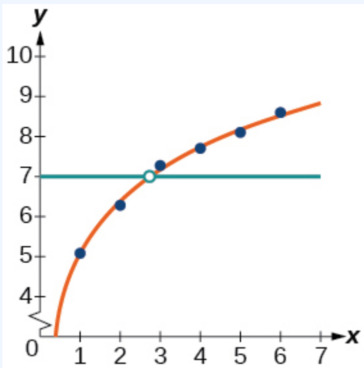
37. Use the logarithmic function to find the value of the function when $x = 10$.

Show Solution
$f(10) \approx 9.5$

38. Graph the logarithmic equation on the scatter diagram.

39. Use the intersect feature to find the value of x for which $f(x) = 7$.

Show Solution
When $f(x) = 7, x \approx 2.7$.



For the following exercises, refer to (Figure).

x	*f(x)*
1	7.5
2	6
3	5.2
4	4.3
5	3.9
6	3.4
7	3.1
8	2.9

40. Use a graphing calculator to create a scatter diagram of the data.

41. Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y=a+b\ln\left(x\right)$ that best fits the data in the table.

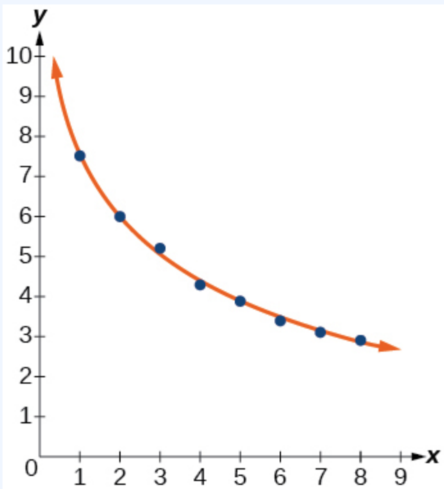
Show Solution

$f\left(x\right)=7.544-2.268\ln\left(x\right)$

42. Use the logarithmic function to find the value of the function when $x=10.$

43. Graph the logarithmic equation on the scatter diagram.

Show Solution



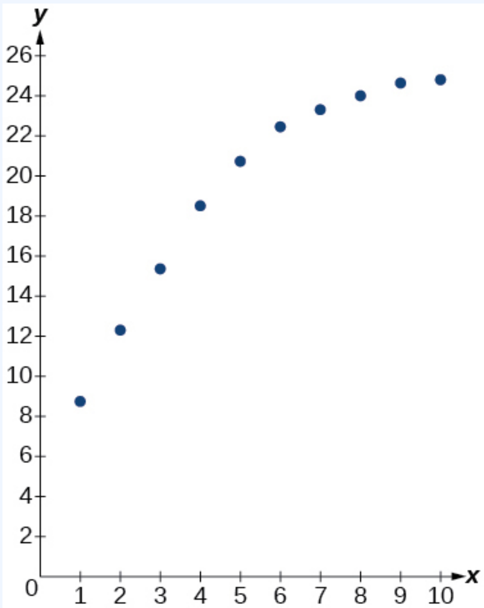
44. Use the intersect feature to find the value of x for which $f\left(x\right)=8.$

For the following exercises, refer to (Figure).

x	f(x)
1	8.7
2	12.3
3	15.4
4	18.5
5	20.7
6	22.5
7	23.3
8	24
9	24.6
10	24.8

45. Use a graphing calculator to create a scatter diagram of the data.

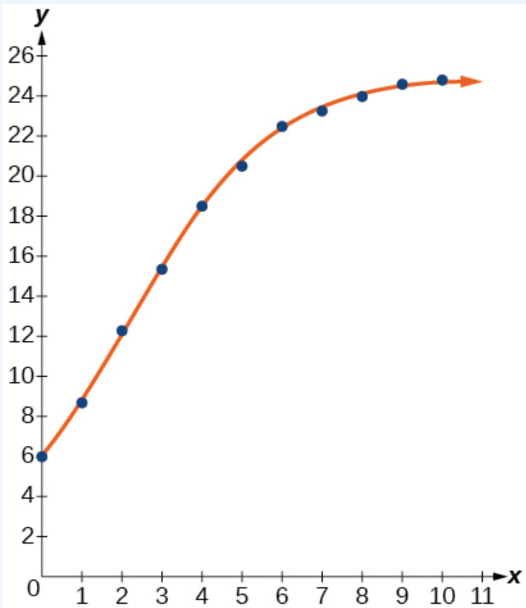
Show Solution



46. Use the LOGISTIC regression option to find a logistic growth model of the form $y = \frac{c}{1+ae^{-bx}}$ that best fits the data in the table.

47. Graph the logistic equation on the scatter diagram.

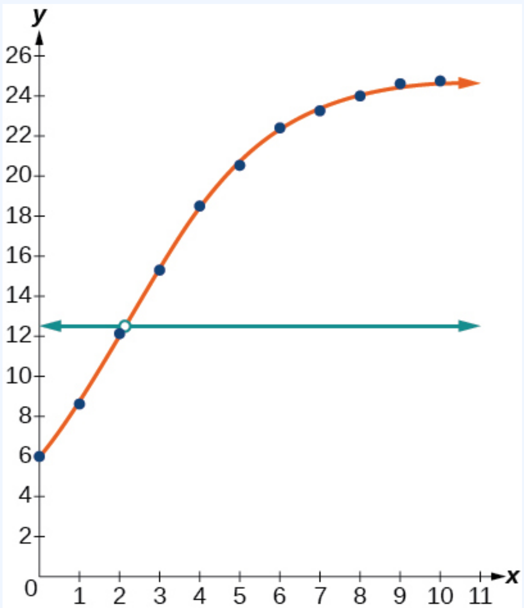
Show Solution



48. To the nearest whole number, what is the predicted carrying capacity of the model?

49. Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

Show Solution
When $f\left(x\right)=12.5, x\approx 2.1.$



For the following exercises, refer to (Figure).

x	$f(x)$
0	12
2	28.6
4	52.8
5	70.3
7	99.9
8	112.5
10	125.8
11	127.9
15	135.1
17	135.9

50. Use a graphing calculator to create a scatter diagram of the data.

51. Use the LOGISTIC regression option to find a logistic growth model of the form $y=\frac{c}{1+ae^{-bx}}$ that best fits the data in the table.

Show Solution
$f\left(x\right)=\frac{136.068}{1+10.324{e}^{-0.480x}}$

52. Graph the logistic equation on the scatter diagram.

53. To the nearest whole number, what is the predicted carrying capacity of the model?

Show Solution
about 136

54. Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

Extensions

55. Recall that the general form of a logistic equation for a population is given by $P\left(t\right)=\frac{c}{1+a{e}^{-bt}}$, such that the initial population at time $t=0$ is $P\left(0\right)={P}_{0}$. Show algebraically that $\frac{c-P\left(t\right)}{P\left(t\right)}=\frac{c-{P}_{0}}{{P}_{0}}{e}^{-bt}$.

Show Solution
Working with the left side of the equation, we see that it can be rewritten as $a{e}^{-bt}$:

$$\frac{c-P\left(t\right)}{P\left(t\right)}=\frac{c-\frac{c}{1+a{e}^{-bt}}}{\frac{c}{1+a{e}^{-bt}}}=\frac{\frac{c\left(1+a{e}^{-bt}\right)-c}{1+a{e}^{-bt}}}{\frac{c}{1+a{e}^{-bt}}}=\frac{\frac{c\left(1+a{e}^{-bt}-1\right)}{1+a{e}^{-bt}}}{\frac{c}{1+a{e}^{-bt}}}=1+a{e}^{-bt}-1=a{e}^{-bt}$$

Working with the right side of the equation we show that it can also be rewritten as $a{e}^{-bt}$. But first note that when $t=0,$
${P}_{0}=\frac{c}{1+a{e}^{-b\left(0\right)}}=\frac{c}{1+a}$. Therefore,

$\frac{c-P_0}{P_0}e^{-bt} = \frac{c-\frac{c}{1+a}}{\frac{c}{1+a}}e^{-bt} = \frac{\frac{c(1+a)-c}{1+a}}{\frac{c}{1+a}}e^{-bt} = \frac{\frac{c(1+a-1)}{1+a}}{\frac{c}{1+a}}e^{-bt} = (1+a-1)\,e^{-bt} = ae^{-bt}$

Thus, $\frac{c-P(t)}{P(t)} = \frac{c-P_0}{P_0}e^{-bt}$.

56. Use a graphing utility to find an exponential regression formula $f\,(x)$ and a logarithmic regression formula $g\,(x)$ for the points $(1.5, 1.5)$ and $(8.5,\ 8.5)$. Round all numbers to 6 decimal places. Graph the points and both formulas along with the line $y = x$ on the same axis. Make a conjecture about the relationship of the regression formulas.

57. Verify the conjecture made in the previous exercise. Round all numbers to six decimal places when necessary.

Show Solution

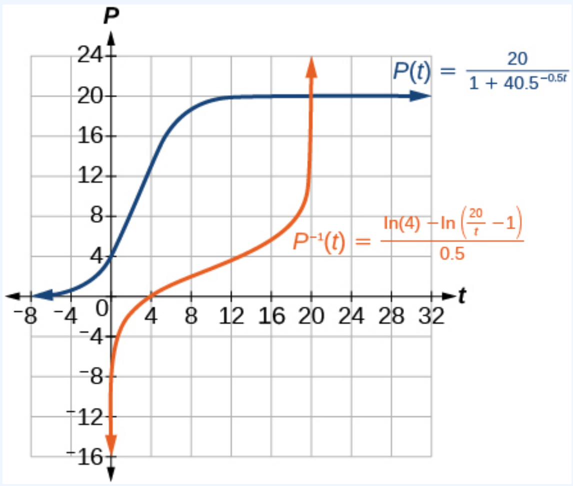
First rewrite the exponential with base *e*: $f\,(x) = 1.034341e^{0.247800\mathrm{x}}$. Then test to verify that $f\,(g\,(x)) = x$, taking rounding error into consideration:

$$\begin{aligned} g\,(f\,(x)) &= 4.035510\ln\left(1.034341e^{0.247800\mathrm{x}}\right) - 0.136259 \\ &= 4.03551\left(\ln\left(1.034341\right) + \ln\left(e^{0.2478x}\right)\right) - 0.136259 \\ &= 4.03551\left(\ln\left(1.034341\right) + 0.2478x\right) - 0.136259 \\ &= 0.136257 + 0.999999x - 0.136259 \\ &= -0.000002 + 0.999999x \\ &\approx 0 + x \\ &= x \end{aligned}$$

58. Find the inverse function $f^{-1}\,(x)$ for the logistic function $f\,(x) = \frac{c}{1+ae^{-bx}}$. Show all steps.

59. Use the result from the previous exercise to graph the logistic model $P\,(t) = \frac{20}{1+4e^{-0.5t}}$ along with its inverse on the same axis. What are the intercepts and asymptotes of each function?

Show Solution



The graph of $P\left(t\right)$ has a *y*-intercept at (0, 4) and horizontal asymptotes at *y* = 0 and *y* = 20. The graph of $P^{-1}\left(t\right)$ has an *x*– intercept at (4, 0) and vertical asymptotes at *x* = 0 and *x* = 20.

Chapter Review Exercises

Exponential Functions

1. Determine whether the function $y = 156(0.825)^t$ represents exponential growth, exponential decay, or neither. Explain

Show Solution

exponential decay; The growth factor, $0.825,$ is between 0 and $1.$

2. The population of a herd of deer is represented by the function $A\left(t\right) = 205(1.13)^t$, where t is given in years. To the nearest whole number, what will the herd population be after 6 years?

3. Find an exponential equation that passes through the points $(2,\ 2.25)$ and $(5, 60.75)$.

Show Solution
$y = 0.25(3)^x$

4. Determine whether (Figure) could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

x	1	2	3	4
f(x)	3	0.9	0.27	0.081

5. A retirement account is opened with an initial deposit of $8,500 and earns 8.12 interest compounded monthly. What will the account be worth in 20 years?

Show Solution
$42,888.18$

6. Hsu-Mei wants to save $5,000 for a down payment on a car. To the nearest dollar, how much will she need to invest in an account now with 7.5 APR, compounded daily, in order to reach her goal in 3 years?

7. Does the equation $y = 2.294e^{-0.654t}$ represent continuous growth, continuous decay, or neither? Explain.

Show Solution
continuous decay; the growth rate is negative.

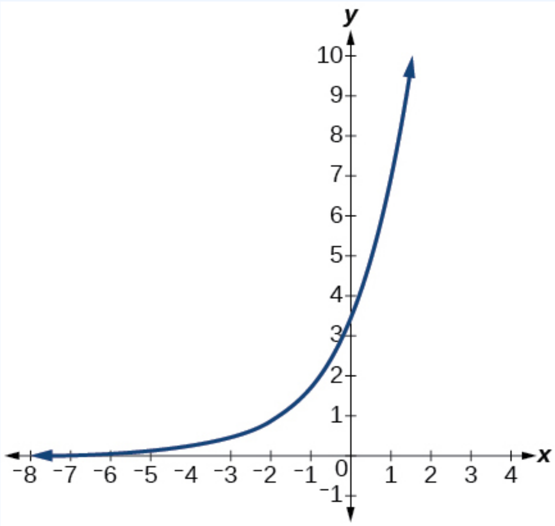
8. Suppose an investment account is opened with an initial deposit of $\$10{,}500$ earning 6.25 interest, compounded continuously. How much will the account be worth after 25 years?

Graphs of Exponential Functions

9. Graph the function $f\left(x\right)=3.5{\left(2\right)}^{x}$. State the domain and range and give the *y*-intercept.

Show Solution

domain: all real numbers; range: all real numbers strictly greater than zero; *y*-intercept: (0, 3.5);



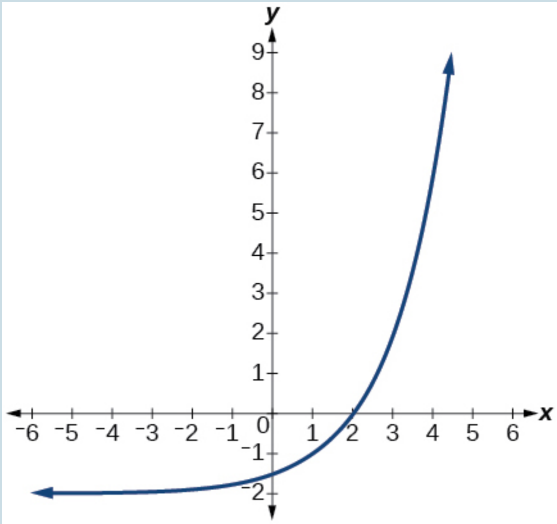
10. Graph the function $f\left(x\right)=4{\left(\frac{1}{8}\right)}^{x}$ and its reflection about the *y*-axis on the same axes, and give the *y*-intercept.

11. The graph of $f\left(x\right)=6.5^{x}$ is reflected about the *y*-axis and stretched vertically by a factor of $7.$ What is the equation of the new function, $g\left(x\right)?$ State its *y*-intercept, domain, and range.

Show Solution

$g\left(x\right)=7{\left(6.5\right)}^{-x}$; *y*-intercept: $\left(0,\ 7\right)$; Domain: all real numbers; Range: all real numbers greater than $0.$

12. The graph below shows transformations of the graph of $f\left(x\right)=2^{x}$. What is the equation for the transformation?



Logarithmic Functions

13. Rewrite $\log_{17}(4913) = x$ as an equivalent exponential equation.

Show Solution
$17^x = 4913$

14. Rewrite $\ln(s) = t$ as an equivalent exponential equation.

15. Rewrite $a^{-\frac{2}{5}} = b$ as an equivalent logarithmic equation.

Show Solution
$\log_a b = -\frac{2}{5}$

16. Rewrite $e^{-3.5} = h$ as an equivalent logarithmic equation.

17. Solve for x if $\log_{64}(x) = \frac{1}{3}$ by converting to exponential form.

Show Solution

$x = 64^{\frac{1}{3}} = 4$

18. Evaluate $\log_5\left(\frac{1}{125}\right)$ without using a calculator.

19. Evaluate $\log\left(0.000001\right)$ without using a calculator.

Show Solution

$\log\left(0.000001\right) = -6$

20. Evaluate $\log\left(4.005\right)$ using a calculator. Round to the nearest thousandth.

21. Evaluate $\ln\left(e^{-0.8648}\right)$ without using a calculator.

Show Solution

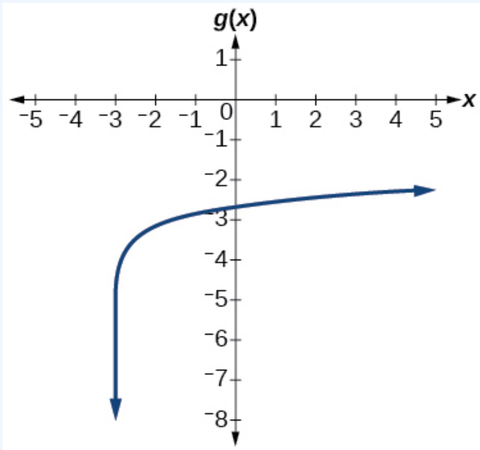
$\ln\left(e^{-0.8648}\right) = -0.8648$

22. Evaluate $\ln\left(\sqrt[3]{18}\right)$ using a calculator. Round to the nearest thousandth.

Graphs of Logarithmic Functions

23. Graph the function $g\left(x\right) = \log\left(7x + 21\right) - 4$.

Show Solution



24. Graph the function $h(x) = 2\ln(9 - 3x) + 1$.

25. State the domain, vertical asymptote, and end behavior of the function $g(x) = \ln(4x + 20) - 17$.

Show Solution

Domain: $x > -5$; Vertical asymptote: $x = -5$; End behavior: as $x \to -5^{+}, f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$.

Logarithmic Properties

26. Rewrite $\ln(7r \cdot 11st)$ in expanded form.

27. Rewrite $\log_8(x) + \log_8(5) + \log_8(y) + \log_8(13)$ in compact form.

Show Solution

$\log_8(65xy)$

28. Rewrite $\log_m\left(\frac{67}{83}\right)$ in expanded form.

29. Rewrite $\ln(z) - \ln(x) - \ln(y)$ in compact form.

Show Solution

$\ln\left(\frac{z}{xy}\right)$

30. Rewrite $\ln\left(\frac{1}{x^5}\right)$ as a product.

31. Rewrite $-\log_y\left(\frac{1}{12}\right)$ as a single logarithm.

Show Solution

$\log_y(12)$

32.Use properties of logarithms to expand $\log\left(\frac{r^2 s^{11}}{t^{14}}\right)$.

33.Use properties of logarithms to expand $\ln\left(2b\sqrt{\frac{b+1}{b-1}}\right)$.

Show Solution

$\ln(2) + \ln(b) + \frac{\ln(b+1) - \ln(b-1)}{2}$

34. Condense the expression $5\ln(b) + \ln(c) + \frac{\ln(4-a)}{2}$ to a single logarithm.

35. Condense the expression $3\log_7 v + 6\log_7 w - \frac{\log_7 u}{3}$ to a single logarithm.

Show Solution

$\log_7\left(\frac{v^3w^6}{\sqrt[3]{u}}\right)$

36. Rewrite $\log_3(12.75)$ to base e.

37. Rewrite $5^{12x-17}=125$ as a logarithm. Then apply the change of base formula to solve for x using the common log. Round to the nearest thousandth.

Show Solution

$x=\frac{\frac{\log(125)}{\log(5)}+17}{12}=\frac{5}{3}$

Exponential and Logarithmic Equations

38. Solve $216^{3x}\cdot 216^{x}=36^{3x+2}$ by rewriting each side with a common base.

39. Solve $\frac{125}{\left(\frac{1}{625}\right)^{-x-3}}=5^3$ by rewriting each side with a common base.

Show Solution

$x=-3$

40. Use logarithms to find the exact solution for $7\cdot 17^{-9x}-7=49$. If there is no solution, write *no solution*.

41. Use logarithms to find the exact solution for $3e^{6n-2}+1=-60$. If there is no solution, write *no solution*.

Show Solution

no solution

42. Find the exact solution for $5e^{3x} - 4 = 6$. If there is no solution, write *no solution*.

43. Find the exact solution for $2e^{5x-2} - 9 = -56.$ If there is no solution, write *no solution*.

Show Solution

no solution

44. Find the exact solution for $5^{2x-3} = 7^{x+1}$. If there is no solution, write *no solution*.

45. Find the exact solution for $e^{2x} - e^{x} - 110 = 0.$ If there is no solution, write *no solution*.

Show Solution

$x = \ln(11)$

46. Use the definition of a logarithm to solve. $-5\log_7(10n) = 5.$

47. Use the definition of a logarithm to find the exact solution for $9 + 6\ln(a + 3) = 33.$

Show Solution

$a = e^4 - 3$

48. Use the one-to-one property of logarithms to find an exact solution for $\log_8(7) + \log_8(-4x) = \log_8(5)$. If there is no solution, write *no solution*.

49. Use the one-to-one property of logarithms to find an exact solution for $\ln(5) + \ln\left(5x^2 - 5\right) = \ln(56)$. If there is no solution, write *no solution*.

Show Solution
$x = \frac{9}{5}$

50. The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a large orchestra with a sound intensity of $6.3 \cdot 10^{-3}$ watts per square meter?

51. The population of a city is modeled by the equation $P\left(t\right) = 256,114e^{0.25t}$ where t is measured in years. If the city continues to grow at this rate, how many years will it take for the population to reach one million?

Show Solution
about 5.45 years

52. Find the inverse function f^{-1} for the exponential function $f\left(x\right) = 2 \cdot e^{x+1} - 5.$

53. Find the inverse function f^{-1} for the logarithmic function $f\left(x\right) = 0.25 \cdot \log_2\left(x^3 + 1\right).$

Show Solution
$f^{-1}\left(x\right) = \sqrt[3]{2^{4x} - 1}$

Exponential and Logarithmic Models

For the following exercises, use this scenario: A doctor prescribes 300 milligrams of a therapeutic drug that decays by about 17 each hour.

54. To the nearest minute, what is the half-life of the drug?

55. Write an exponential model representing the amount of the drug remaining in the patient's

system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 24 hours. Round to the nearest hundredth of a gram.

Show Solution
$f\,(t) = 300(0.83)^t; f\,(24) \approx 3.43g$

For the following exercises, use this scenario: A soup with an internal temperature of 350 Fahrenheit was taken off the stove to cool in a $71\mathrm{F}$ room. After fifteen minutes, the internal temperature of the soup was $175\mathrm{F}.$

56. Use Newton's Law of Cooling to write a formula that models this situation.

57. How many minutes will it take the soup to cool to $85\mathrm{F}?$

Show Solution
about 45 minutes

For the following exercises, use this scenario: The equation $N\,(t) = \frac{1200}{1+199e^{-0.625t}}$ models the number of people in a school who have heard a rumor after t days.

58. How many people started the rumor?

59. To the nearest tenth, how many days will it be before the rumor spreads to half the carrying capacity?

Show Solution
about 8.5 days

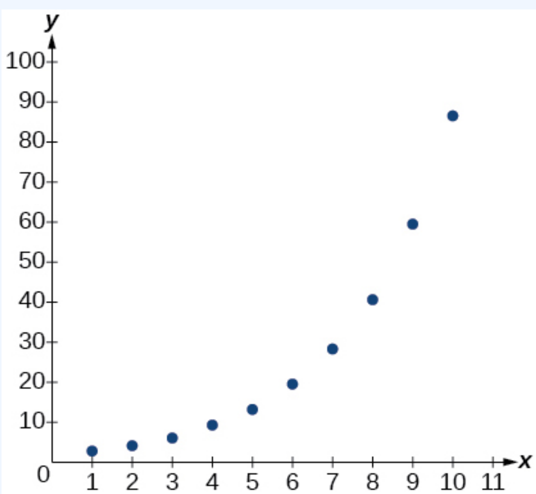
60. What is the carrying capacity?

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table would likely represent a function that is linear, exponential, or logarithmic.

61.

x	*f(x)*
1	3.05
2	4.42
3	6.4
4	9.28
5	13.46
6	19.52
7	28.3
8	41.04
9	59.5
10	86.28

Show Solution
exponential



62.

x	*f(x)*
0.5	18.05
1	17
3	15.33
5	14.55
7	14.04
10	13.5
12	13.22
13	13.1
15	12.88
17	12.69
20	12.45

63. Find a formula for an exponential equation that goes through the points $(-2, 100)$ and $(0, 4)$. Then express the formula as an equivalent equation with base *e*.

Show Solution

$y = 4(0.2)^x; y = 4e^{-1.609438x}$

Fitting Exponential Models to Data

64. What is the carrying capacity for a population modeled by the logistic equation $P(t) = \frac{250{,}000}{1 + 499e^{-0.45t}}$? What is the initial population for the model?

65. The population of a culture of bacteria is modeled by the logistic equation $P(t) = \frac{14{,}250}{1 + 29e^{-0.62t}}$, where t is in days. To the nearest tenth, how many days will it take the culture to reach 75 of its carrying capacity?

Show Solution

about 7.2 days

For the following exercises, use a graphing utility to create a scatter diagram of the data given in the table. Observe the shape of the scatter diagram to determine whether the data is best described by an exponential, logarithmic, or logistic model. Then use the appropriate regression feature to find an equation that models the data. When necessary, round values to five decimal places.

66.

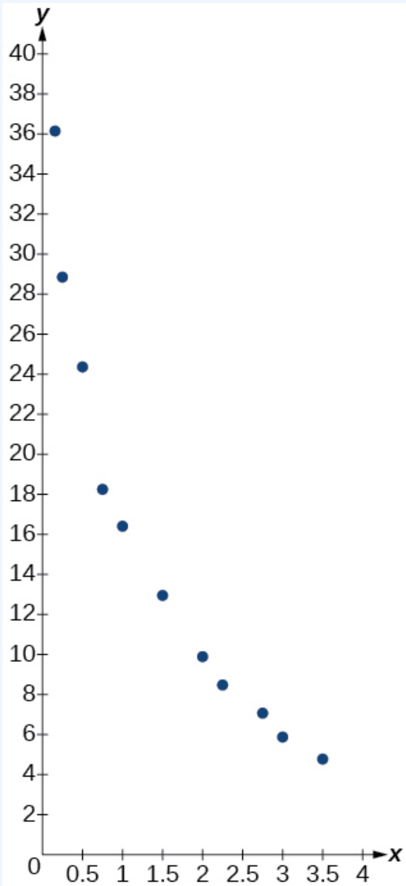
x	*f(x)*
1	409.4
2	260.7
3	170.4
4	110.6
5	74
6	44.7
7	32.4
8	19.5
9	12.7
10	8.1

67.

x	*f(x)*
0.15	36.21
0.25	28.88
0.5	24.39
0.75	18.28
1	16.5
1.5	12.99
2	9.91
2.25	8.57
2.75	7.23
3	5.99
3.5	4.81

Show Solution

logarithmic; $y = 16.68718 - 9.71860\ln\left(x\right)$


y
40
38
36
34
32
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
0.5
1
1.5
2
2.5
3
3.5
4
x

68.

x	$f(x)$
0	9
2	22.6
4	44.2
5	62.1
7	96.9
8	113.4
10	133.4
11	137.6
15	148.4
17	149.3

CHAPTER 10: COMPLEX NUMBERS

CHAPTER 10.1: COMPLEX NUMBERS

Learning Objectives

In this section you will:

- Add and subtract complex numbers.
- Multiply and divide complex numbers.
- Simplify powers of i .

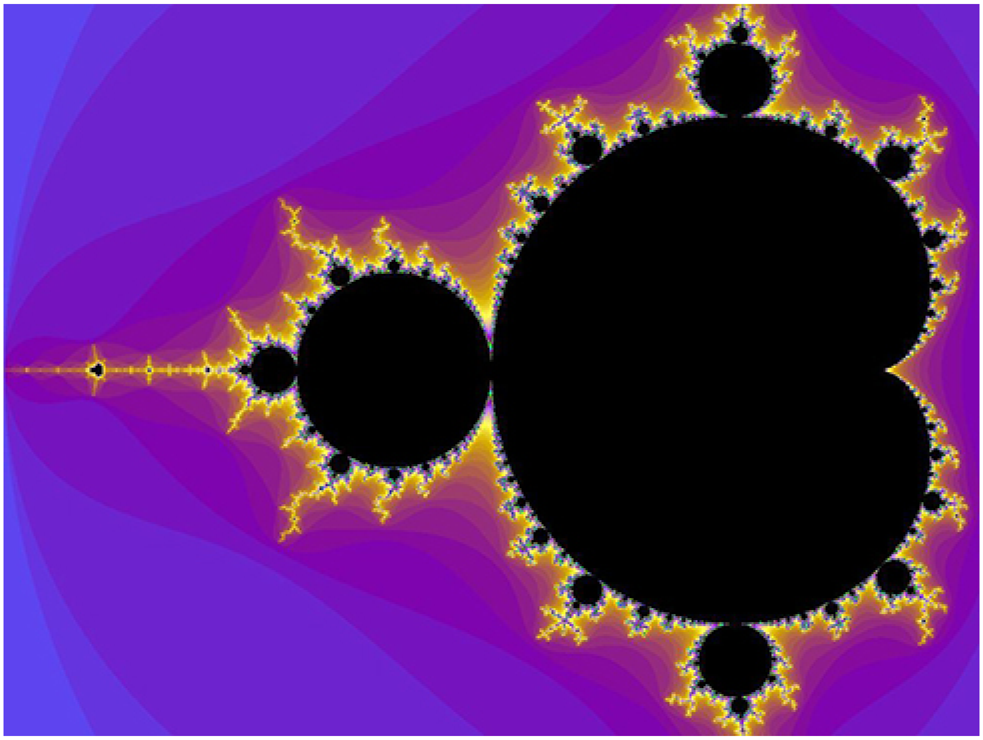


Figure 1.

Discovered by Benoit Mandelbrot around 1980, the Mandelbrot Set is one of the most recognizable fractal images. The image is built on the theory of self-similarity and the operation of iteration. Zooming in on a fractal image brings many surprises, particularly in the high level of repetition of detail that appears as magnification increases. The equation that generates this image turns out to be rather simple.

In order to better understand it, we need to become familiar with a new set of numbers. Keep in mind that the study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. In this section, we will explore a set of numbers that fills voids in the set of real numbers and find out how to work within it.

Expressing Square Roots of Negative Numbers as Multiples of i

We know how to find the square root of any positive real number. In a similar way, we can find the square root of any negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number**.** The imaginary number i is defined as the square root of $-1.$

$$\sqrt{-1} = i$$

So, using properties of radicals,

$$i^2 = \left(\sqrt{-1}\right)^2 = -1$$

We can write the square root of any negative number as a multiple of i. Consider the square root of $-49.$

$$\begin{aligned} \sqrt{-49} &= \sqrt{49 \cdot (-1)} \\ &= \sqrt{49}\sqrt{-1} \\ &= 7i \end{aligned}$$

We use $7i$ and not $-7i$ because the principal root of 49 is the positive root.

A complex number is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a + bi$ where a is the real part and b is the imaginary part. For example, $5 + 2i$ is a complex number. So, too, is $3 + 4i\sqrt{3}$.

Imaginary numbers differ from real numbers in that a squared imaginary number produces a negative real number. Recall that when a positive real number is squared, the result is a positive real number and when a negative real number is squared, the result is also a positive real number. Complex numbers consist of real and imaginary numbers.

Imaginary and Complex Numbers

A complex number is a number of the form $a + bi$ where

- a is the real part of the complex number.
- b is the imaginary part of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and b is not equal to 0, the complex number is called a pure imaginary number. An imaginary number is an even root of a negative number.

How To

Given an imaginary number, express it in the standard form of a complex number.

1. Write $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$.
2. Express $\sqrt{-1}$ as i.
3. Write $\sqrt{a} \cdot i$ in simplest form.

Expressing an Imaginary Number in Standard Form

Express $\sqrt{-9}$ in standard form.

Show Solution

$$\begin{aligned} \sqrt{-9} &= \sqrt{9}\sqrt{-1} \\ &= 3i \end{aligned}$$

In standard form, this is $0 + 3i$.

Try It

Express $\sqrt{-24}$ in standard form.

Show Solution
$\sqrt{-24} = 0 + 2i\sqrt{6}$

Plotting a Complex Number on the Complex Plane

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number, we need to address the two components of the number. We use the complex plane, which is a coordinate system in which the horizontal axis represents the real component and the vertical axis represents the imaginary component. Complex numbers are the points on the plane, expressed as ordered pairs (a, b) , where a represents the coordinate for the horizontal axis and b represents the coordinate for the vertical axis.

Let's consider the number $-2 + 3i$. The real part of the complex number is -2 and the imaginary part is 3. We plot the ordered pair $(-2, 3)$ to represent the complex number $-2 + 3i$, as shown in (Figure).

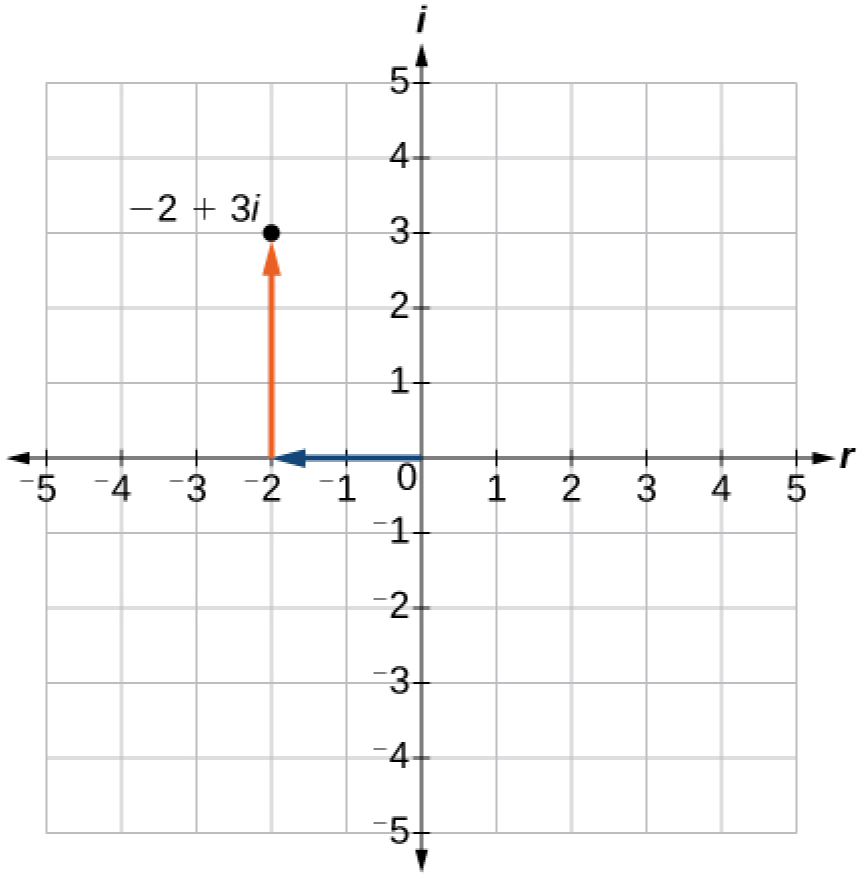


Figure 2.

Complex Plane

In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis, as shown in (Figure).

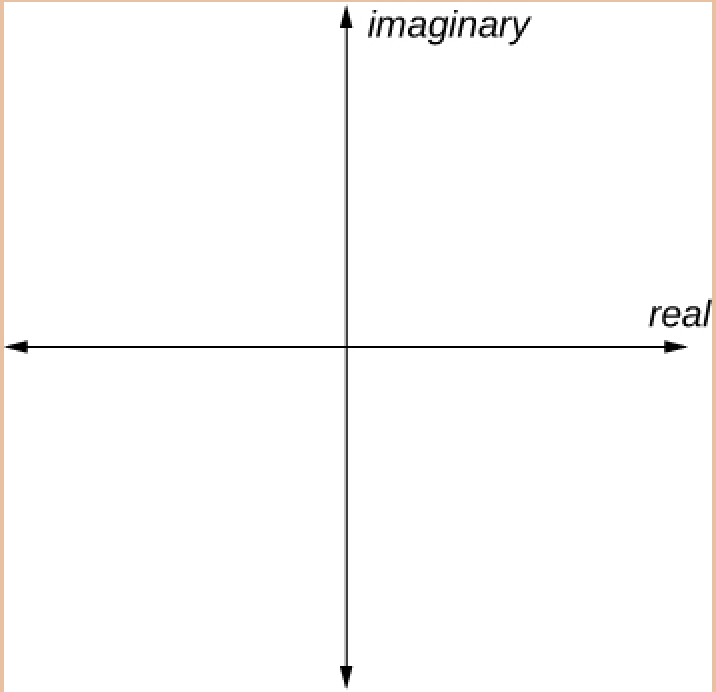


Figure 3.

How To

Given a complex number, represent its components on the complex plane.

1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

Plotting a Complex Number on the Complex Plane

Plot the complex number $3-4i$ on the complex plane.

Show Solution

The real part of the complex number is $3,$ and the imaginary part is –4. We plot the ordered pair $(3,-4)$ as shown in (Figure).

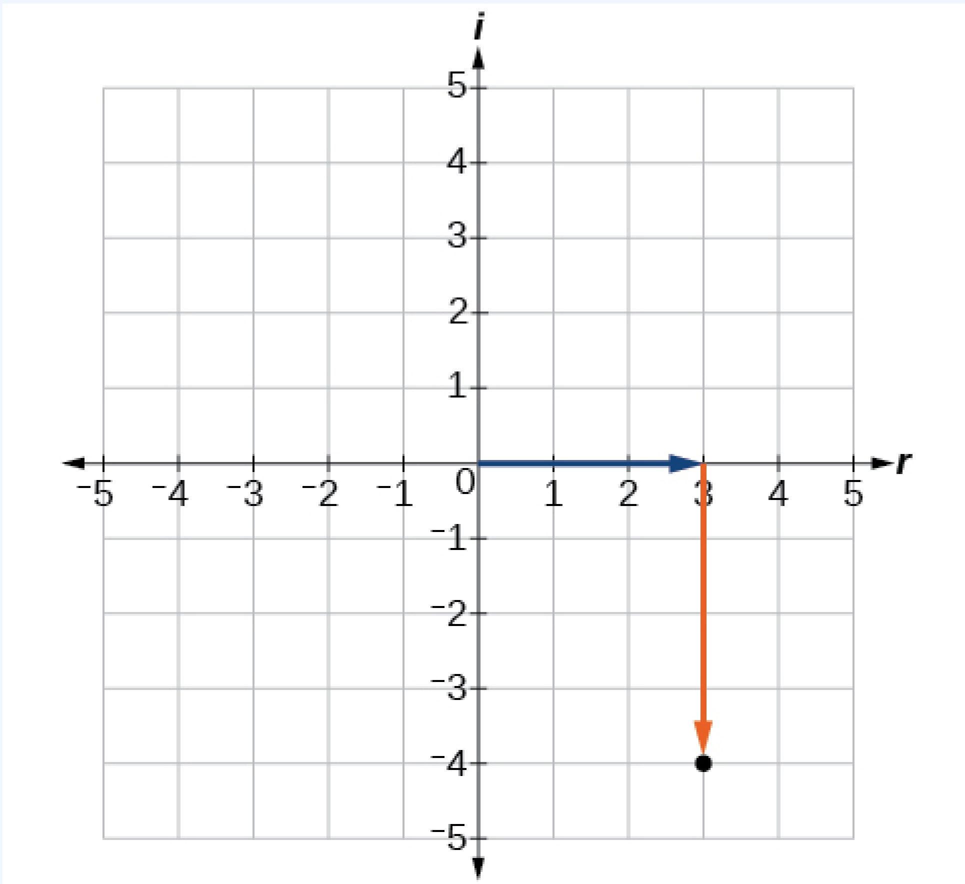
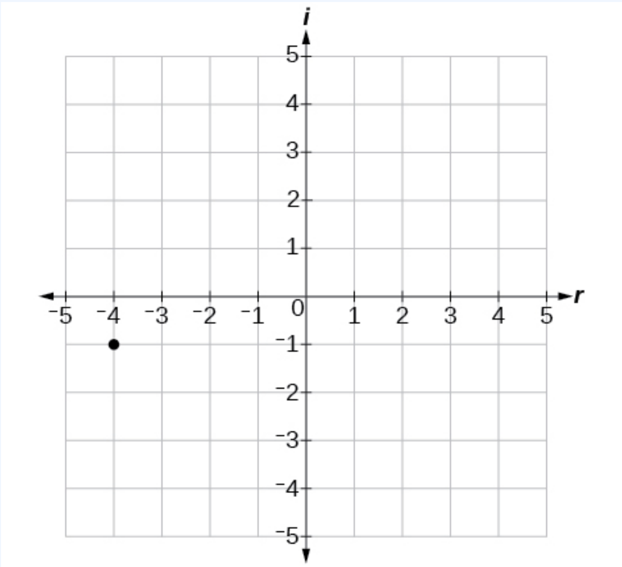


Figure 4.

Try It

Plot the complex number $-4 - i$ on the complex plane.

Show Solution



Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and then combine the imaginary parts.

Complex Numbers: Addition and Subtraction

Adding complex numbers:

$$(a + bi) + (c + di) = (a + c) + (b + d)\,i$$

Subtracting complex numbers:

$$(a+bi)-(c+di)=(a-c)+(b-d)\,i$$

How To

Given two complex numbers, find the sum or difference.

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

Adding and Subtracting Complex Numbers

Add or subtract as indicated.

1. $(3-4i)+(2+5i)$
2. $(-5+7i)-(-11+2i)$

Show Solution

We add the real parts and add the imaginary parts.

1.
$$\begin{aligned}(3-4i)+(2+5i) &= 3-4i+2+5i\\ &= 3+2+(-4i)+5i\\ &= (3+2)+(-4+5)\,i\\ &= 5+i\end{aligned}$$

2.
$$\begin{aligned}(-5+7i)-(-11+2i) &= -5+7i+11-2i\\ &= -5+11+7i-2i\\ &= (-5+11)+(7-2)\,i\\ &= 6+5i\end{aligned}$$

Try It

Subtract $2 + 5i$ from $3 - 4i.$

Show Solution
$(3 - 4i) - (2 + 5i) = 1 - 9i$

Multiplying Complex Numbers

Multiplying complex numbers is much like multiplying binomials. The major difference is that we work with the real and imaginary parts separately.

Multiplying a Complex Number by a Real Number

Lets begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. Consider, for example, $3\left(6 + 2i\right)$:

$$\begin{aligned} 3(6 + 2i) &= (3 \cdot 6) + (3 \cdot 2i) && \text{Distribute.} \\ &= 18 + 6i && \text{Simplify.} \end{aligned}$$

How To

Given a complex number and a real number, multiply to find the product.

1. Use the distributive property.
2. Simplify.

Multiplying a Complex Number by a Real Number

Find the product $4\left(2+5i\right)$.

Show Solution

Distribute the 4.

$$\begin{array}{lll}4\left(2+5i\right) & = & \left(4\cdot 2\right)+\left(4\cdot 5i\right)\\ & = & 8+20i\end{array}$$

Try It

Find the product: $\frac{1}{2}\left(5-2i\right)$.

Show Solution

$\frac{5}{2}-i$

Multiplying Complex Numbers Together

Now, let's multiply two complex numbers. We can use either the distributive property or more specifically the FOIL method because we are dealing with binomials. Recall that FOIL is an acronym for multiplying First, Inner, Outer, and Last terms together. The difference with complex numbers is that when we get a squared term, i^2, it equals -1.

$$\begin{array}{llll}\left(a+bi\right)\left(c+di\right) & = & ac+adi+bci+bdi^2 & \\ & = & ac+adi+bci-bd & i^2=-1\\ & = & \left(ac-bd\right)+\left(ad+bc\right)i & \text{Group real terms and imaginary terms.}\end{array}$$

How To

Given two complex numbers, multiply to find the product.

1. Use the distributive property or the FOIL method.
2. Remember that $i^2 = -1.$
3. Group together the real terms and the imaginary terms

Multiplying a Complex Number by a Complex Number

Multiply: $(4 + 3i)(2 - 5i)$.

Show Solution

$$\begin{array}{lll} (4+3i)(2-5i) & = & 4(2) - 4(5i) + 3i(2) - (3i)(5i) \\ & = & 8 - 20i + 6i - 15\left(i^2\right) \\ & = & (8+15) + (-20+6)i \\ & = & 23 - 14i \end{array}$$

Try It

Multiply: $(3 - 4i)(2 + 3i)$.

Show Solution

$18+i$

Dividing Complex Numbers

Dividing two complex numbers is more complicated than adding, subtracting, or multiplying because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator to write the answer in standard form $a+bi$. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the complex conjugate of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of $a+bi$ is $a-bi$. For example, the product of $a+bi$ and $a-bi$ is

$$\begin{aligned}(a+bi)(a-bi) &= a^2-abi+abi-b^2i^2\\ &= a^2+b^2\end{aligned}$$

The result is a real number.

Note that complex conjugates have an opposite relationship: The complex conjugate of $a+bi$ is $a-bi$, and the complex conjugate of $a-bi$ is $a+bi$. Further, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide $c+di$ by $a+bi$, where neither a nor b equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.

$$\frac{c+di}{a+bi} \text{ where } a\neq 0 \text{ and } b\neq 0$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(c+di)}{(a+bi)}\cdot\frac{(a-bi)}{(a-bi)}=\frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$

Apply the distributive property.

$$=\frac{ca-cbi+adi-bdi^2}{a^2-abi+abi-b^2i^2}$$

Simplify, remembering that $i^2=-1$.

$$=\frac{ca-cbi+adi-bd(-1)}{a^2-abi+abi-b^2(-1)}$$
$$=\frac{(ca+bd)+(ad-cb)i}{a^2+b^2}$$

The Complex Conjugate

The complex conjugate of a complex number $a + bi$ is $a - bi.$ It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

Finding Complex Conjugates

Find the complex conjugate of each number.

1. $2 + i\sqrt{5}$
2. $-\frac{1}{2}i$

Show Solution

1. The number is already in the form $a + bi.$ The complex conjugate is $a - bi,$ or $2 - i\sqrt{5}.$
2. We can rewrite this number in the form $a + bi$ as $0 - \frac{1}{2}i.$ The complex conjugate is $a - bi,$ or $0 + \frac{1}{2}i.$ This can be written simply as $\frac{1}{2}i.$

Analysis

Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by $i.$

Try It

Find the complex conjugate of $-3+4i$.

Show Solution
$-3-4i$

How To

Given two complex numbers, divide one by the other.

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

Dividing Complex Numbers

Divide: $(2+5i)$ by $(4-i)$.

Show Solution

We begin by writing the problem as a fraction.

$\frac{(2+5i)}{(4-i)}$

Then we multiply the numerator and denominator by the complex conjugate of the denominator.

$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}$

To multiply two complex numbers, we expand the product as we would with polynomials (using FOIL).

$$\begin{aligned}
\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)} &= \frac{8+2i+20i+5i^2}{16+4i-4i-i^2} \\
&= \frac{8+2i+20i+5(-1)}{16+4i-4i-(-1)} && \text{Because} i^2 = -1. \\
&= \frac{3+22i}{17} \\
&= \frac{3}{17} + \frac{22}{17}i && \text{Separate real and imaginary parts.}
\end{aligned}$$

Note that this expresses the quotient in standard form.

Simplifying Powers of *i*

The powers of i are cyclic. Let's look at what happens when we raise i to increasing powers.

$i^1 = i$
$i^2 = -1$
$i^3 = i^2 \cdot i = -1 \cdot i = -i$
$i^4 = i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1$
$i^5 = i^4 \cdot i = 1 \cdot i = i$

We can see that when we get to the fifth power of i, it is equal to the first power. As we continue to multiply i by increasing powers, we will see a cycle of four. Let's examine the next four powers of i.

$i^6 = i^5 \cdot i = i \cdot i = i^2 = -1$
$i^7 = i^6 \cdot i = i^2 \cdot i = i^3 = -i$
$i^8 = i^7 \cdot i = i^3 \cdot i = i^4 = 1$
$i^9 = i^8 \cdot i = i^4 \cdot i = i^5 = i$

The cycle is repeated continuously: $i, -1, -i, 1,$ every four powers.

Simplifying Powers of i

Evaluate: i^{35}.

Show Solution

Since $i^4 = 1,$ we can simplify the problem by factoring out as many factors of i^4 as possible. To do so, first determine how many times 4 goes into 35: $35 = 4 \cdot 8 + 3.$

$$i^{35} = i^{4 \cdot 8+3} = i^{4 \cdot 8} \cdot i^3 = \left(i^4\right)^8 \cdot i^3 = 1^8 \cdot i^3 = i^3 = -i$$

Try It

Evaluate: i^{18}

Show Solution

-1

Can we write i^{35} in other helpful ways?

As we saw in (Figure), we reduced i^{35} to i^3 by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of i^{35} may be more useful. (Figure) shows some other possible factorizations.

Factorization of i^{35}	$i^{34} \cdot i$	$i^{33} \cdot i^{2}$	$i^{31} \cdot i^{4}$	$i^{19} \cdot i^{16}$
Reduced form	$\left(i^{2}\right)^{17} \cdot i$	$i^{33} \cdot (-1)$	$i^{31} \cdot 1$	$i^{19} \cdot \left(i^{4}\right)^{4}$
Simplified form	$(-1)^{17} \cdot i$	$-i^{33}$	i^{31}	i^{19}

Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

Access these online resources for additional instruction and practice with complex numbers.

- Adding and Subtracting Complex Numbers
- Multiply Complex Numbers
- Multiplying Complex Conjugates
- Raising i to Powers

Key Concepts

- The square root of any negative number can be written as a multiple of i. See (Figure).
- To plot a complex number, we use two number lines, crossed to form the complex plane. The horizontal axis is the real axis, and the vertical axis is the imaginary axis. See (Figure).
- Complex numbers can be added and subtracted by combining the real parts and combining the imaginary parts. See (Figure).
- Complex numbers can be multiplied and divided.
    - To multiply complex numbers, distribute just as with polynomials. See (Figure) and (Figure).
    - To divide complex numbers, multiply both numerator and denominator by the complex conjugate of the denominator to eliminate the complex number from the denominator. See (Figure) and (Figure).
- The powers of i are cyclic, repeating every fourth one. See (Figure).

Section Exercises

Verbal

1. Explain how to add complex numbers.

Show Solution
Add the real parts together and the imaginary parts together.

2. What is the basic principle in multiplication of complex numbers?

3. Give an example to show that the product of two imaginary numbers is not always imaginary.

Show Solution
Possible answer: i times i equals -1, which is not imaginary.

4. What is a characteristic of the plot of a real number in the complex plane?

Algebraic

For the following exercises, evaluate the algebraic expressions.

5. If $y = x^2 + x - 4,$ evaluate y given $x = 2i.$

Show Solution
$-8 + 2i$

6. If $y = x^3 - 2,$ evaluate y given $x = i.$

7. If $y = x^2 + 3x + 5,$ evaluate y given $x = 2 + i.$

Show Solution

$14 + 7i$

8. If $y = 2x^2 + x - 3$, evaluate y given $x = 2 - 3i$.

9. If $y = \frac{x+1}{2-x}$, evaluate y given $x = 5i$.

Show Solution

$-\frac{23}{29} + \frac{15}{29}i$

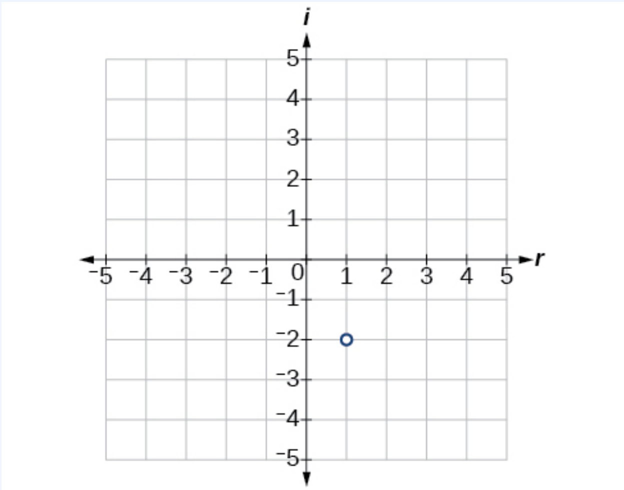
10. If $y = \frac{1+2x}{x+3}$, evaluate y given $x = 4i$.

Graphical

For the following exercises, plot the complex numbers on the complex plane.

11. $1 - 2i$

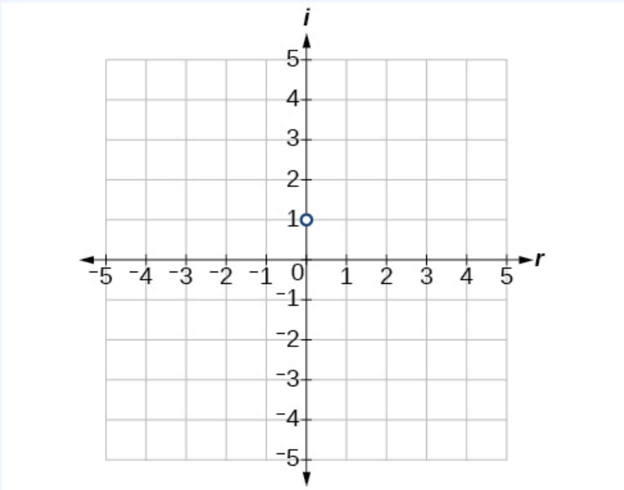
Show Solution



12. $-2+3i$

13. i

Show Solution



14. $-3-4i$

Numeric

For the following exercises, perform the indicated operation and express the result as a simplified complex number.

15. $(3 + 2i) + (5 - 3i)$

Show Solution
$8 - i$

16. $(-2 - 4i) + (1 + 6i)$

17. $(-5 + 3i) - (6 - i)$

Show Solution
$-11 + 4i$

18. $(2 - 3i) - (3 + 2i)$

19. $(-4 + 4i) - (-6 + 9i)$

Show Solution
$2 - 5i$

20. $(2 + 3i)(4i)$

21. $(5 - 2i)(3i)$

Show Solution
$6+15i$

22. $(6-2i)(5)$

23. $(-2+4i)(8)$

Show Solution
$-16+32i$

24. $(2+3i)(4-i)$

25. $(-1+2i)(-2+3i)$

Show Solution
$-4-7i$

26. $(4-2i)(4+2i)$

27. $(3+4i)(3-4i)$

Show Solution
25

28. $\frac{3+4i}{2}$

29. $\frac{6-2i}{3}$

Show Solution
$2 - \frac{2}{3}i$

30. $\frac{-5+3i}{2i}$

31. $\frac{6+4i}{i}$

Show Solution
$4 - 6i$

32. $\frac{2-3i}{4+3i}$

33. $\frac{3+4i}{2-i}$

Show Solution
$\frac{2}{5} + \frac{11}{5}i$

34. $\frac{2+3i}{2-3i}$

35. $\sqrt{-9} + 3\sqrt{-16}$

Show Solution
$15i$

36. $-\sqrt{-4} - 4\sqrt{-25}$

37. $\frac{2+\sqrt{-12}}{2}$

Show Solution
$1+i\sqrt{3}$

38. $\frac{4+\sqrt{-20}}{2}$

39. i^{8}

Show Solution
1

40. i^{15}

41. i^{22}

Show Solution
-1

Technology

For the following exercises, use a calculator to help answer the questions.

42. Evaluate $(1+i)^{k}$ for $k=4, 8, \text{and } 12.$ Predict the value if $k=16.$

43. Evaluate $(1-i)^{k}$ for $k=2, 6, \text{and } 10.$ Predict the value if $k=14.$

Show Solution
128i

44. Evaluate $(1+i)^k-(1-i)^k$ for $k=4, 8, \text{and } 12.$ Predict the value for $k=16.$

45. Show that a solution of $x^6+1=0$ is $\frac{\sqrt{3}}{2}+\frac{1}{2}i.$

Show Solution
$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^6=-1$

46. Show that a solution of $x^8-1=0$ is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i.$

Extensions

For the following exercises, evaluate the expressions, writing the result as a simplified complex number.

47. $\frac{1}{i}+\frac{4}{i^3}$

Show Solution
$3i$

48. $\frac{1}{i^{11}}-\frac{1}{i^{21}}$

49. $i^7\left(1+i^2\right)$

Show Solution
0

50. $i^{-3}+5i^7$

51. $\frac{(2+i)(4-2i)}{(1+i)}$

Show Solution
$5 - 5i$

52. $\frac{(1+3i)(2-4i)}{(1+2i)}$

53. $\frac{(3+i)^2}{(1+2i)^2}$

Show Solution
$-2i$

54. $\frac{3+2i}{2+i} + (4+3i)$

55. $\frac{4+i}{i} + \frac{3-4i}{1-i}$

Show Solution
$\frac{9}{2} - \frac{9}{2}i$

56. $\frac{3+2i}{1+2i} - \frac{2-3i}{3+i}$

Glossary

complex conjugate
: a complex number containing the same terms as another complex number, but with the opposite operator. Multiplying a complex number by its conjugate yields a real number.

complex number
: the sum of a real number and an imaginary number; the standard form is $a+bi,$ where *a* is the real part and b is the complex part.

complex plane
: the coordinate plane in which the horizontal axis represents the real component of a complex number, and the vertical axis represents the imaginary component, labeled *i*.

imaginary number
: the square root of $-1: i=\sqrt{-1}.$

CHAPTER 10.2: POLAR FORM OF COMPLEX NUMBERS

Learning Objectives

In this section, you will:

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

"God made the integers; all else is the work of man." This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in Complex Numbers. In this section, we will focus on the mechanics of working with complex numbers: translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre's Theorem.

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a + bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

How To

Given a complex number $a + bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis.*
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

Plotting a Complex Number in the Complex Plane

Plot the complex number $2 - 3i$ in the complex plane.

Show Solution

From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See (Figure).

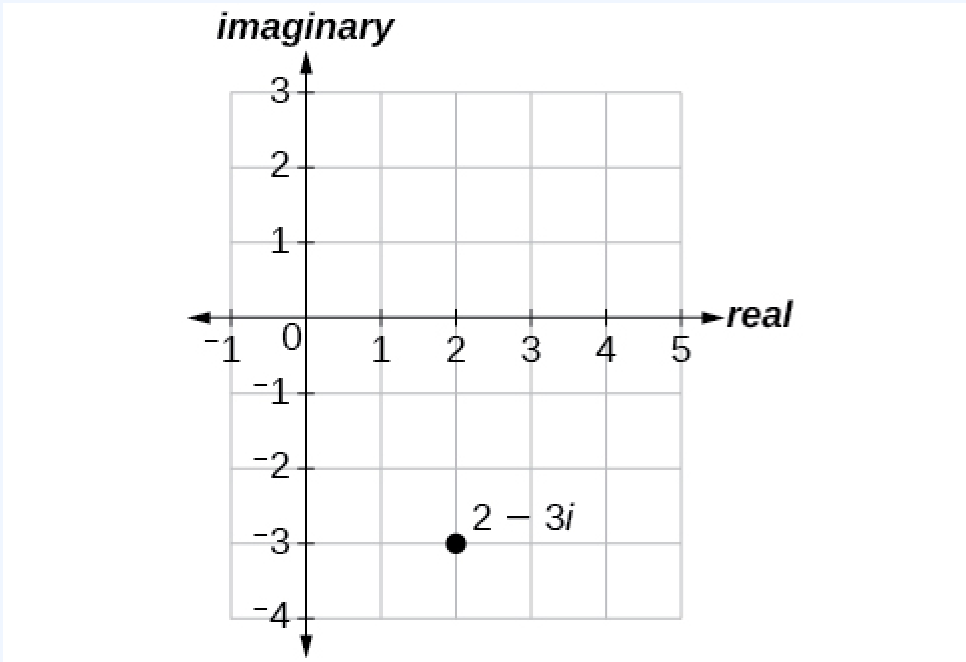
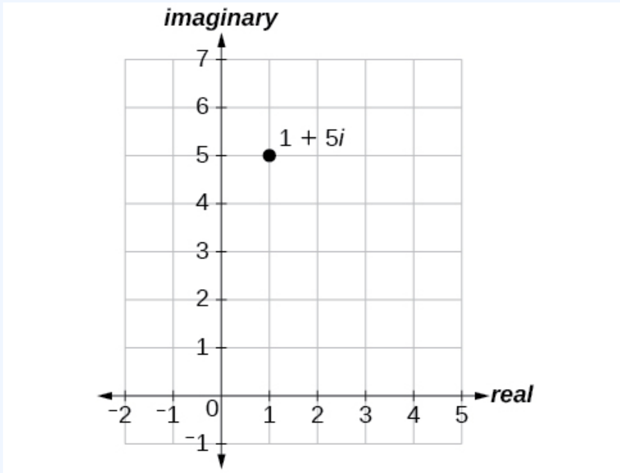


Figure 1.

Try It

Plot the point $1 + 5i$ in the complex plane.

Show Solution



Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z = 2 + 4i$, in (Figure), shows $|z|$.

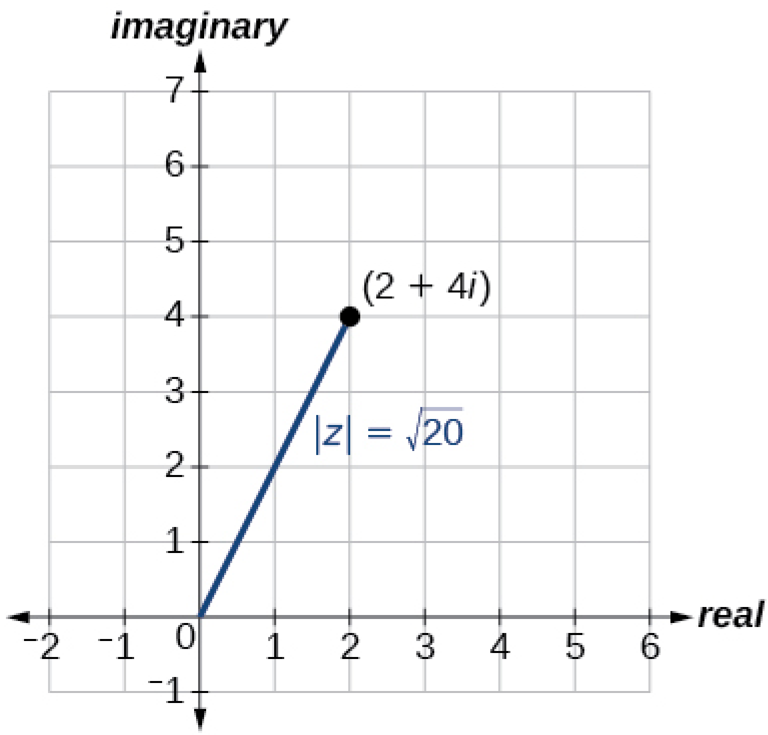


Figure 2.

Absolute Value of a Complex Number

Given $z = x + yi$, a complex number, the absolute value of z is defined as

$|z| = \sqrt{x^2 + y^2}$

It is the distance from the origin to the point (x, y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, $(0,\ 0)$.

Finding the Absolute Value of a Complex Number with a Radical

Find the absolute value of $z = \sqrt{5} - i$.

Show Solution

Using the formula, we have

$|z| = \sqrt{x^2 + y^2}$
$|z| = \sqrt{\left(\sqrt{5}\right)^2 + (-1)^2}$
$|z| = \sqrt{5 + 1}$
$|z| = \sqrt{6}$

See (Figure).

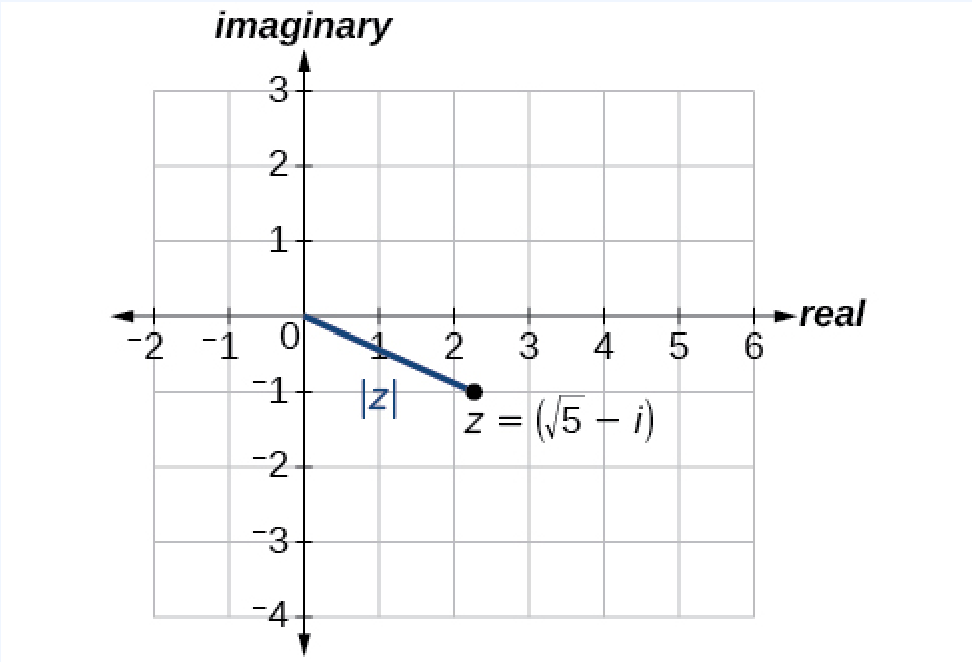


Figure 3.

Try It

Find the absolute value of the complex number $z = 12 - 5i$.

Show Solution

13

Finding the Absolute Value of a Complex Number

Given $z = 3 - 4i$, find $|z|$.

Show Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(3)^2 + (-4)^2}$$
$$|z| = \sqrt{9 + 16}$$
$$|z| = \sqrt{25}$$
$$|z| = 5$$

The absolute value z is 5. See (Figure).

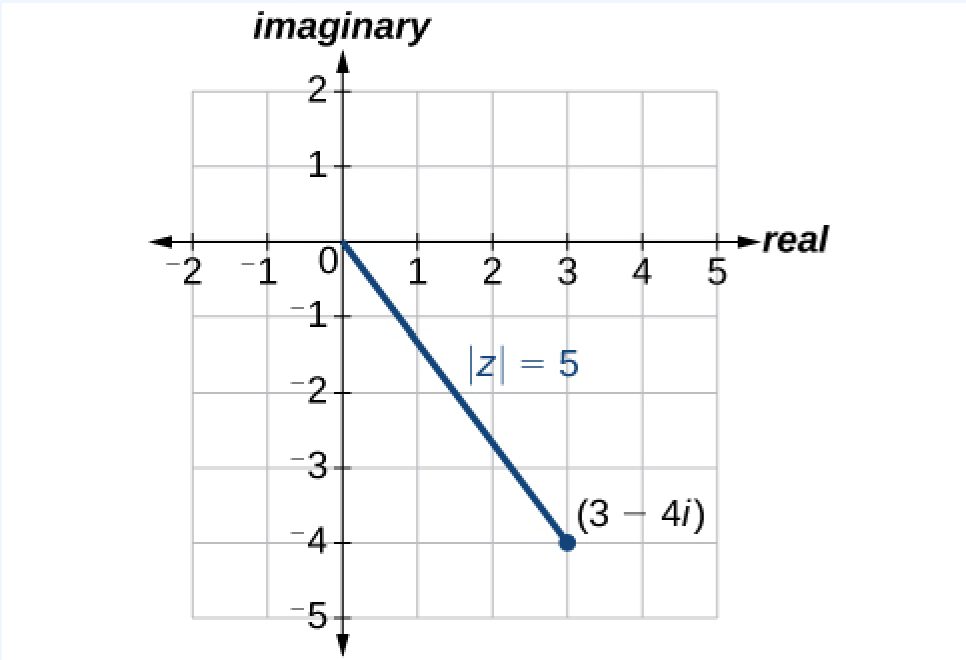


Figure 4.

Try It

Given $z = 1 - 7i$, find $|z|$.

Show Solution

$|z| = \sqrt{50} = 5\sqrt{2}$

Writing Complex Numbers in Polar Form

The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$

We review these relationships in (Figure).

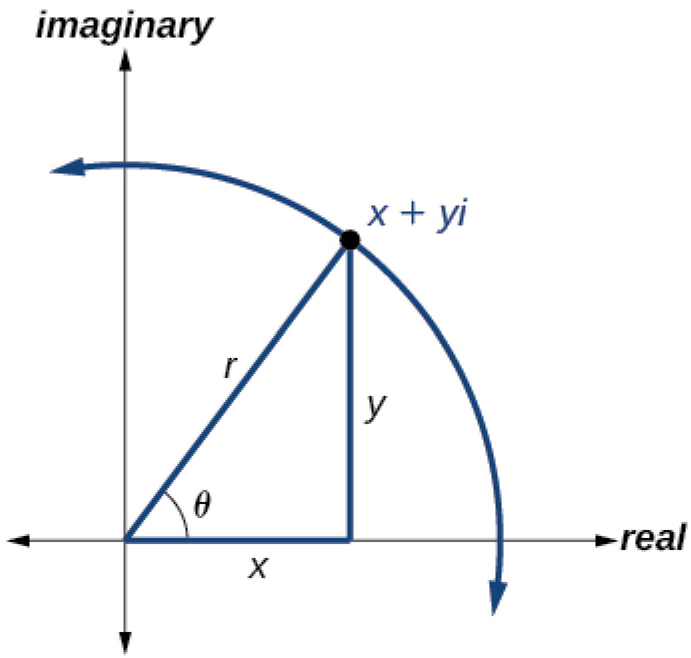


Figure 5.

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x, y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$z = x + yi$$
$$z = r\cos\theta + (r\sin\theta)\, i$$
$$z = r\,(\cos\theta + i\sin\theta)$$

Polar Form of a Complex Number

Writing a complex number in polar form involves the following conversion formulas:

$x = r\cos\theta$
$y = r\sin\theta$
$r = \sqrt{x^2 + y^2}$

Making a direct substitution, we have
$z = x + yi$
$z = (r\cos\theta) + i\,(r\sin\theta)$
$z = r\,(\cos\theta + i\sin\theta)$

where r is the modulus and θ is the argument. We often use the abbreviation $r\text{cis}\,\theta$ to represent $r\,(\cos\theta + i\sin\theta)$.

Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Show Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$r = \sqrt{x^2 + y^2}$
$r = \sqrt{0^2 + 4^2}$
$r = \sqrt{16}$
$r = 4$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \frac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ or $4\text{cis}\left(\frac{\pi}{2}\right)$. See (Figure).

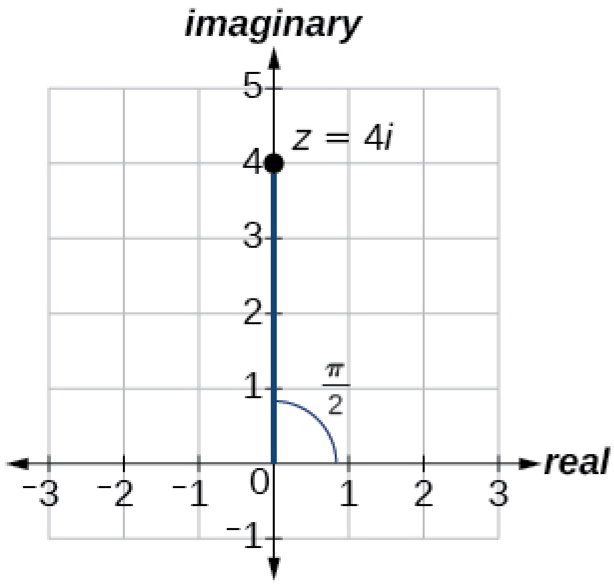


Figure 6.

Try It

Express $z = 3i$ as $r\,\text{cis}\,\theta$ in polar form.

Show Solution

$z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

Finding the Polar Form of a Complex Number

Find the polar form of $-4+4i$.

Show Solution

First, find the value of r.

$$r=\sqrt{x^2+y^2}$$
$$r=\sqrt{(-4)^2+(4^2)}$$
$$r=\sqrt{32}$$
$$r=4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta=\frac{x}{r}$$
$$\cos\theta=\frac{-4}{4\sqrt{2}}$$
$$\cos\theta=-\frac{1}{\sqrt{2}}$$
$$\theta=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\frac{3\pi}{4}$$

Thus, the solution is $4\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$.

Try It

Write $z=\sqrt{3}+i$ in polar form.

Show Solution

$$z=2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given $z = r\left(\cos\theta + i\sin\theta\right)$, first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$. Then, multiply through by r.

Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:

$z = 12\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

Show Solution

We begin by evaluating the trigonometric expressions.

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

After substitution, the complex number is

$z = 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

We apply the distributive property:

$$\begin{aligned} z &= 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= (12)\frac{\sqrt{3}}{2} + (12)\frac{1}{2}i \\ &= 6\sqrt{3} + 6i \end{aligned}$$

The rectangular form of the given point in complex form is $6\sqrt{3} + 6i$.

Finding the Rectangular Form of a Complex Number

Find the rectangular form of the complex number given $r = 13$ and $\tan\theta = \frac{5}{12}$.

Show Solution

If $\tan\theta = \frac{5}{12}$, and $\tan\theta = \frac{y}{x}$, we first determine $r = \sqrt{x^2 + y^2} = \sqrt{12^2 + 5^2} = 13$. We then find $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

$$z = 13\left(\cos\theta + i\sin\theta\right)$$
$$= 13\left(\frac{12}{13} + \frac{5}{13}i\right)$$
$$= 12 + 5i$$

The rectangular form of the given number in complex form is $12 + 5i$.

Try It

Convert the complex number to rectangular form:

$$z = 4\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

Show Solution

$$z = 2\sqrt{3} - 2i$$

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham de Moivre (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

Products of Complex Numbers in Polar Form

If $z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$
$$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$$

Notice that the product calls for multiplying the moduli and adding the angles.

Finding the Product of Two Complex Numbers in Polar Form

Find the product of $z_1 z_2$, given $z_1 = 4 (\cos(80) + i\sin(80))$ and $z_2 = 2 (\cos(145) + i\sin(145))$.

Show Solution

Follow the formula

$$z_1 z_2 = 4 \cdot 2 [\cos(80 + 145) + i\sin(80 + 145)]$$
$$z_1 z_2 = 8 [\cos(225) + i\sin(225)]$$
$$z_1 z_2 = 8 \left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$z_1 z_2 = 8 \left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$z_1 z_2 = -4\sqrt{2} - 4i\sqrt{2}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

Quotients of Complex Numbers in Polar Form

If $z_1 = r_1 \left(\cos \theta_1 + i\sin \theta_1\right)$ and $z_2 = r_2 \left(\cos \theta_2 + i\sin \theta_2\right)$, then the quotient of these numbers is

$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left[\cos \left(\theta_1 - \theta_2\right) + i\sin \left(\theta_1 - \theta_2\right)\right], \;\; z_2 \neq 0$
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis} \left(\theta_1 - \theta_2\right), \;\; z_2 \neq 0$

Notice that the moduli are divided, and the angles are subtracted.

How To

Given two complex numbers in polar form, find the quotient.

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r \left(\cos \theta + i\sin \theta\right)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2 \left(\cos \left(213\right) + i\sin \left(213\right)\right)$ and $z_2 = 4 \left(\cos \left(33\right) + i\sin \left(33\right)\right)$.

Show Solution

Using the formula, we have

$$\frac{z_1}{z_2} = \frac{2}{4}\left[\cos\left(213 - 33\right) + i\sin\left(213 - 33\right)\right]$$
$$\frac{z_1}{z_2} = \frac{1}{2}\left[\cos\left(180\right) + i\sin\left(180\right)\right]$$
$$\frac{z_1}{z_2} = \frac{1}{2}\left[-1 + 0i\right]$$
$$\frac{z_1}{z_2} = -\frac{1}{2} + 0i$$
$$\frac{z_1}{z_2} = -\frac{1}{2}$$

Try It

Find the product and the quotient of $z_1 = 2\sqrt{3}\left(\cos\left(150\right) + i\sin\left(150\right)\right)$ and $z_2 = 2\left(\cos\left(30\right) + i\sin\left(30\right)\right).$

Show Solution

$z_1 z_2 = -4\sqrt{3};\ \frac{z_1}{z_2} = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using De Moivre's Theorem. It states that, for a positive integer n, z^n is found by raising the modulus to the $n\mathrm{th}$ power and multiplying the argument by n. It is the standard method used in modern mathematics.

De Moivre's Theorem

If $z = r\left(\cos\theta + i\sin\theta\right)$ is a complex number, then

$z^n = r^n\left[\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right]$
$z^n = r^n \text{cis}\left(n\theta\right)$
where n
is a positive integer.

Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $\left(1+i\right)^5$ using De Moivre's Theorem.

Show Solution

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $\left(1+i\right)$ in polar form. Let us find r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{\left(1\right)^2 + \left(1\right)^2}$$
$$r = \sqrt{2}$$

Then we find θ. Using the formula $\tan\theta = \frac{y}{x}$ gives

$$\tan\theta = \frac{1}{1}$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Use De Moivre's Theorem to evaluate the expression.

$$\left(a+bi\right)^n = r^n\left[\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right]$$
$$\left(1+i\right)^5 = \left(\sqrt{2}\right)^5\left[\cos\left(5\cdot\frac{\pi}{4}\right) + i\sin\left(5\cdot\frac{\pi}{4}\right)\right]$$
$$\left(1+i\right)^5 = 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$\left(1+i\right)^5 = 4\sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$\left(1+i\right)^5 = -4 - 4i$$

Finding Roots of Complex Numbers in Polar Form

To find the *n*th root of a complex number in polar form, we use the $n\text{th}$ Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding $n\text{th}$ roots of complex numbers in polar form.

The *n*th Root Theorem

To find the $n\text{th}$ root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

where $k = 0,\ 1,\ 2,\ 3,\ .\ .\ .\ ,\ n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

Finding the *n*th Root of a Complex Number

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$.

Show Solution

We have

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right)\right]$$
$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)\right]$$

There will be three roots: $k = 0,\ 1,\ 2.$ When $k = 0,$ we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)\right)$$

When $k = 1,$ we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9} + \frac{6\pi}{9}\right) + i\sin\left(\frac{2\pi}{9} + \frac{6\pi}{9}\right)\right] \qquad \text{Add } \frac{2(1)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9}\right)\right)$$

When $k = 2,$ we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9} + \frac{12\pi}{9}\right) + i\sin\left(\frac{2\pi}{9} + \frac{12\pi}{9}\right)\right] \qquad \text{Add } \frac{2(2)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{14\pi}{9}\right) + i\sin\left(\frac{14\pi}{9}\right)\right)$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1,$ the angle simplification is

$$\begin{aligned}\frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3} &= \frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right)\\ &= \frac{2\pi}{9}+\frac{6\pi}{9}\\ &= \frac{8\pi}{9}\end{aligned}$$

Try It

Find the four fourth roots of $16\left(\cos\left(120\right)+i\sin\left(120\right)\right)$.

Show Solution

$z_0 = 2\left(\cos\left(30\right)+i\sin\left(30\right)\right)$

$z_1 = 2\left(\cos\left(120\right)+i\sin\left(120\right)\right)$

$z_2 = 2\left(\cos\left(210\right)+i\sin\left(210\right)\right)$

$z_3 = 2\left(\cos\left(300\right)+i\sin\left(300\right)\right)$

Access these online resources for additional instruction and practice with polar forms of complex numbers.

- The Product and Quotient of Complex Numbers in Trigonometric Form
- De Moivre's Theorem

Key Concepts

- Complex numbers in the form $a+bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the *x*-axis as the *real* axis and the *y*-axis as the *imaginary* axis. See (Figure).

- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2 + b^2}$. See (Figure) and (Figure).
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta, y = r\sin\theta,$ and $r = \sqrt{x^2 + y^2}$. Then, $z = r\left(\cos\theta + i\sin\theta\right)$. See (Figure) and (Figure).
- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See (Figure) and (Figure).
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See (Figure).
- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See (Figure).
- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See (Figure).
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See (Figure).

Section Exercises

Verbal

1. A complex number is $a + bi$. Explain each part.

Show Solution
a is the real part, *b* is the imaginary part, and $i = \sqrt{-1}$

2. What does the absolute value of a complex number represent?

3. How is a complex number converted to polar form?

Show Solution
Polar form converts the real and imaginary part of the complex number in polar form using $x = r\cos\theta$ and $y = r\sin\theta$.

4. How do we find the product of two complex numbers?

5. What is De Moivre's Theorem and what is it used for?

Show Solution
$z^n = r^n \left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$ It is used to simplify polar form when a number has been raised to a power.

Algebraic

For the following exercises, find the absolute value of the given complex number.

6. $5 + 3i$

7. $-7 + i$

Show Solution
$5\sqrt{2}$

8. $-3 - 3i$

9. $\sqrt{2} - 6i$

Show Solution

$\sqrt{38}$

10. $2i$

11. $2.2-3.1i$

Show Solution
$\sqrt{14.45}$

For the following exercises, write the complex number in polar form.

12. $2+2i$

13. $8-4i$

Show Solution
$4\sqrt{5}\text{cis}\,(333.4)$

14. $-\frac{1}{2}-\frac{1}{2}i$

15. $\sqrt{3}+i$

Show Solution
$2\text{cis}\left(\frac{\pi}{6}\right)$

16. $3i$

For the following exercises, convert the complex number from polar to rectangular form.

17. $z = 7\text{cis}\left(\frac{\pi}{6}\right)$

Show Solution
$\frac{7\sqrt{3}}{2} + i\frac{7}{2}$

18. $z = 2\text{cis}\left(\frac{\pi}{3}\right)$

19. $z = 4\text{cis}\left(\frac{7\pi}{6}\right)$

Show Solution
$-2\sqrt{3} - 2i$

20. $z = 7\text{cis}\left(25\right)$

21. $z = 3\text{cis}\left(240\right)$

Show Solution
$-1.5 - i\frac{3\sqrt{3}}{2}$

22. $z = \sqrt{2}\text{cis}\left(100\right)$

For the following exercises, find $z_1 z_2$ in polar form.

23. $z_1 = 2\sqrt{3}\text{cis}\left(116\right); \quad z_2 = 2\text{cis}\left(82\right)$

Show Solution
$4\sqrt{3}\mathrm{cis}\left(198\right)$

24. $z_1=\sqrt{2}\mathrm{cis}\left(205\right);\;\;z_2=2\sqrt{2}\mathrm{cis}\left(118\right)$
25. $z_1=3\mathrm{cis}\left(120\right);\;\;z_2=\frac{1}{4}\mathrm{cis}\left(60\right)$

Show Solution
$\frac{3}{4}\mathrm{cis}\left(180\right)$

26. $z_1=3\mathrm{cis}\left(\frac{\pi}{4}\right);\;\;z_2=5\mathrm{cis}\left(\frac{\pi}{6}\right)$
27. $z_1=\sqrt{5}\mathrm{cis}\left(\frac{5\pi}{8}\right);\;\;z_2=\sqrt{15}\mathrm{cis}\left(\frac{\pi}{12}\right)$

Show Solution
$5\sqrt{3}\mathrm{cis}\left(\frac{17\pi}{24}\right)$

28. $z_1=4\mathrm{cis}\left(\frac{\pi}{2}\right);\;\;z_2=2\mathrm{cis}\left(\frac{\pi}{4}\right)$

For the following exercises, find $\frac{z_1}{z_2}$ in polar form.

29. $z_1=21\mathrm{cis}\left(135\right);\;\;z_2=3\mathrm{cis}\left(65\right)$

Show Solution
$7\mathrm{cis}\left(70\right)$

30. $z_1=\sqrt{2}\mathrm{cis}\left(90\right);\;\;z_2=2\mathrm{cis}\left(60\right)$

31. $z_1 = 15\text{cis}\left(120\right);\ \ z_2 = 3\text{cis}\left(40\right)$

Show Solution
$5\text{cis}\left(80\right)$

32. $z_1 = 6\text{cis}\left(\frac{\pi}{3}\right);\ \ z_2 = 2\text{cis}\left(\frac{\pi}{4}\right)$

33. $z_1 = 5\sqrt{2}\text{cis}\left(\pi\right);\ \ z_2 = \sqrt{2}\text{cis}\left(\frac{2\pi}{3}\right)$

Show Solution
$5\text{cis}\left(\frac{\pi}{3}\right)$

34. $z_1 = 2\text{cis}\left(\frac{3\pi}{5}\right);\ \ z_2 = 3\text{cis}\left(\frac{\pi}{4}\right)$

For the following exercises, find the powers of each complex number in polar form.

35. Find z^3 when $z = 5\text{cis}\left(45\right)$.

Show Solution
$125\text{cis}\left(135\right)$

36. Find z^4 when $z = 2\text{cis}\left(70\right)$.

37. Find z^2 when $z = 3\text{cis}\left(120\right)$.

Show Solution

$9\text{cis}\left(240\right)$

38. Find z^2 when $z = 4\text{cis}\left(\frac{\pi}{4}\right)$.

39. Find z^4 when $z = \text{cis}\left(\frac{3\pi}{16}\right)$.

Show Solution

$\text{cis}\left(\frac{3\pi}{4}\right)$

40. Find z^3 when $z = 3\text{cis}\left(\frac{5\pi}{3}\right)$.

For the following exercises, evaluate each root.

41. Evaluate the cube root of z when $z = 27\text{cis}\left(240\right)$.

Show Solution

$3\text{cis}\left(80\right), 3\text{cis}\left(200\right), 3\text{cis}\left(320\right)$

42. Evaluate the square root of z when $z = 16\text{cis}\left(100\right)$.

43. Evaluate the cube root of z when $z = 32\text{cis}\left(\frac{2\pi}{3}\right)$.

Show Solution

$2\sqrt[3]{4}\text{cis}\left(\frac{2\pi}{9}\right), 2\sqrt[3]{4}\text{cis}\left(\frac{8\pi}{9}\right), 2\sqrt[3]{4}\text{cis}\left(\frac{14\pi}{9}\right)$

44. Evaluate the square root of z when $z = 32\text{cis}\left(\pi\right)$.

45. Evaluate the cube root of z when $z = 8\text{cis}\left(\frac{7\pi}{4}\right)$.

Show Solution

$2\sqrt{2}\text{cis}\left(\frac{7\pi}{8}\right), 2\sqrt{2}\text{cis}\left(\frac{15\pi}{8}\right)$

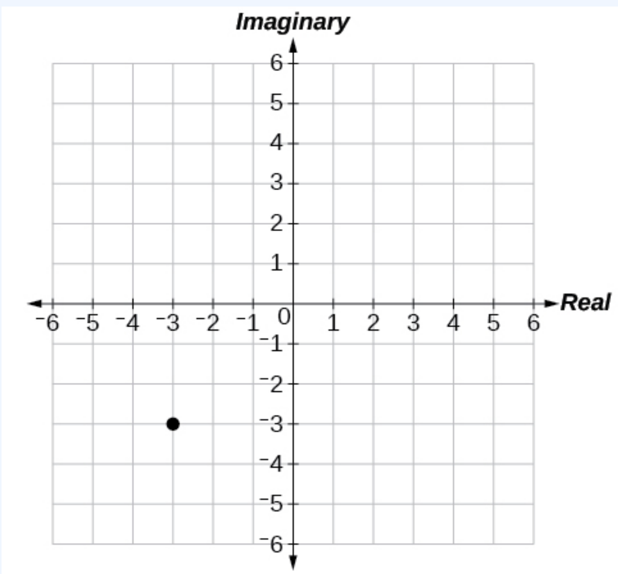
Graphical

For the following exercises, plot the complex number in the complex plane.

46. $2 + 4i$

47. $-3 - 3i$

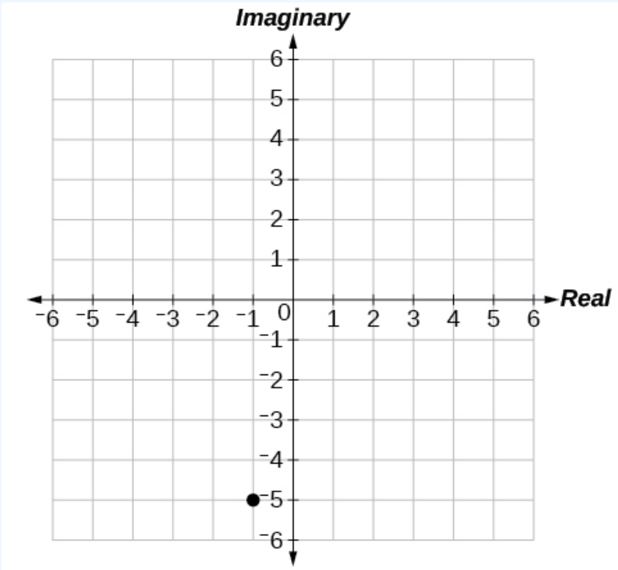
Show Solution



48. $5 - 4i$

49. $-1 - 5i$

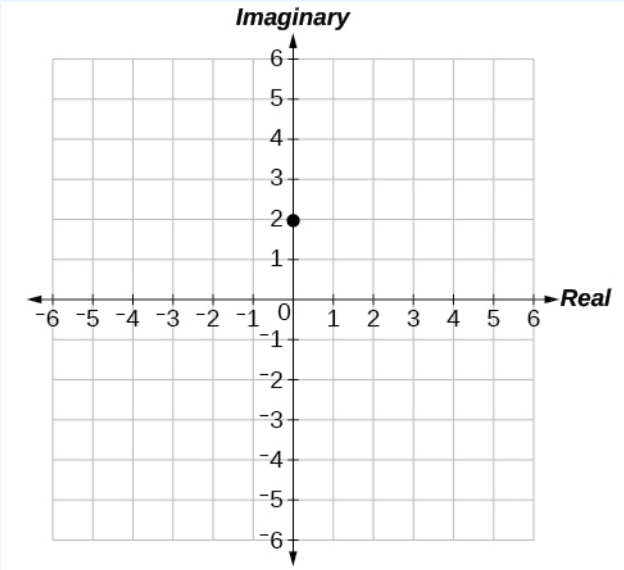
Show Solution



50. $3+2i$

51. $2i$

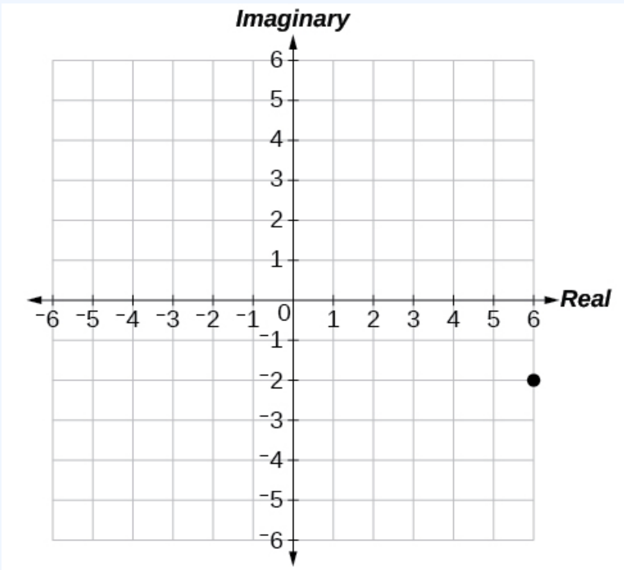
Show Solution



52. -4

53. $6 - 2i$

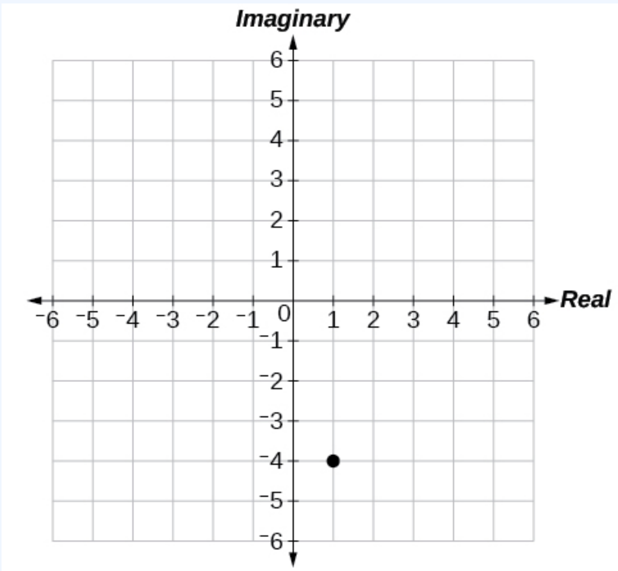
Show Solution



54. $-2 + i$

55. $1 - 4i$

Show Solution



Technology

For the following exercises, find all answers rounded to the nearest hundredth.

56. Use the rectangular to polar feature on the graphing calculator to change $5 + 5i$ to polar form.

57. Use the rectangular to polar feature on the graphing calculator to change $3 - 2i$ to polar form.

Show Solution

$3.61e^{-0.59i}$

58. Use the rectangular to polar feature on the graphing calculator to change $-3 - 8i$ to polar form.

59. Use the polar to rectangular feature on the graphing calculator to change $4\text{cis}\,(120)$ to rectangular form.

Show Solution
$-2 + 3.46i$

60. Use the polar to rectangular feature on the graphing calculator to change $2\text{cis}\,(45)$ to rectangular form.

61. Use the polar to rectangular feature on the graphing calculator to change $5\text{cis}\,(210)$ to rectangular form.

Show Solution
$-4.33 - 2.50i$

Glossary

argument
: the angle associated with a complex number; the angle between the line from the origin to the point and the positive real axis

De Moivre's Theorem
: formula used to find the $n\text{th}$ power or *n*th roots of a complex number; states that, for a positive integer n, z^n is found by raising the modulus to the $n\text{th}$ power and multiplying the angles by n

modulus
: the absolute value of a complex number, or the distance from the origin to the point (x, y) ; also called the amplitude

polar form of a complex number
: a complex number expressed in terms of an angle θ and its distance from the origin r; can be found by using conversion formulas $x = r\cos\theta,\ y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$

REFERENCE SECTION

REFERENCE SECTION

Symbols & Abbreviations
Common Powers
SI Unit Prefixes
Greek Alphabet
Linear Inequalities
Properties of Absolute Values
Metric to English (US) Conversions
Plane Geometry Formula
Solid Geometry Formula
Pythagorean Theorem (Variations)
Linear Equations
Conic Sections
Polynomials, Pascals Triangle
Properties of Complex Numbers
Exponents & Radicals
Trigonometric Functions & Values
Trigonometric Identities
Basic Trigonometric Ratios Graphs
Trigonometric Tables
Properties of Logarithmic Functions
Common Logarithmic Tables

Common Powers

(and not so common)

Squares Cubes 4th Power 5th Power 6th Power 7th Power

22 = 4 23 = 8 24 = 16 25 = 32 26 = 64 27 = 128

32 = 9 33 = 27 34 = 81 35 = 243 36 = 729 37 = 2187

42 = 16 43 = 64 44 = 256 45 = 1024 46 = 4096 47 = 16384

52 = 25 53 = 125 54 = 625 55 = 3125 56 = 15625 57 = 78125

62 = 36 63 = 216 64 = 1296 65 = 7776 66 = 46656 67 = 279936

72 = 49 73 = 343 74 = 2401 75 = 16807 76 = 117649 77 = 823543

82 = 64 83 = 512 84 = 4096 85 = 32768 86 = 262144 87 = 2097152

92 = 81 93 = 729 94 = 6561 95 = 59049 96 = 531441 97 = 4782969

102 = 100 103 = 1000 104 = 10000 105 = 100000 106 = 1000000 107 = 10000000

112 = 121 122 = 144 132 = 169 142 = 196 152 = 225 202 = 400

Greek Alphabet

A α Alpha N ν Nu

B β Beta Ξ ξ Xi

Γ γ Gamma O ο Omicron

Δ δ Delta Π π Pi

E ε Epsilon P ρ Rho

Z ζ Zeta Σ σ Sigma

H η Eta T τ Tau

Θ θ Theta ϒ υ Upsilon

I ι Iota Φ φ Phi

K κ Kappa X χ Chi

Λ λ Lambda Ψ ψ Psi

M μ Mu Ω ω Omega

SI Unit Prefixes

Factor Name Symbol Factor Name Symbol

10-18 atto a 10-1 deci d

10-15 femto f 10 deca da

10-12 pico p 102 hecto h

10-9 nano n 103 kilo k

10-6 micro μ 106 mega M

10-3 milli m 109 giga G

10-2 centi c 1012 tera T

Linear Inequalities

Interval Notation Set Builder Notation Graph of the Inequality

(a, + ∞) {x | x > a}

[a, + ∞) {x | x ≥ a}

(- ∞, a) {x | x < a}

(- ∞, a] {x | x ≤ a}

[a, b] {x | a ≤ x ≤ b}

(a, b) {x | a < x < b}

[a, b) {x | a ≤ x < b}

(a, b] {x | a < x ≤ b}

(- ∞, + ∞) {x | x ∈ R}

(- ∞, b) or (a, + ∞) {x | x < a or x > b}

$(-\infty, a]$ or $[a, +\infty)$ $\{x \mid x < a \text{ or } x > b\}$

$(-\infty, a]$ or $[a, +\infty)$ $\{x \mid x < a \text{ or } x > b\}$

$(-\infty, a]$ or $[a, +\infty)$ $\{x \mid x < a \text{ or } x > b\}$

Properties of Absolute Values

If $| X | = k$, then $X = k$ or $X = -k$

If $| X | < k$, then $-k < X < k$

If $| X | > k$, then $X > k$ or $X < -k$Metric to English (US) Conversions

Distance:

12 in = 1 ft

3 ft = 1 yd 10 mm = 1 cm

1760 yds = 1 mi 100 cm = 1 m

5280 ft = 1 mi 1000m = 1 km

(English-Metric conversions: 1 inch = 2.54 cm; 1 mile = 1.61 km)

Area:

144 in2 = 1 ft2 10,000 cm2 = 1 m2

43,560 ft2 = 1 acre 10,000 m2 = 1 hectare

640 acres = 1 mi2 100 hectare = 1 km2

(English-Metric conversions: 1 in2 = 6.45 cm2; 1 mi2 = 2.59 km2)

Volume:

57.75 in3 = 1 qt 1 cm3 = 1 ml

4 qt = 1 gal 1000 ml = 1 liter

42 gal (petroleum) = 1 barrel 1000 liter = 1 m3

(English-Metric conversions: 16.39 cm3 = 1 in3; 3.79 liters = 1 gal)

Mass:

437.5 grains = 1 oz 1000 mg = 1 g

16 oz = 1 lb 1000 g = 1 kg

2000 lb = 1 short ton 1000 kg = 1 metric ton

(English-Metric conversions: 453 g = 1 lb; 2.2 lb = 1 kg)

Temperature:

(Fahrenheit – Celsius Conversions: °C = 5/9 (°F – 32) and °F = 9/5 °C + 32)

Plane Geometry Formula

Circle Square Rectangle

Area = π r2 Area = s2 Area = l w

Perimeter = 2 π r Perimeter = 4 s Perimeter = 2l + 2w

Triangle Rhombus Trapezoid

Area = 1/2 b h Area = b h Area = 1/2 (l1 + l2) h

Perimeter = s1 + s2 + s3 Perimeter = 4 b Perimeter = l1 + l2 + h1 + h2

Parallelogram Regular Polygon (n-gon)

Area = b h Area = (1/2 s h) (number of sides)

Perimeter = 2 h1 + 2 b Perimeter = s (number of sides)

Solid Geometry Formula

Cube Right Rectangular Prism Right Cylindrical Prism

Volume = s3 Volume = l w h Volume = π r2 h

S. A. = 6 s2 S. A. = 2 l w + 2 h w + 2 l h S. A. = 2 π r h + 2 π r2

Sphere Torus Right Triangular Prism

Volume = 4/3 π r3 Volume = 2 π2 r2 R Volume = (1/2 b h) l

S. A. = 4 π r2 S. A. = 4 π2 r R S. A. = b h + 2 l s + l b

Right Circular Cone General Cone/Pyramid Square Pyramid

Volume = 1/3 (π r2) h Volume = 1/3 (base area) h Volume = 1/3 (s2) h

S. A. = π r (r2 + h2)1/2 + π r2 S.A. = s [s + (s2 + 4h2)

Pythagorean Theorem (Variations)

For any right triangle a, b and c:

a2 + b2 = c2

For any non-right triangle a, b and c:

a2 = b2 + c2 – 2bc cos A

b2 = a2 + c2 – 2ac cos B

c2 = a2 + b2 – 2ab cos C

For any rectangular prism a, b and c, the diagonal (d) length is:

d2 = a2 + b2 + c2

Linear Equations

An Ordered Pair: (x, y)

Distance between Two Ordered Pairs: d2 = Δ x2 + Δ y2 or d2 = (x2 – x1)2 + (y2 – y1)2

Midpoint between Two Ordered Pairs: [(x1 + x2) , (y1 + y2)]

2 2

Slope: m = Δ y or m = (y2 – y1) ... where Δy = y2 – y1, Δ x = x2 – x1 and Δ x ≠ 0

Δ x (x2 – x1)

The slope for Two Parallel Lines: m1 = m2

The slope for Two Perpendicular Lines: m1 . m2 = -1 or m1 = -1/m2

To find the Linear Equation Using Two Ordered Pairs: (x2 – x1) m = (y2 – y1)

General Form of a Linear Equation: Ax + By + C = 0 (A, B, C are integers, A is positive)

Slope Intercept Form of a Linear Equation: y = mx + b

Conic Sections

Conic Equations (Standard Form):

Circle: (x – h)2 + (y – k)2 = r2 (h, k) is the center point, r is the radius from the center to the circles (x, y) coordinates

Parabolas: y – k = a(x – h)2 Parabolas, commonly written as y = ax2 + bx + c
x – h = a(y – k)2

Ellipse: (x – h)2 + (y – k)2 = 1 (h, k) is the center point, rx is the radius length in rx2 ry2 the ± x direction, ry is the radius length in the ± y direction

Hyperbola: (x – h)2 – (y – k)2 = 1 (h, k) is the center point, rx is the distance from the rx2 ry2 center to the hyperbola's ± x asymptote. ry is the distance from the center to the hyperbola's ± x asymptote.Polynomials

Quadratic Solutions:

The solution for x from a quadratic equation ax2 + bx + c = 0, (where a ≠ 0), can be found from:

Factoring:

a2 – b2 = (a + b)(a – b) a2 + b2 … cannot be factored
a3 – b3 = (a – b)(a2 + ab + b2) a3 + b3 = (a + b)(a2 – ab + b2)

Binomial Expansions:

(a + b)0 = 1 (a – b)0 = 1
(a + b)1 = a + b (a – b)1 = a – b
(a + b)2 = a2 + 2ab + b2 (a – b)2 = a2 – 2ab + b2
(a + b)3 = a3 + 3a2b + 3ab2 + b3 (a – b)3 = a3 – 3a2b + 3ab2 – b3
(a + b)4 = a4 + 4a3b + 6a2b2 + 4ab3 + b4 (a – b)4 = a4 – 4a3b + 6a2b2 – 4ab3 + b4
(a + b)5 = a5 + 5a4b + 10a3b2 + 10a2b3 + 5ab4 + b5
(a – b)5 = a5 – 5a4b + 10a3b2 – 10a2b3 + 5ab4 – b5
(a + b)6 = a6 + 6a5b + 15a4b2 + 20a3b3 + 15a2b4 + 6ab5 + b5
(a – b)6 = a6 – 6a5b + 15a4b2 – 20a3b3 + 15a2b4 – 6ab5 + b5

Properties of Complex Numbers

(a + bi) + (c + di) = a + c + (b + d)i (a + bi) – (c + di) = a – c + (b – d)i

(a + bi) (c + di) = ac – bd + (ab + bd)i (a + bi) (a – bi) = a2 + b2

(-a)1/2 = i(a)1/2, a ≥ 0

Properties of Exponents

Properties of Rational Exponents and Radicals

Basic Trigonometric Functions & Values

Basic Trigonometric Ratios

Sin = Opposite Cos = Adjacent Tan = Opposite
Hypotenuse Hypotenuse Adjacent

Sec = Hypotenuse Csc = Hypotenuse Cot = Adjacent
Opposite Adjacent Opposite

Trigonometric Identities

Reciprocal Identities:

sin θ = 1/csc θ tan θ = 1/cot θ cos θ = 1/sec θ

csc θ = 1/sin θ cot θ = 1/tan θ sec θ = 1/cos θ

Tangent and Cotangent Identities:

tan θ = sin θ / cos θ cot θ = cos θ / sin θ

Pythagorean Identities:

sin2 θ + cos2 θ =1 tan2 θ + 1 = sec2 θ 1 + cot2 θ = csc2 θ

Double Angle Formulas:

sin 2θ = 2 sin θ cos θ

cos 2θ = cos2 θ – sin2 θ cos 2θ = 2 cos2 θ – 1 cos 2θ = 1– 2sin2 θ

tan 2θ = 2 tan θ / 1– tan2 θ

Sum and Difference Formulas:

sin (α + β) = sin α cos β + cos α sin β sin (α – β) = sin α cos β – cos α sin β

cos (α + β) = cos α cos β – sin α sin β cos (α – β) = cos α cos β + sin α sin β

tan(α + β) = tan α + tan β / 1 – tan α tan β tan(α – β) = tan α – tan β / 1 + tan α tan β

Graphs of Basic Trigonometric Ratios

Trigonometric Tables

Angle	Sin	Cos	Tan	Csc	Angle	Sin	Cos	Tan	Csc
1	0.017	1.000	0.017	57.299	46	0.719	0.695	1.036	1.390
2	0.035	0.999	0.035	28.654	47	0.731	0.682	1.072	1.36
3	0.052	0.999	0.052	19.107	48	0.743	0.669	1.111	1.346
4	0.070	0.998	0.070	14.336	49	0.755	0.656	1.150	1.325
5	0.087	0.996	0.087	11.474	50	0.766	0.643	1.192	1.305
6	0.105	0.995	0.105	9.567	51	0.777	0.629	1.235	1.287
7	0.122	0.993	0.123	8.206	52	0.788	0.616	1.280	1.269
8	0.139	0.990	0.141	7.185	53	0.799	0.602	1.327	1.252
9	0.156	0.988	0.158	6.392	54	0.809	0.588	1.376	1.236
10	0.174	0.985	0.176	5.759	55	0.819	0.574	1.428	1.221
11	0.191	0.982	0.194	5.241	56	0.829	0.559	1.483	1.206
12	0.208	0.978	0.213	4.810	57	0.839	0.545	1.540	1.192
13	0.225	0.974	0.231	4.445	58	0.848	0.530	1.600	1.179
14	0.242	0.970	0.249	4.134	59	0.857	0.515	1.664	1.167
15	0.259	0.966	0.268	3.864	60	0.866	0.500	1.732	1.155
16	0.276	0.961	0.287	3.628	61	0.875	0.485	1.804	1.143
17	0.292	0.956	0.306	3.420	62	0.883	0.469	1.881	1.133
18	0.309	0.951	0.325	3.236	63	0.891	0.454	1.963	1.122
19	0.326	0.946	0.344	3.072	64	0.899	0.438	2.050	1.113

20	0.342	0.940	0.364	2.924	65	0.906	0.423	2.145	1.103
21	0.358	0.934	0.384	2.790	66	0.914	0.407	2.246	1.095
22	0.375	0.927	0.404	2.669	67	0.921	0.391	2.356	1.086
23	0.391	0.921	0.424	2.559	68	0.927	0.375	2.475	1.079
24	0.407	0.914	0.445	2.459	69	0.934	0.358	2.605	1.071
25	0.423	0.906	0.466	2.366	70	0.940	0.342	2.747	1.064
26	0.438	0.899	0.488	2.281	71	0.946	0.326	2.904	1.058
27	0.454	0.891	0.510	2.203	72	0.951	0.309	3.078	1.051
28	0.469	0.883	0.532	2.130	73	0.956	0.292	3.271	1.046
29	0.485	0.875	0.554	2.063	74	0.961	0.276	3.487	1.040
30	0.500	0.866	0.577	2.000	75	0.966	0.259	3.732	1.035
31	0.515	0.857	0.601	1.942	76	0.970	0.242	4.011	1.031
32	0.530	0.848	0.625	1.887	77	0.974	0.225	4.331	1.026
33	0.545	0.839	0.649	1.836	78	0.978	0.208	4.705	1.022
34	0.559	0.829	0.675	1.788	79	0.982	0.191	5.145	1.019
35	0.574	0.819	0.700	1.743	80	0.985	0.174	5.671	1.015
36	0.588	0.809	0.727	1.701	81	0.988	0.156	6.314	1.012
37	0.602	0.799	0.754	1.662	82	0.990	0.139	7.115	1.010
38	0.616	0.788	0.781	1.624	83	0.993	0.122	8.144	1.008
39	0.629	0.777	0.810	1.589	84	0.995	0.105	9.514	1.006
40	0.643	0.766	0.839	1.556	85	0.996	0.087	11.430	1.004
41	0.656	0.755	0.869	1.524	86	0.998	0.070	14.301	1.002
42	0.669	0.743	0.900	1.494	87	0.999	0.052	19.081	1.001
43	0.682	0.731	0.933	1.466	88	0.999	0.035	28.636	1.001
44	0.695	0.719	0.966	1.440	89	1.000	0.017	57.290	1.000
45	0.707	0.707	1.000	1.414	90	1.000	0.000		1.000

Properties of Logarithmic Functions

x = ay is equivalent to y = loga x ex = y is equivalent to ln y = x

loga (xy) = loga x + loga y loga (x/y) = loga x – loga y loga (1/x) = – loga x

ln (xy) = ln x – ln y ln (x/y) = ln x – ln y ln (1/x) = – ln x

loga x = log x/ log a loga a = 1 loga 1 = 0 loga xy = y loga x

loga x = ln x/ ln a ln e = 1 ln 1 = 0 ln xy = y ln x

Common Logarithm Table

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170	0.0212	0.0253	0.0294	0.0334	0.0374
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430

1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732
1.5	0.1761	0.1790	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014
1.6	0.2041	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279
1.7	0.2304	0.2330	0.2355	0.2380	0.2405	0.2430	0.2455	0.2480	0.2504	0.2529
1.8	0.2553	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765
1.9	0.2788	0.2810	0.2833	0.2856	0.2878	0.2900	0.2923	0.2945	0.2967	0.2989
2.0	0.3010	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.3160	0.3181	0.3201
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.3560	0.3579	0.3598
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784
2.4	0.3802	0.3820	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133
2.6	0.4150	0.4166	0.4183	0.4200	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298
2.7	0.4314	0.4330	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.4440	0.4456
2.8	0.4472	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609
2.9	0.4624	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757
3.0	0.4771	0.4786	0.4800	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.4900
N	0	1	2	3	4	5	6	7	8	9
3.1	0.4914	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038
3.2	0.5051	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172
3.3	0.5185	0.5198	0.5211	0.5224	0.5237	0.5250	0.5263	0.5276	0.5289	0.5302
3.4	0.5315	0.5328	0.5340	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428
3.5	0.5441	0.5453	0.5465	0.5478	0.5490	0.5502	0.5514	0.5527	0.5539	0.5551
3.6	0.5563	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.5670
3.7	0.5682	0.5694	0.5705	0.5717	0.5729	0.5740	0.5752	0.5763	0.5775	0.5786
3.8	0.5798	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899
3.9	0.5911	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.6010
4.0	0.6021	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117
4.1	0.6128	0.6138	0.6149	0.6160	0.6170	0.6180	0.6191	0.6201	0.6212	0.6222
4.2	0.6232	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325
4.3	0.6335	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425
4.4	0.6435	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522
4.5	0.6532	0.6542	0.6551	0.6561	0.6571	0.6580	0.6590	0.6599	0.6609	0.6618
4.6	0.6628	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712
4.7	0.6721	0.6730	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803
4.8	0.6812	0.6821	0.6830	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893
4.9	0.6902	0.6911	0.6920	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981

5.0	0.6990	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.7050	0.7059	0.7067
5.1	0.7076	0.7084	0.7093	0.7101	0.7110	0.7118	0.7126	0.7135	0.7143	0.7152
5.2	0.7160	0.7168	0.7177	0.7185	0.7193	0.7202	0.7210	0.7218	0.7226	0.7235
5.3	0.7243	0.7251	0.7259	0.7267	0.7275	0.7284	0.7292	0.7300	0.7308	0.7316
N	0	1	2	3	4	5	6	7	8	9
5.4	0.7324	0.7332	0.7340	0.7348	0.7356	0.7364	0.7372	0.7380	0.7388	0.7396
5.5	0.7404	0.7412	0.7419	0.7427	0.7435	0.7443	0.7451	0.7459	0.7466	0.7474
5.6	0.7482	0.7490	0.7497	0.7505	0.7513	0.7520	0.7528	0.7536	0.7543	0.7551
5.7	0.7559	0.7566	0.7574	0.7582	0.7589	0.7597	0.7604	0.7612	0.7619	0.7627
5.8	0.7634	0.7642	0.7649	0.7657	0.7664	0.7672	0.7679	0.7686	0.7694	0.7701
5.9	0.7709	0.7716	0.7723	0.7731	0.7738	0.7745	0.7752	0.7760	0.7767	0.7774
6.0	0.7782	0.7789	0.7796	0.7803	0.7810	0.7818	0.7825	0.7832	0.7839	0.7846
6.1	0.7853	0.7860	0.7868	0.7875	0.7882	0.7889	0.7896	0.7903	0.7910	0.7917
6.2	0.7924	0.7931	0.7938	0.7945	0.7952	0.7959	0.7966	0.7973	0.7980	0.7987
6.3	0.7993	0.8000	0.8007	0.8014	0.8021	0.8028	0.8035	0.8041	0.8048	0.8055
6.4	0.8062	0.8069	0.8075	0.8082	0.8089	0.8096	0.8102	0.8109	0.8116	0.8122
6.5	0.8129	0.8136	0.8142	0.8149	0.8156	0.8162	0.8169	0.8176	0.8182	0.8189
6.6	0.8195	0.8202	0.8209	0.8215	0.8222	0.8228	0.8235	0.8241	0.8248	0.8254
6.7	0.8261	0.8267	0.8274	0.8280	0.8287	0.8293	0.8299	0.8306	0.8312	0.8319
6.8	0.8325	0.8331	0.8338	0.8344	0.8351	0.8357	0.8363	0.8370	0.8376	0.8382
6.9	0.8388	0.8395	0.8401	0.8407	0.8414	0.8420	0.8426	0.8432	0.8439	0.8445
7.0	0.8451	0.8457	0.8463	0.8470	0.8476	0.8482	0.8488	0.8494	0.8500	0.8506
7.1	0.8513	0.8519	0.8525	0.8531	0.8537	0.8543	0.8549	0.8555	0.8561	0.8567
7.2	0.8573	0.8579	0.8585	0.8591	0.8597	0.8603	0.8609	0.8615	0.8621	0.8627
7.3	0.8633	0.8639	0.8645	0.8651	0.8657	0.8663	0.8669	0.8675	0.8681	0.8686
7.4	0.8692	0.8698	0.8704	0.8710	0.8716	0.8722	0.8727	0.8733	0.8739	0.8745
7.5	0.8751	0.8756	0.8762	0.8768	0.8774	0.8779	0.8785	0.8791	0.8797	0.8802
7.6	0.8808	0.8814	0.8820	0.8825	0.8831	0.8837	0.8842	0.8848	0.8854	0.8859
N	0	1	2	3	4	5	6	7	8	9
7.7	0.8865	0.8871	0.8876	0.8882	0.8887	0.8893	0.8899	0.8904	0.8910	0.8915
7.8	0.8921	0.8927	0.8932	0.8938	0.8943	0.8949	0.8954	0.8960	0.8965	0.8971
7.9	0.8976	0.8982	0.8987	0.8993	0.8998	0.9004	0.9009	0.9015	0.9020	0.9025
8.0	0.9031	0.9036	0.9042	0.9047	0.9053	0.9058	0.9063	0.9069	0.9074	0.9079
8.1	0.9085	0.9090	0.9096	0.9101	0.9106	0.9112	0.9117	0.9122	0.9128	0.9133
8.2	0.9138	0.9143	0.9149	0.9154	0.9159	0.9165	0.9170	0.9175	0.9180	0.9186
8.3	0.9191	0.9196	0.9201	0.9206	0.9212	0.9217	0.9222	0.9227	0.9232	0.9238
8.4	0.9243	0.9248	0.9253	0.9258	0.9263	0.9269	0.9274	0.9279	0.9284	0.9289

8.5	0.9294	0.9299	0.9304	0.9309	0.9315	0.9320	0.9325	0.9330	0.9335	0.9340
8.6	0.9345	0.9350	0.9355	0.9360	0.9365	0.9370	0.9375	0.9380	0.9385	0.9390
8.7	0.9395	0.9400	0.9405	0.9410	0.9415	0.9420	0.9425	0.9430	0.9435	0.9440
8.8	0.9445	0.9450	0.9455	0.9460	0.9465	0.9469	0.9474	0.9479	0.9484	0.9489
8.9	0.9494	0.9499	0.9504	0.9509	0.9513	0.9518	0.9523	0.9528	0.9533	0.9538
9.0	0.9542	0.9547	0.9552	0.9557	0.9562	0.9566	0.9571	0.9576	0.9581	0.9586
9.1	0.9590	0.9595	0.9600	0.9605	0.9609	0.9614	0.9619	0.9624	0.9628	0.9633
9.2	0.9638	0.9643	0.9647	0.9652	0.9657	0.9661	0.9666	0.9671	0.9675	0.9680
9.3	0.9685	0.9689	0.9694	0.9699	0.9703	0.9708	0.9713	0.9717	0.9722	0.9727
9.4	0.9731	0.9736	0.9741	0.9745	0.9750	0.9754	0.9759	0.9763	0.9768	0.9773
9.5	0.9777	0.9782	0.9786	0.9791	0.9795	0.9800	0.9805	0.9809	0.9814	0.9818
9.6	0.9823	0.9827	0.9832	0.9836	0.9841	0.9845	0.9850	0.9854	0.9859	0.9863
9.7	0.9868	0.9872	0.9877	0.9881	0.9886	0.9890	0.9894	0.9899	0.9903	0.9908
9.8	0.9912	0.9917	0.9921	0.9926	0.9930	0.9934	0.9939	0.9943	0.9948	0.9952
9.9	0.9956	0.9961	0.9965	0.9969	0.9974	0.9978	0.9983	0.9987	0.9991	0.9996