Math 3080 Preparation

Math 3080 Preparation

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FANSHAWE COLLEGE PRESSBOOKS
LONDON

Contents

Acknowledgements

This open textbook has been compiled, edited and partially adapted by Erin Kox, M.Sc., B. Ed. in partnership with the OER Design Studio and the Library Learning Commons at Fanshawe College in London, Ontario

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Collaborators

This project was a collaboration between the author and the team in the OER Design Studio at Fanshawe. The following staff and students were involved in the creation of this project:

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About this Book

This textbook is intended as preparation material for students who previously took College Qualifying Mathematics and are moving onto Advanced Functions. It has been edited by Fanshawe College from its original version. For a list of changes, please see the table below.

The textbook reviews functions, domain and range, transformation of functions, and factoring polynomials.

Changes

Chapter 1	*Reoganized and added information*. Reorganized contents of section 1.1 from Precalculus by J. Abramson. YouTube video from Math Antics added.
Chapter 2	*Reoganized*. Reorganized contents of section 1.2 from Precalculus by J. Abramson.
Chapter 3	*Reoganized*. Reorganized contents of section 1.5 from Precalculus by J. Abramson.
Chapter 4	*Reoganized and added information*. Reorganized contents of section 1.5 from Algebra and Trigonometry by J. Abramson. Added a slideshow video that introduces polynomial functions and their characteristics.
Overall	

Feedback

To provide feedback on this text please contact oer@fanshwec.ca .

Introduction

This package contains preparation material for MATH 3080 for students who previously took MATH 3069. The content is optional and provides some material to review before moving from MATH 3069 to MATH 3080.

MATH 3080 is broken down into five distinct parts: an introduction to functions and then four specific types of functions that we will study.

Introduction to Functions	Definition of a function Toolkit functions Characteristics of functions (repeated in each chapter)
Polynomial Functions	Characteristics Graphing Solving
Rational Functions	Characteristics Graphing Solving
Exponential and Logarithmic Functions	Characteristics Graphing Solving
Trigonometric Functions	Characteristics Graphing Solving Identities

In MATH 3080, there is a lot of content to cover, so it will move quicker than MATH 3069. For most of the course, we will cover three sections every week, starting the very first week of classes. Other than the pace, the course set-up is very similar to MATH 3069, so you will have a lab due at the end of each chapter, two assignments and two tests.

We will be using Vretta again this semester, so please go ahead and purchase your e-textbook and access code on Vretta, through FOL, as soon as you have access to the course. Unlike MATH 3069, the textbook at the bookstore DOES NOT contain your access code, so if you choose to purchase from the bookstore, you will still need to make the e-textbook purchase through Vretta on FOL.

The following chapters will help you to prepare for the first few weeks of class.

CHAPTER 1: FUNCTIONS AND FUNCTION NOTATION

1.1 Introduction to Functions and Function Notation

Learning Objectives

In this section, you will:

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Introduction to Functions

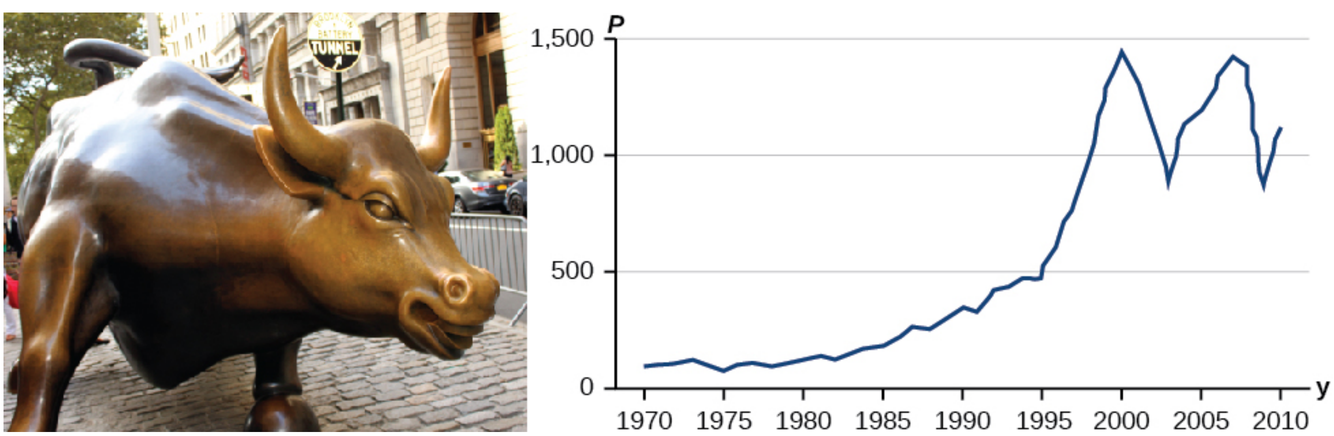

Figure 1-1: Standard and Poor's Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)

Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. Figure 1-1 tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1,100 by the beginning of 2000. That five-year period became known as the “dot-com bubble” because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

Overview

The following video (11:33) provides a short overview of functions, domain and range which will be covered in this section (1 – Introduction to Functions) and the next section (2 – Domain and Range).

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://ecampusontario.pressbooks.pub/math3080prep/?p=5#oembed-1

Access for free at https://openstax.org/books/precalculus/pages/1-introduction-to-functions

1.2 Determining Whether a Relation Represents a Function

A **relation** is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,\ 2),(2,\ 4),(3,\ 6),(4,\ 8),(5,\ 10)\}$$

The domain is $\{1, 2, 3, 4, 5\}$. The range is $\{2, 4, 6, 8, 10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an **output** value, or dependent variable, and is often labeled lowercase letter y.

A function, f, is a relation that assigns a single value in the range to each value in the domain. In other words, no *x*-values are repeated. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, $\{1, 2, 3, 4, 5\}$, is paired with exactly one element in the range, $\{2, 4, 6, 8, 10\}$.

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},\ 1),(\text{even},\ 2),(\text{odd},\ 3),(\text{even},\ 4),(\text{odd},\ 5)\}$$

Notice that each element in the domain, $\{\text{even,odd}\}$ is *not* paired with exactly one element in the range, $\{1, 2, 3, 4, 5\}$. For example, the term "odd" corresponds to three values from the domain, $\{1, 3, 5\}$ and the term "even" corresponds to two values from the range, $\{2, 4\}$. This violates the definition of a function, so this relation is not a function.

Figure 1-2 compares relations that are functions and not functions.

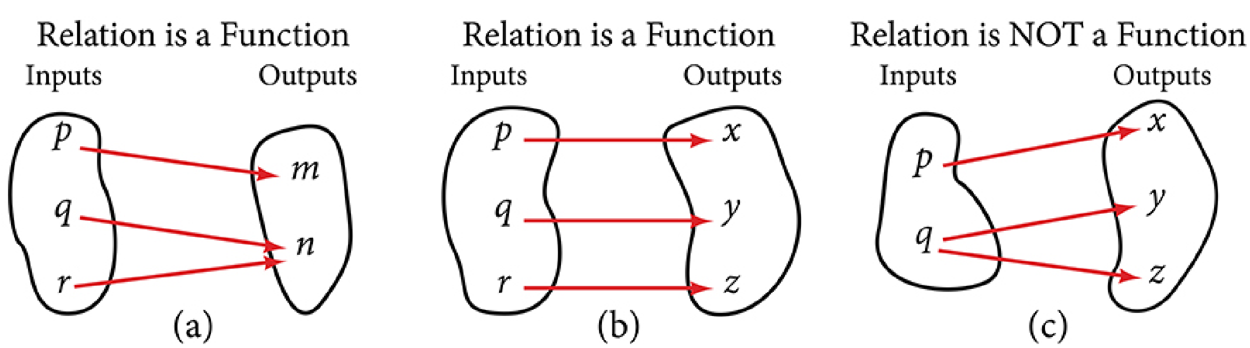

Figure 1-2: (a) This relationship is a function because each input is associated with a single output. Note that input [latex]\text{}q\text{}[/latex] and [latex]\text{}r\text{}[/latex] both give output [latex]\text{}n\text{}[/latex]. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input [latex]\text{}\text{}q\text{}[/latex] is associated with two different outputs.

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say “the output is a function of the input.”

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To

Given a relationship between two quantities, determine whether the relationship is a function.

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 1: Determining if Menu Price Lists are Functions

The coffee shop menu, shown in Figure 1-3 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

Figure 1-3

Solution

Example 2: Determining if Class Grade Rules are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 1-1 shows a possible rule for assigning grade points.

Table 1-1

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

Example 3: Determining if a Function is Present

Table 1-2 lists the five greatest baseball players of all time in order of rank.

Table 1-2

Player	**Rank**
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Solution

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1.3 Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters $f, g,$ and h are often used to represent functions just as we use $x, y,$ and z to represent numbers and $A, B,$ and C to represent sets.

$$\begin{array}{ll} h \text{ is } f \text{ of } a & \text{We name the function } f;\ \text{height is a function of age.} \\ h = f(a) & \text{We use parentheses to indicate the function input.} \\ f(a) & \text{We name the function } f;\ \text{the expression is read as ``} f \text{ of } a\text{.''} \end{array}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add *a* and *b*, and the result is the input for the function *f*." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x". The letter x represents the input value, or independent variable. The letter y,or $f(x)$, represents the output value, or dependent variable.

Example 1: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Solution

Example 2: Interpreting Function Notation

A function $N = f\left(y\right)$ gives the number of police officers, N, in a town in year y. What does $f\left(2005\right) = 300$ represent?

Solution

Example 3: Using Function Notation

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Solution

Question & Answer

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "*y* is a function of *x*"?

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

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1.4 Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 1-3 lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table 1-3

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table 1-4 defines a function $Q = g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table 1-4

n	1	2	3	4	5
Q	8	6	7	6	8

Table 1-5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table 1-5

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

How To

Given a table of input and output values, determine whether the table represents a function.

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example 1: Identifying Tables that Represent Functions

Which table, Table 1-6, Table 1-7, or Table 1-8, represents a function (if any)?

Table 1-6

Input	Output
2	1
5	3
8	6

Table 1-7

Input	Output
–3	5
0	1
4	5

Table 1-8

Input	Output
1	0
5	2
5	4

Solution

Example 2: Identifying Functions

Does Table 1-9 represent a function?

Table 1-9

Input	Output
1	10
2	100
3	1000

Solution

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1.5 Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 1: Evaluating Functions at Specific Values

Evaluating Functions at Specific Values

Evaluate $f(x) = x^2 + 3x - 4$ at

a. 2

b. a
c. $a+h$
d. $\dfrac{f\left(a+h\right)-f\left(a\right)}{h}$

Solution

Example 2: Evaluating Functions

a. Given the function $h\left(p\right)=p^2+2p$, evaluate $h\left(4\right)$.
b. Given the function $g\left(m\right)=\sqrt{m-4}$, evaluate $g\left(5\right)$.

Solution

Example 3: Solving Functions

1. Given the function $h\left(p\right)=p^2+2p$ solve for $h\left(p\right)=3$.
2. Given the function $g\left(m\right)=\sqrt{m-4},$ solve $g\left(m\right)=2$.

Solution

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n + 6p = 12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

How To

Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves *only* the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example 4: Finding an Equation of a Function

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Solution

Example 5: Expressing the Equation of a Circle as a Function

a. Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

b. If $x - 8y^3 = 0,$ express y as a function of x.

Solution

Question & Answer

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table. See Table 1-10.[1]

Table 1-10

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P\left(\text{goldfish}\right)=2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

How To

Given a function represented by a table, identify specific output and input values.

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

1. http://www.kgbanswers.com/how-long-is-a-dogs-memory-span/4221590. Accessed 3/24/2014.

Example 6: Evaluating and Solving a Tabular Function

Using Table 1-11,

a. Evaluate $g\left(3\right)$.
b. Solve $g\left(n\right)=6$.
c. Evaluate $g\left(1\right)$.

Table 1-11

n	1	2	3	4	5
$g\left(n\right)$	8	6	7	6	8

Solution

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 7: Reading Function Values from a Graph

Given the graph in Figure 1-4,

1. Evaluate $\left(f\left(2\right)\right.$.
2. Solve $f\left(x\right)=4$.
3. Solve $f\left(x\right)=1$.

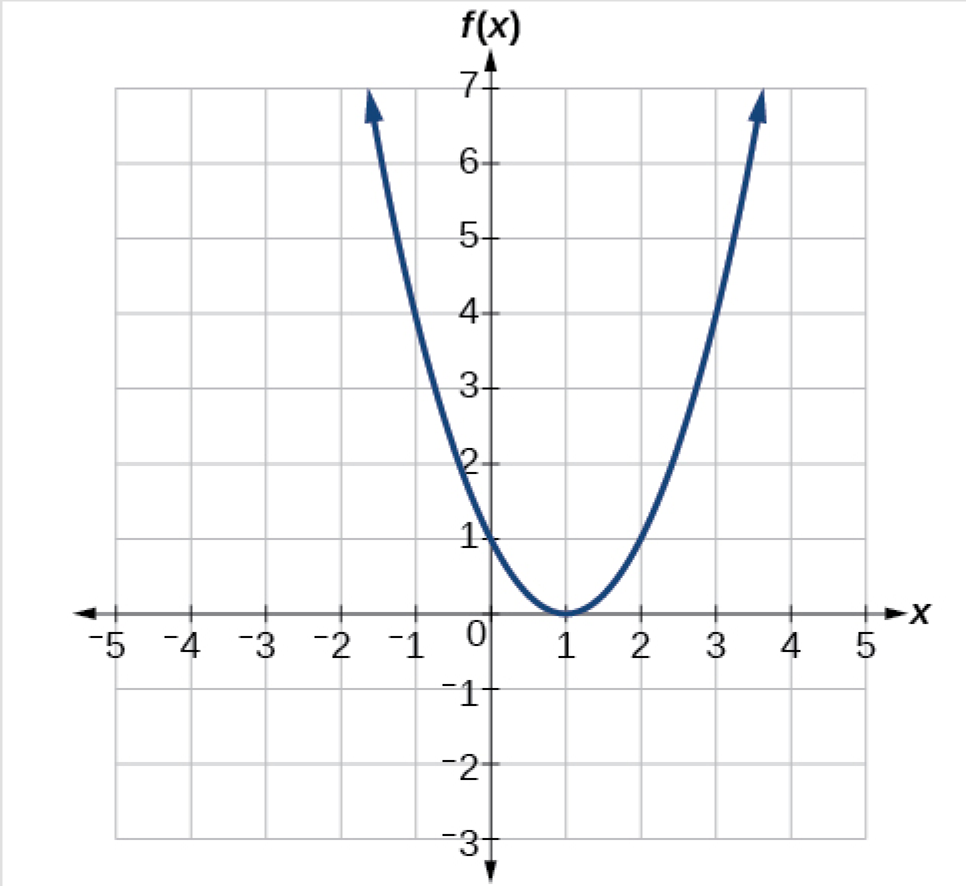

Figure 1-4

Solution

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1.6 Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 1-12.

Table 1-12

Letter grade	Grade point average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figure 1-5 **(a)** and Figure 1 **(b)**. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

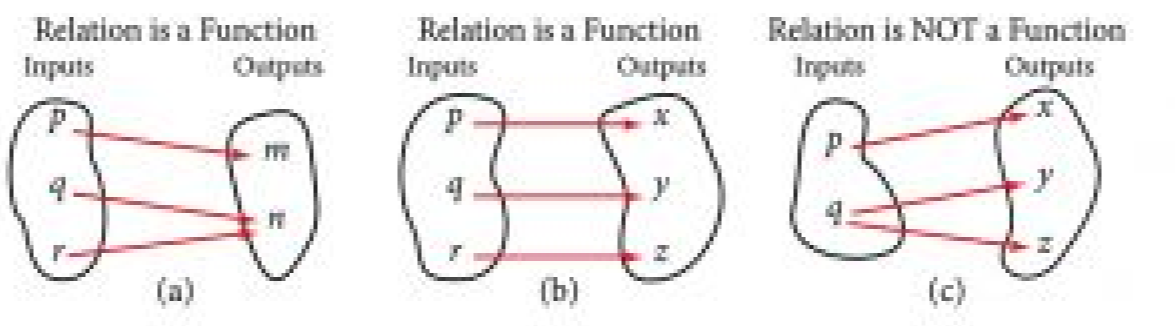


Figure 1-5

One-to-One Function

A one-to-one function is a function in which each output value corresponds to exactly one input value.

Example 1: Determining Whether a Relationship is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

Example 2: Determining Whether a Relationship is a One-to-One Function

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Solution

Example 3: Determining Whether a Relationship is a One-to-One Function

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Solution

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value y and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the *x*-coordinate of each point is an input value and the *y*-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 1-6 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points.

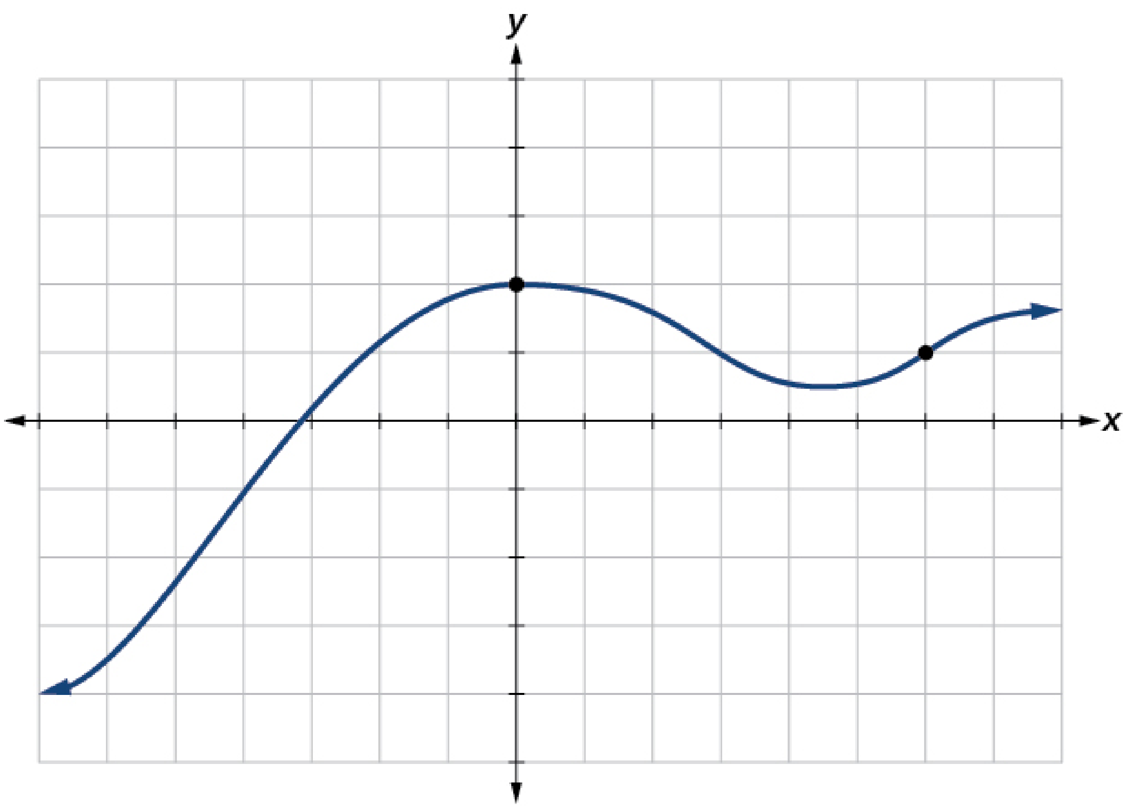


Figure 1-6

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does *not* define a function because a function has only one output value for each input value. See Figure 1-7.

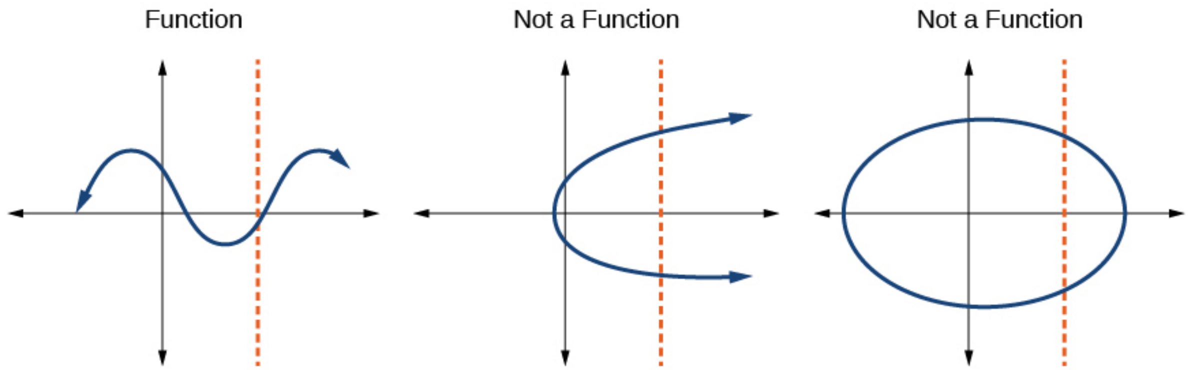


Figure 1-7

How To

Given a graph, use the vertical line test to determine if the graph represents a function.

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 4: Applying the Vertical Line Test

a. Which of the graphs in Figure 1-8 represent(s) a function $y = f(x)$?

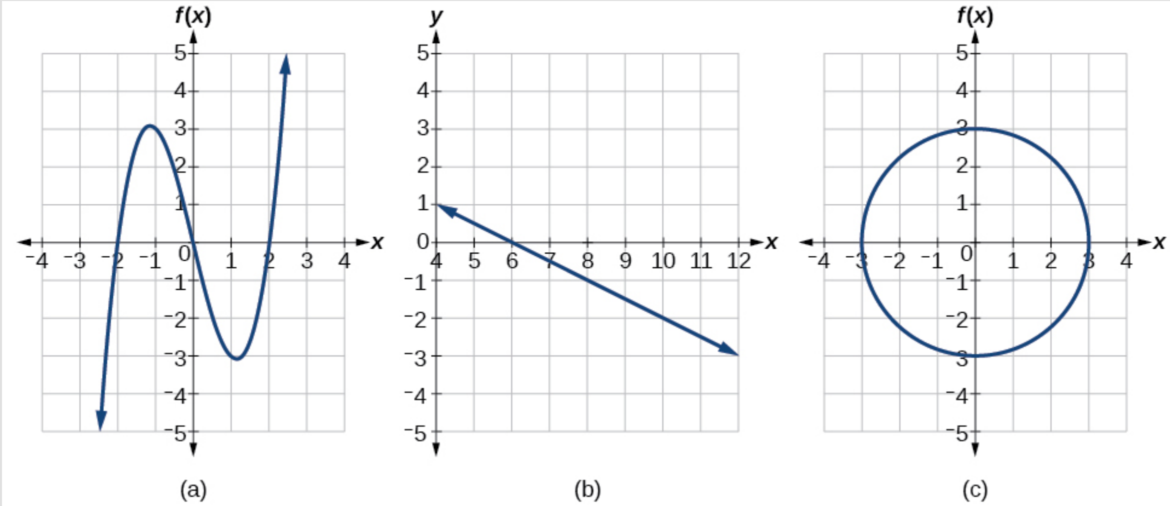


Figure 1-8

b. Does the graph in Figure 1-9 represent a function?

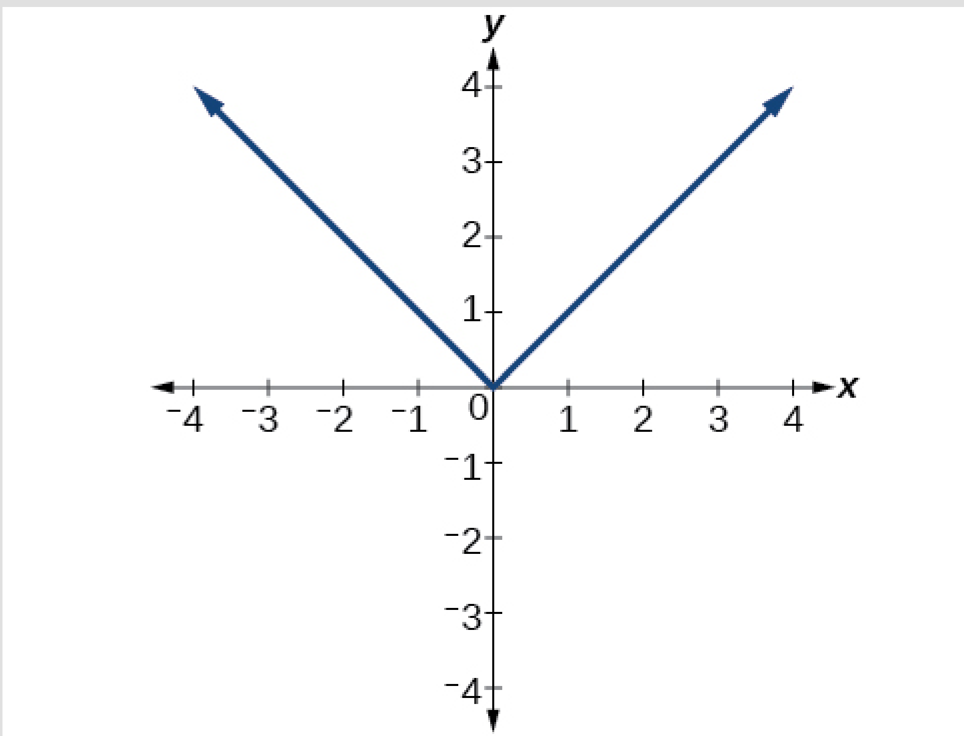


Figure 1-9

[Solution]

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

How To

Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Example 5: Applying the Horizontal Line Test

a. Consider the functions shown in Figure 1-10 **(a)** and Figure 1-10 **(b)**. Are either of the functions one-to-one?

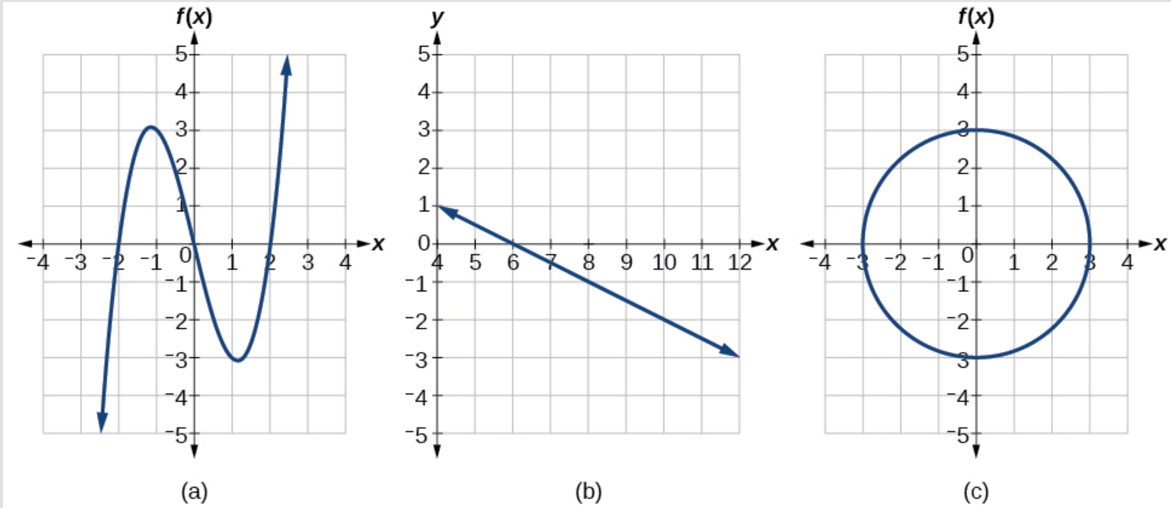


Figure 1-10

b. Is the graph shown in Figure 1-10 **(c)** one-to-one?

Solution

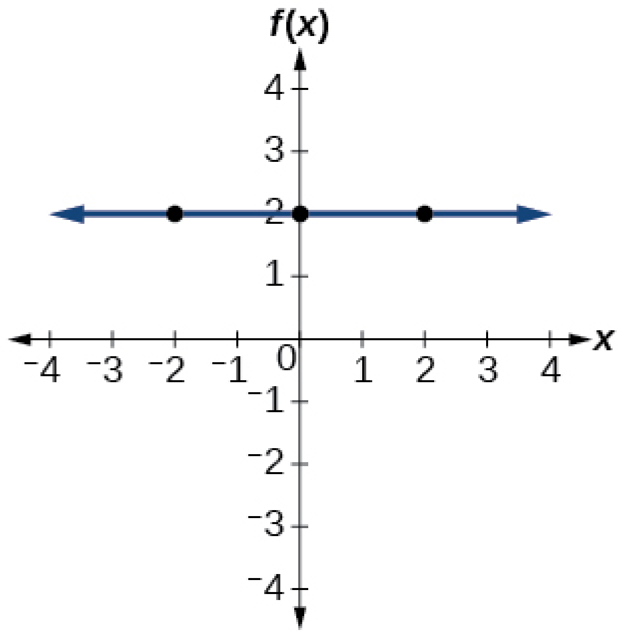
Access for free at https://openstax.org/books/precalculus/pages/1-introduction-to-functions

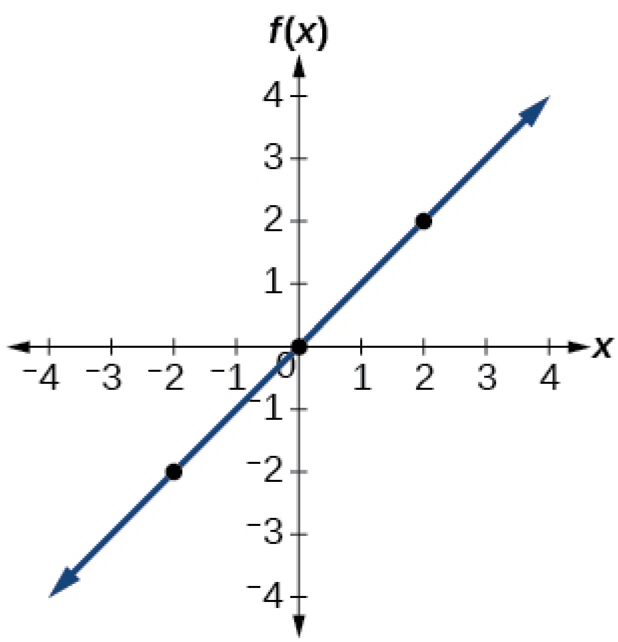
1.7 Identifying Basic Toolkit Functions

In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “toolkit functions,” which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

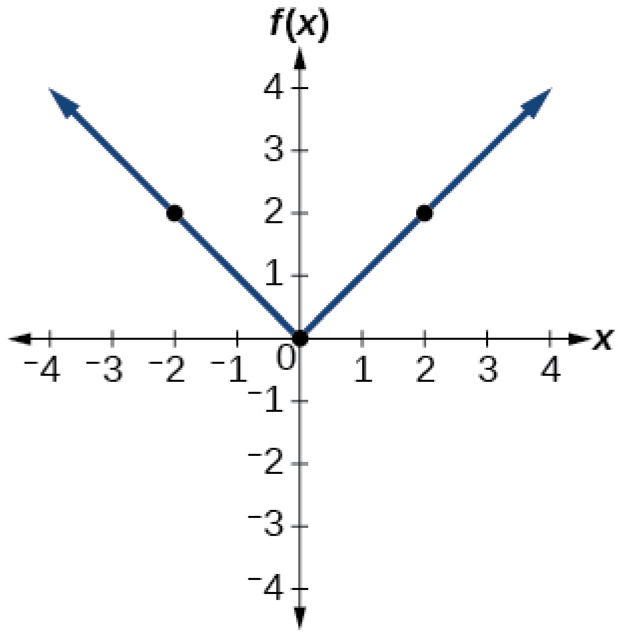
We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table 1-13.

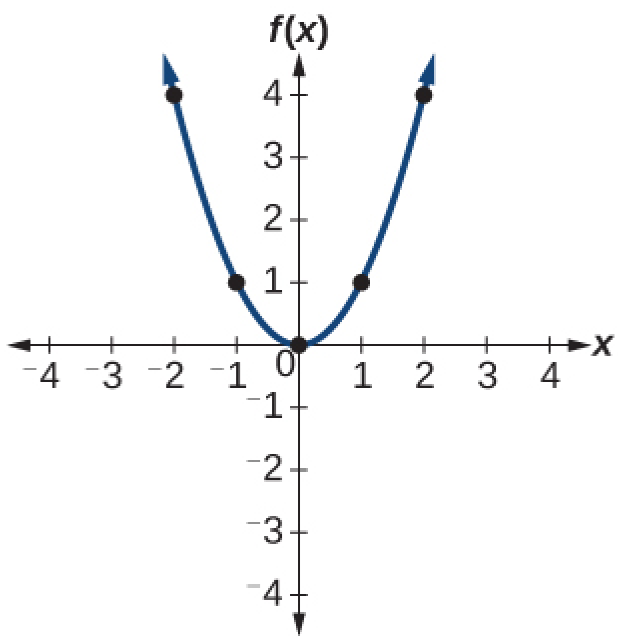
Table 1-13

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Constant</td><td>$f\left(x\right)=c,$ where c is a constant</td><td>

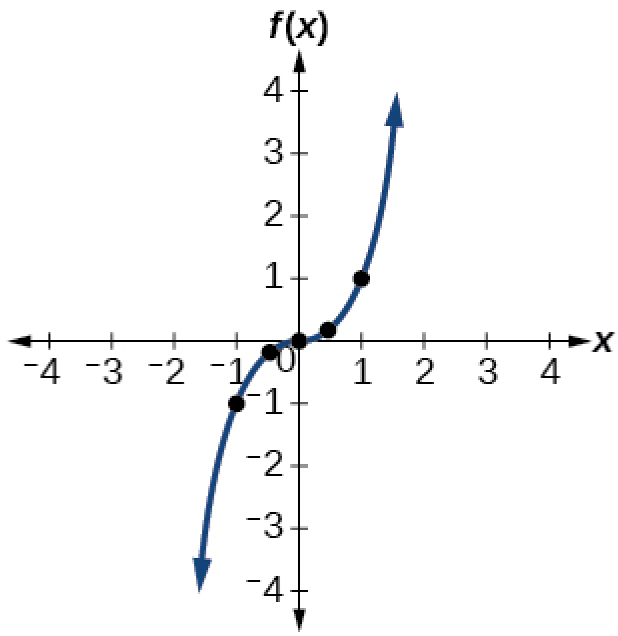
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Identity</td><td>$f\left(x\right)=x$</td><td>

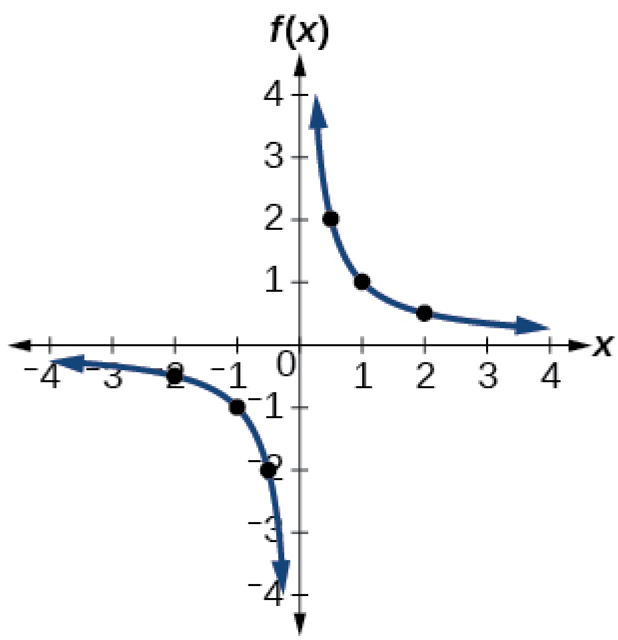
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
</table>

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Absolute value</td><td>$f\left(x\right)=\left|x\right|$</td><td>

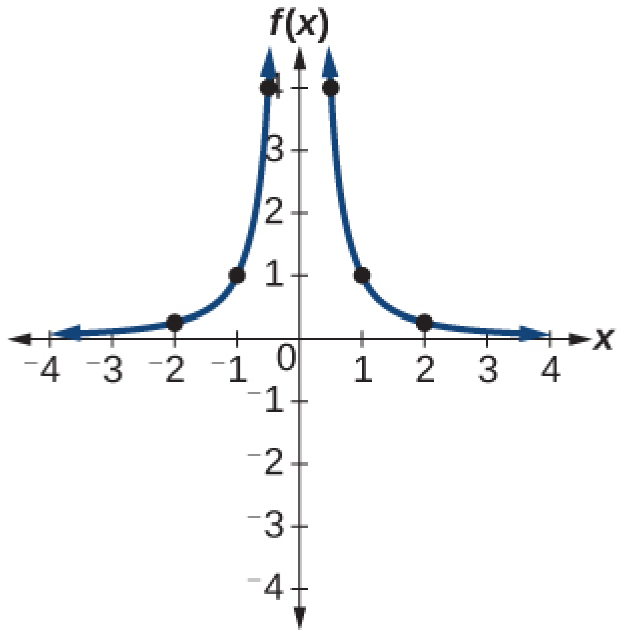
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
<tr><td>Quadratic</td><td>$f\left(x\right)=x^2$</td><td>

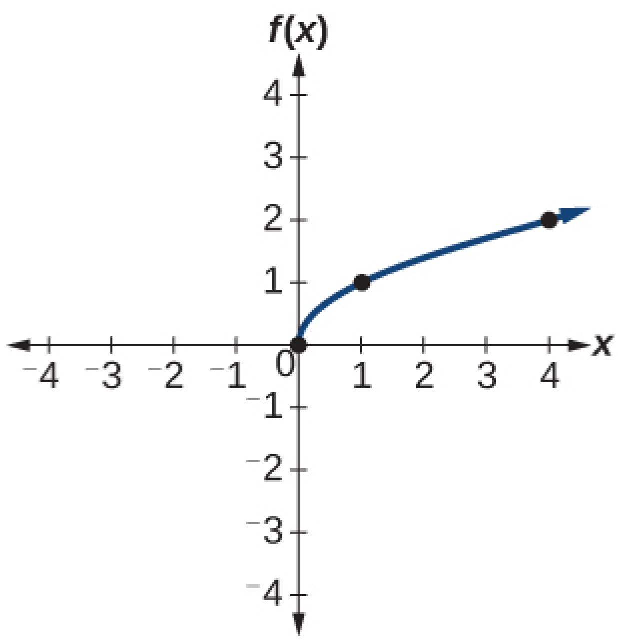
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>4</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
</table></td></tr>
</table>

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Cubic</td><td>$f\left(x\right)=x^{3}$</td><td>


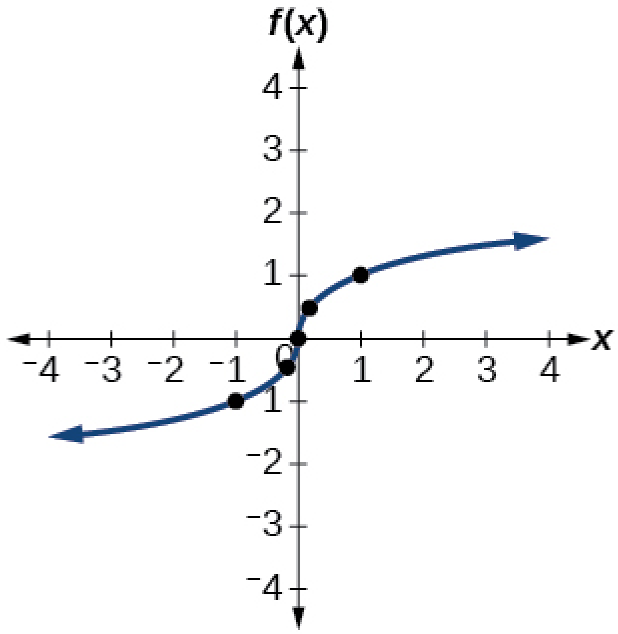
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table>
</td></tr>
<tr><td>Reciprocal</td><td>$f\left(x\right)=\frac{1}{x}$</td><td>


<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table>
</td></tr>
</table>

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Reciprocal squared</td><td>$f\left(x\right)=\dfrac{1}{x^{2}}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>0.25</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>−0.5</td><td>4</td></tr>
<tr><td>0.5</td><td>4</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.25</td></tr>
</table></td></tr>
<tr><td>Square root</td><td>$f\left(x\right)=\sqrt{x}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>4</td><td>2</td></tr>
</table></td></tr>
</table>

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><td>Name</td><td>Function</td><td>Graph</td></tr>
<tr><td>Cube root</td><td>$f\left(x\right)=\sqrt[3]{x}$</td><td>

x	f(x)
−1	−1
−0.125	−0.5
0	0
0.125	0.5
1	1
</td></tr>
</table>

Access for free at https://openstax.org/books/precalculus/pages/1-introduction-to-functions

1.8 Review and Summary

Additional Resources

Access the following online video resources for additional instruction and practice with functions.

- Determine if a Relation is a Function
- Vertical Line Test
- Introduction to Functions
- Vertical Line Test on Graph
- One-to-one Functions
- Graphs as One-to-one Functions

Key Equations

Constant function	$f\left(x\right)=c$, where c is a constant
Identity function	$f\left(x\right)=x$
Absolute value function	$f\left(x\right)=\lvert x\rvert$
Quadratic function	$f\left(x\right)=x^{2}$
Cubic function	$f\left(x\right)=x^{3}$
Reciprocal function	$f\left(x\right)=\dfrac{1}{x}$
Reciprocal squared function	$f\left(x\right)=\dfrac{1}{x^{2}}$
Square root function	$f\left(x\right)=\sqrt{x}$
Cube root function	$f\left(x\right)=\sqrt[3]{x}$

Key Terms

Dependent variable – an output variable

Domain – the set of all possible input values for a relation

Function – a relation in which each input value yields a unique output value

Horizontal Line Test – a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

Independent Variable – an input variable

Input – each object or value in a domain that relates to another object or value by a relationship known as a function

One-to-One Function – a function for which each value of the output is associated with a unique input value

Output – each object or value in the range that is produced when an input value is entered into a function

Range – the set of output values that result from the input values in a relation

Relation – a set of ordered pairs

Vertical Line Test – a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output. See 1.2 Determining Whether a Relation Represents a Function.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$. See 1.3 Using Function Notation.
- In tabular form, a function can be represented by rows or columns that relate to input and output values. See 1.4 Representing Functions Using Tables.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value. See 1.5 Finding Input and Output Values of a Function.
- To solve for a specific function value, we determine the input values that yield the specific output value. See 1.5 Finding Input and Output Values of a Function.
- An algebraic form of a function can be written from an equation. See 1.5 Finding Input and Output Values of a Function.
- Input and output values of a function can be identified from a table. See 1.5 Finding Input and Output

Values of a Function.
- Relating input values to output values on a graph is another way to evaluate a function. See 1.5 Finding Input and Output Values of a Function.
- A function is one-to-one if each output value corresponds to only one input value. See 1.6 Determining Whether a Function is One-to-One.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point. See 1.6 Determining Whether a Function is One-to-One.
- The graph of a one-to-one function passes the horizontal line test. See 1.6 Determining Whether a Function is One-to-One.

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1.9 Practice Questions

Verbal Questions

1. What is the difference between a relation and a function?
2. What is the difference between the input and the output of a function?
3. Why does the vertical line test tell us whether the graph of a relation represents a function?
4. How can you determine if a relation is a one-to-one function?
5. Why does the horizontal line test tell us whether the graph of a function is one-to-one?

Odd Number Verbal Solutions

Algebraic Questions

For the following exercises, determine whether the relation represents a function.

6. $\{(a,b),\ (c,d),\ (a,c)\}$
7. $\{(a,b),(b,c),(c,c)\}$

For the following exercises, determine whether the relation represents y as a function of x.

8. $5x+2y=10$
9. $y=x^2$
10. $x=y^2$
11. $3x^2+y=14$
12. $2x+y^2=6$
13. $y=-2x^2+40x$
14. $y=\dfrac{1}{x}$

15. $x = \dfrac{3y+5}{7y-1}$
16. $x = \sqrt{1-y^2}$
17. $y = \dfrac{3x+5}{7x-1}$
18. $x^2 + y^2 = 9$
19. $2xy = 1$
20. $x = y^3$
21. $y = x^3$
22. $y = \sqrt{1-x^2}$
23. $x = \pm\sqrt{1-y}$
24. $y = \pm\sqrt{1-x}$
25. $y^2 = x^2$
26. $y^3 = x^2$

For the following exercises, evaluate the function f at the indicated values $f(-3), f(2), f(-a), -f(a), f(a+h)$.

27. $f(x) = 2x - 5$
28. $f(x) = -5x^2 + 2x - 1$
29. $f(x) = \sqrt{2-x} + 5$
30. $f(x) = \dfrac{6x-1}{5x+2}$
31. $f(x) = |x-1| - |x+1|$
32. Given the function $g(x) = 5 - x^2$, evaluate $\dfrac{g(x+h) - g(x)}{h}, h \neq 0$
33. Given the function $g(x) = x^2 + 2x$, evaluate $\dfrac{g(x) - g(a)}{x-a}, x \neq a$
34. Given the function $k(t) = 2t - 1$:
    a. Evaluate $k(2)$
    b. Solve $k(t) = 7$
35. Given the function $f(x) = 8 - 3x$:
    a. Evaluate $f(-2)$
    b. Solve $f(x) = -1$

36. Given the function $p(c) = c^2 + c$:

a. Evaluate $p(-3)$
b. Solve $p(c) = 2$

37. Given the function $f(x) = x^2 - 3x$:

a. Evaluate $f(5)$
b. Solve $f(x) = 4$

38. Given the function $f(x) = \sqrt{x+2}$:

a. Evaluate $f(7)$
b. Solve $f(x) = 4$

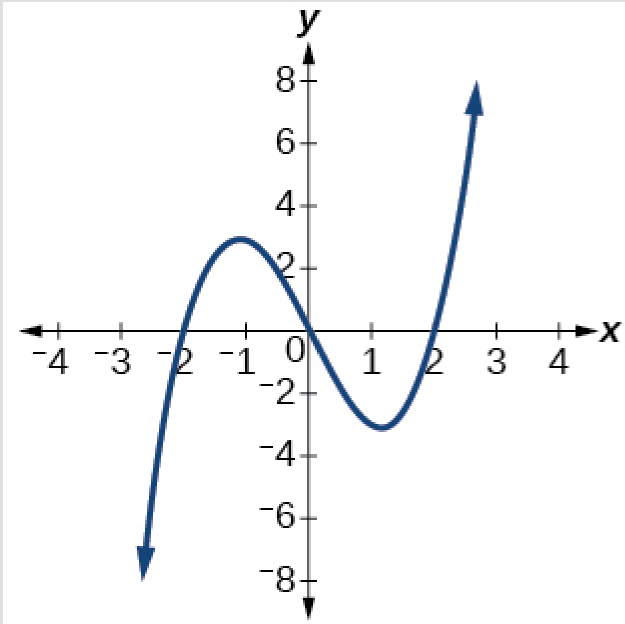
39. Consider the relationship $3r + 2t = 18$

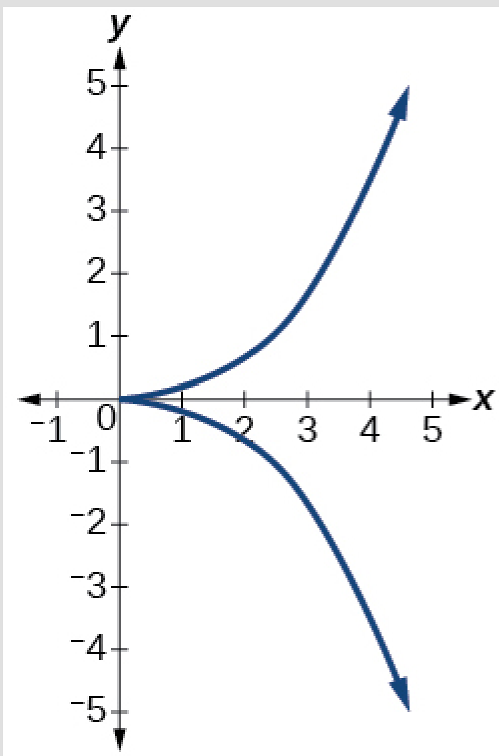
a. Write the relationship as a function $r = f(t)$
b. Evaluate $f(-3)$
c. Solve $f(t) = 2$

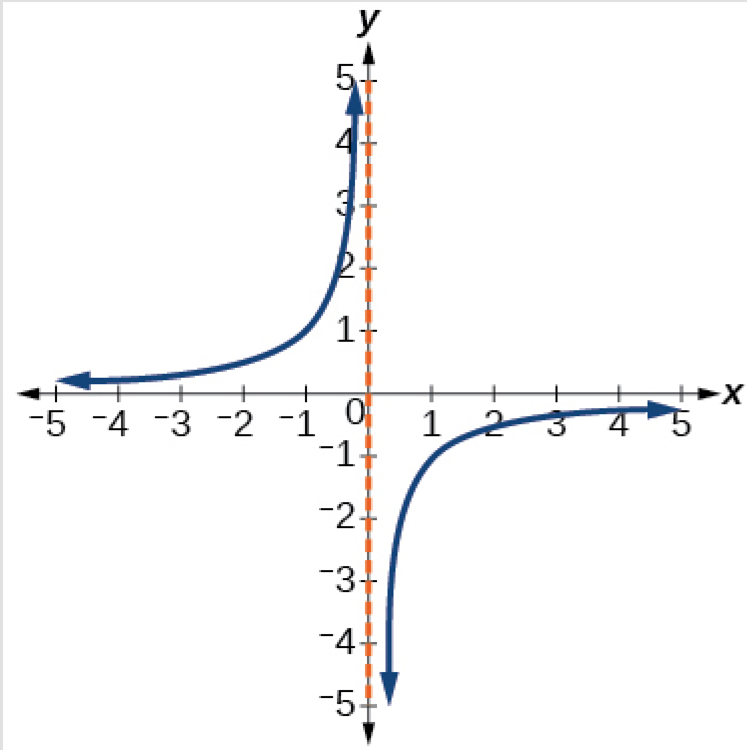
Odd Number Algebraic Solutions

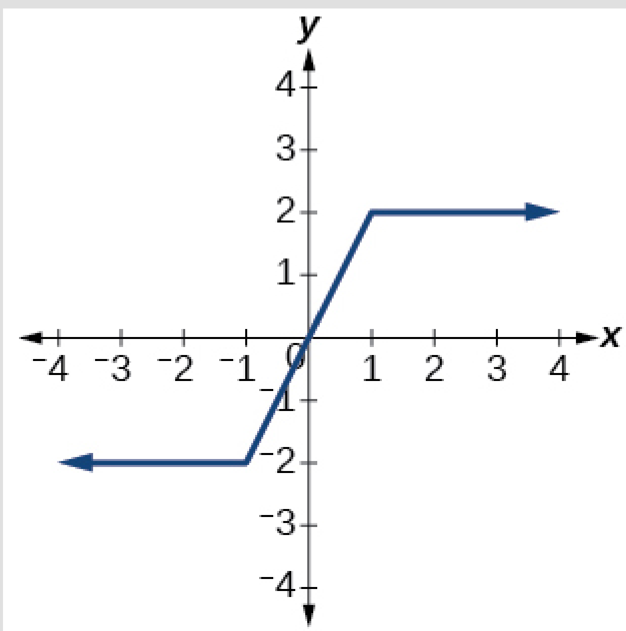
Graphical Questions

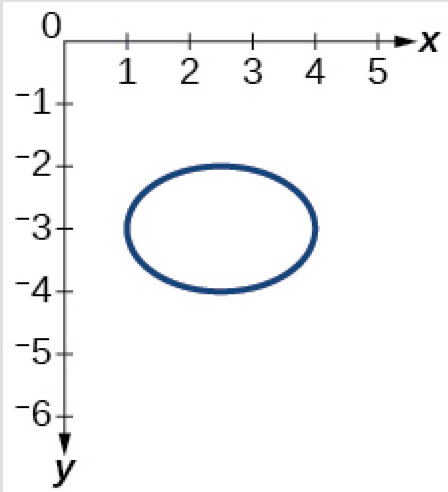
For the following exercises, use the vertical line test to determine which graphs show relations that are functions.

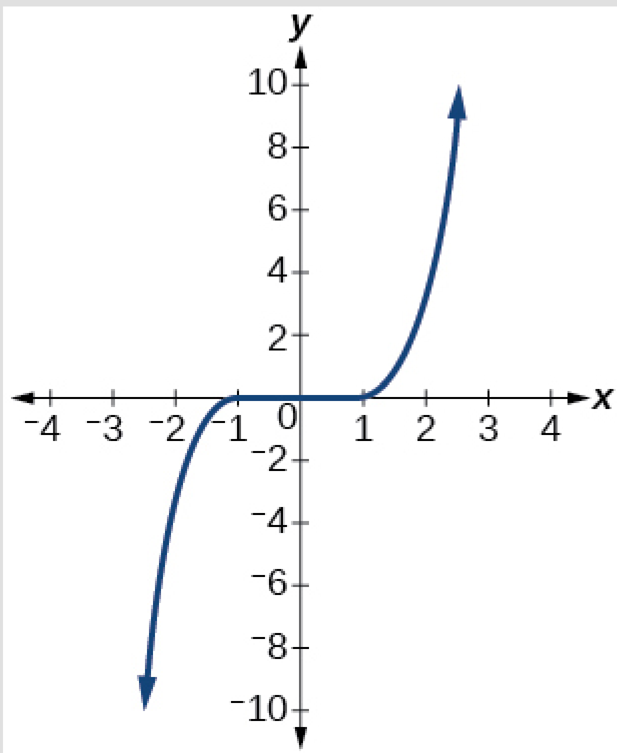
40. 

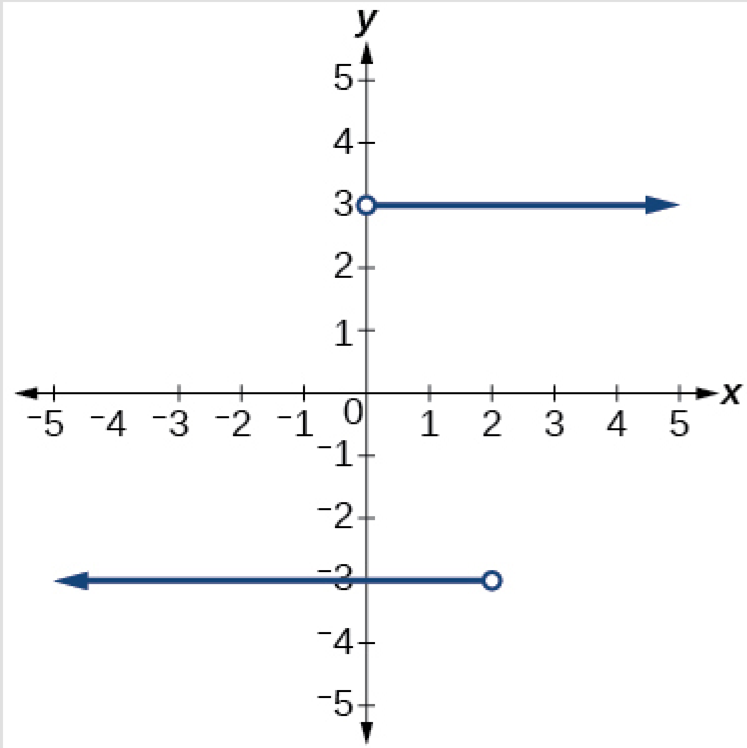
41. 

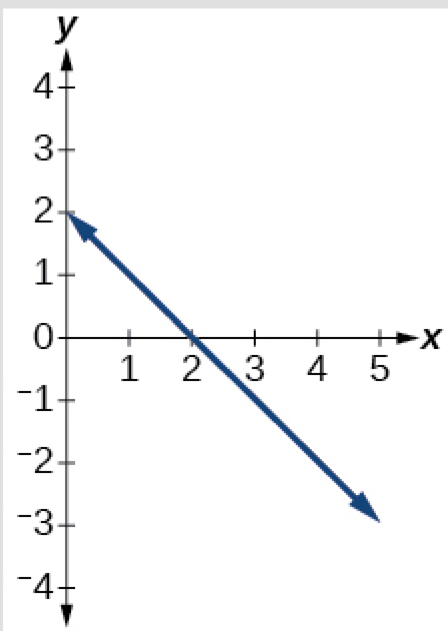
42. 

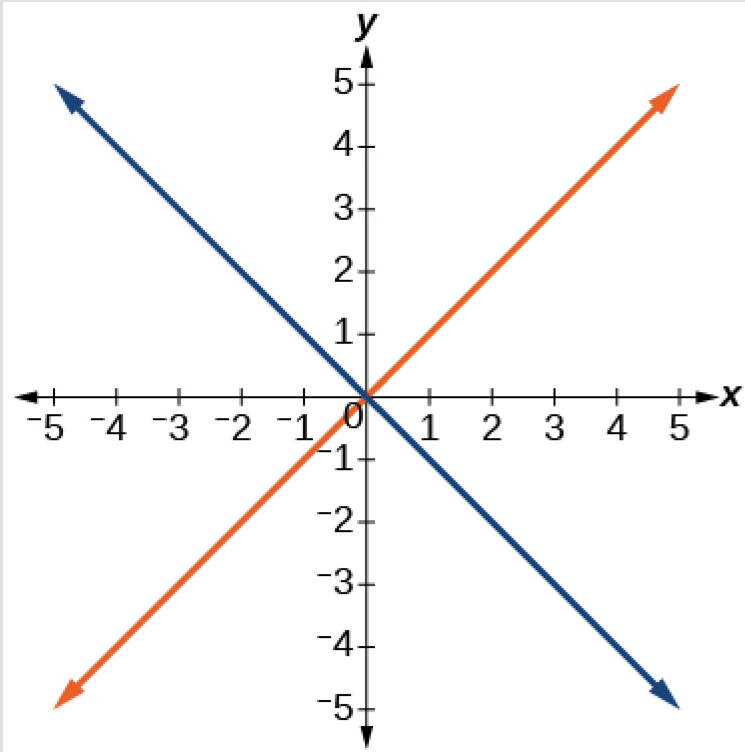
43. 

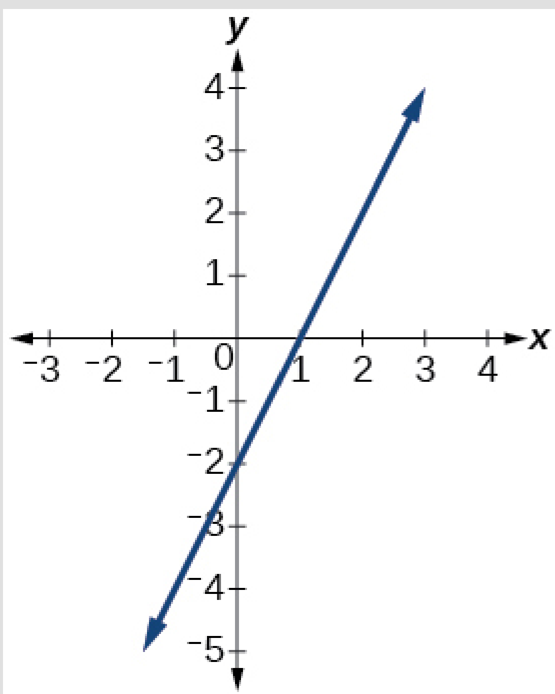
44. 

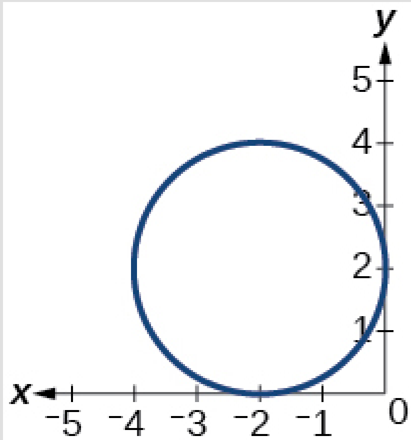
45. 

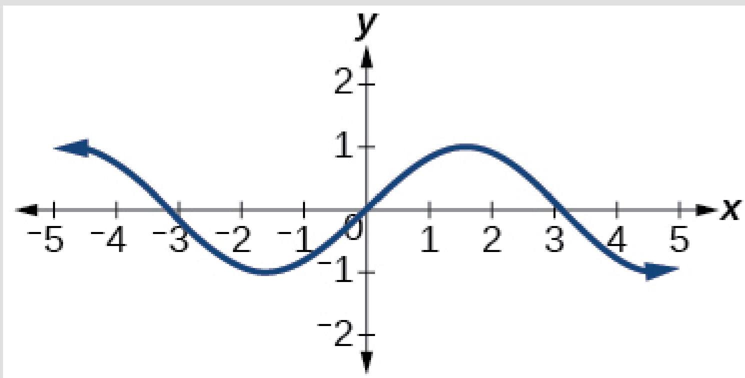
46. 

47. 

48. 

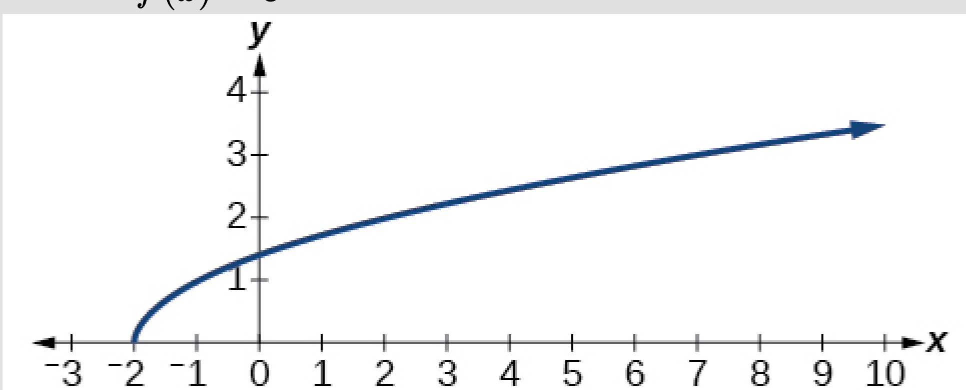
49. 

50. 

51. 

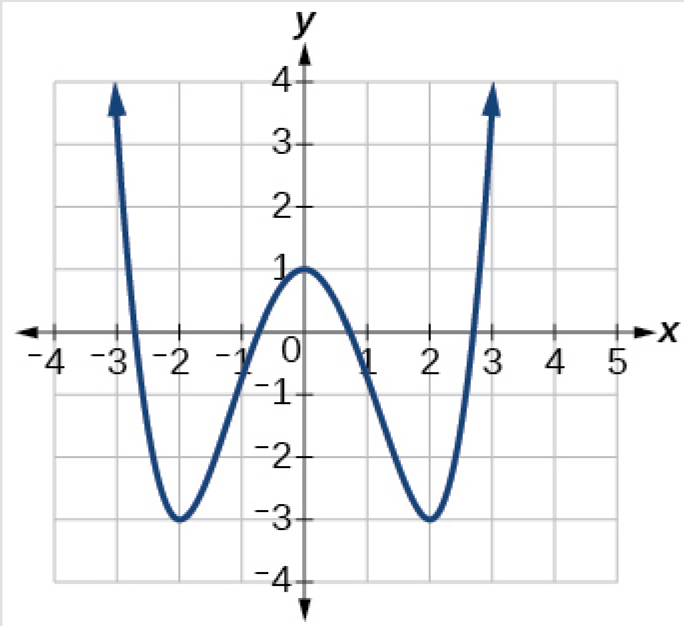
52. Given the following graph,

a. Evaluate $f(-1)$.
b. Solve for $f(x) = 3$.



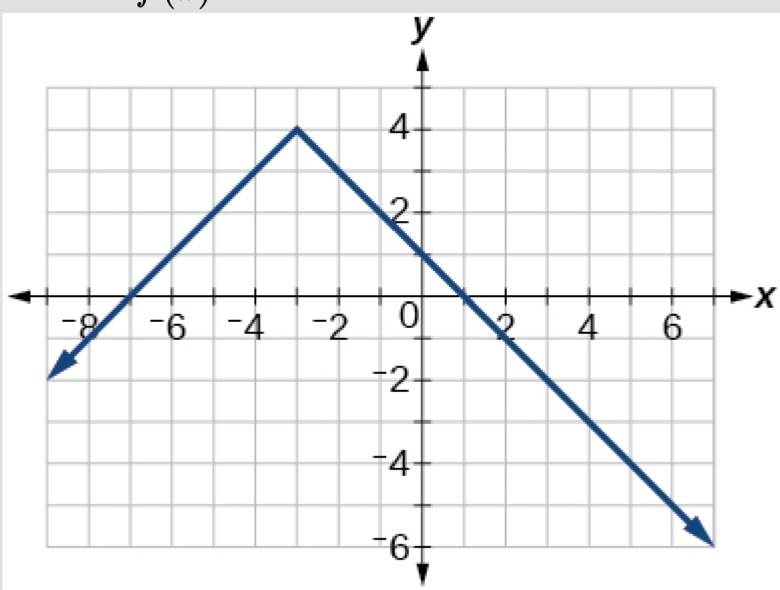
53. Given the following graph,

a. Evaluate $f(0)$.
b. Solve for $f(x) = -3$.

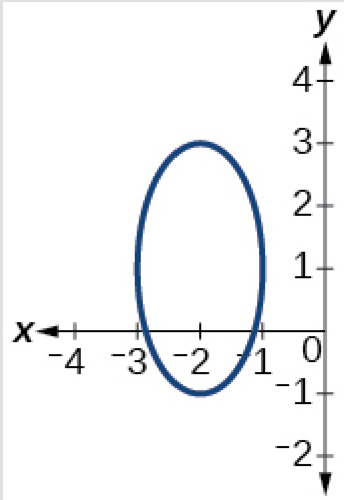


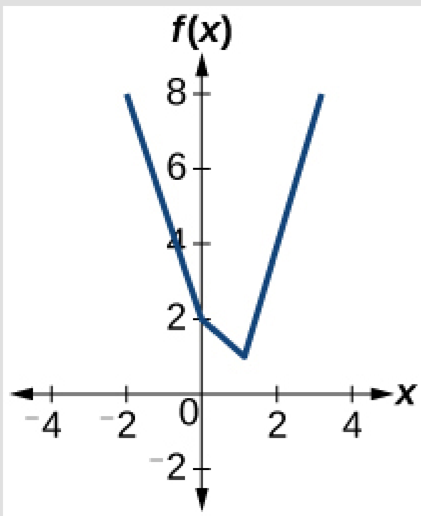
54. Given the following graph,

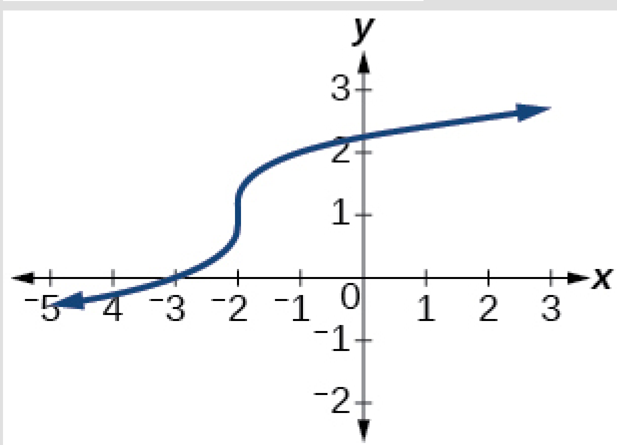
   a. Evaluate $f\left(4\right)$.
   b. Solve for $f\left(x\right)=1$.

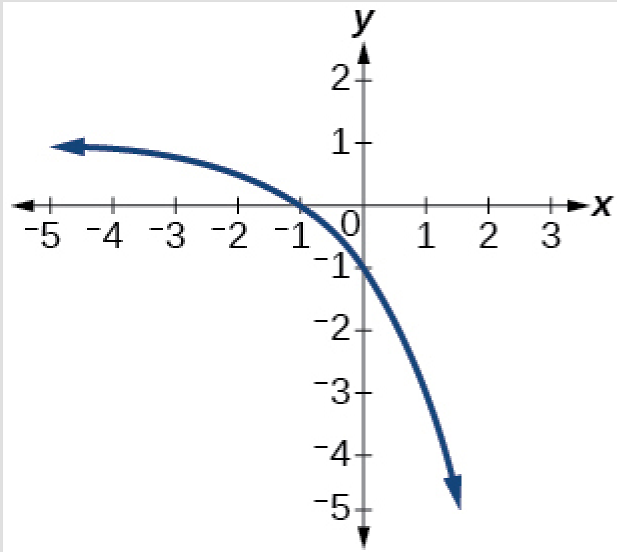


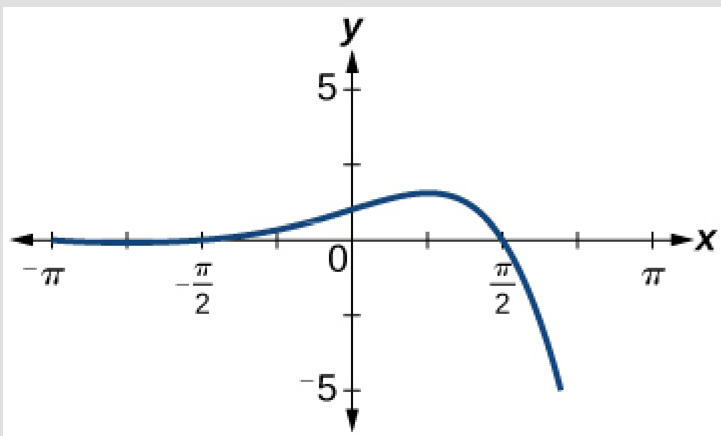
For the following exercises, determine if the given graph is a one-to-one function.

55. 

56. 

57. 

58. 

59. 

[Odd Number Graphical Solutions]

Numeric Questions

For the following exercises, determine whether the relation represents a function.

60. $\{(-1,-1),(-2,-2),(-3,-3)\}$
61. $\{(3,4),(4,5),(5,6)\}$
62. $\{(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

63.

x	5	10	15
y	3	8	14

64.

x	5	10	15
y	3	8	8

65.

x	5	10	10
y	3	8	14

For the following exercises, use the function f represented in Table 1-14.

Table 1-14

x	$f(x)$
0	74
1	28
2	1
3	53
4	56
5	3
6	36
7	45
8	14
9	47

66. Evaluate $f(3)$.

67. Solve $f(x) = 1$.

For the following exercises, evaluate the function f at the values $f(-2), f(-1), f(0), f(1)$ and $f(2)$.

68. $f(x) = 4 - 2x$

69. $f(x) = 8 - 3x$

70. $f(x) = 8x^2 - 7x + 3$

71. $f(x) = 3 + \sqrt{x + 3}$

72. $f(x)=\dfrac{x-2}{x+3}$
73. $f(x)=3^x$

For the following exercises, evaluate the expressions, given functions f, g, and h :

- $f(x)=3x-2$
- $g(x)=5-x^2$
- $h(x)=-2x^2+3x-1$

74. $3f(1)-4g(-2)$
75. $f\left(\dfrac{7}{3}\right)-h(-2)$

Odd Number Numeric Solutions

Technology Questions

For the following exercises, graph $y=x^2$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

76. $[-0.1,\ 0.1]$
77. $[-10,\ 10]$
78. $[-100, 100]$

For the following exercises, graph $y=x^3$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

79. $[-0.1,\ 0.1]$
80. $[-10,\ 10]$
81. $[-100,\ 100]$

For the following exercises, graph $y=\sqrt{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

82. $[0,\ 0.01]$

83. $[0,\ 100]$
84. $[0,\ 10{,}000]$

For the following exercises, graph $y = \sqrt[3]{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

85. $[-0.001, 0.001]$
86. $[-1000, 1000]$
87. $[-1{,}000{,}000, 1{,}000{,}000]$

Odd Number Technology Solutions

Real-World Applications Questions

88. The amount of garbage, $G,$ produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

    a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
    b. Explain the meaning of the statement $f(5) = 2$.

89. The number of cubic yards of dirt, $D,$ needed to cover a garden with area a square feet is given by $D = g(a)$.

    a. A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function g.
    b. Explain the meaning of the statement $g(100) = 1$.

90. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

    a. $f(5) = 30$
    b. $f(10) = 40$

91. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:

    a. $h(1) = 200$

b. $h(2) = 350$

92. Show that the function $f(x) = 3(x-5)^2 + 7$ is not one-to-one.

Odd Number Real-World Application Solutions

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1.10 Chapter 1 Example Solutions

1.2 Solutions

Example 1: Determining If Menu Price Lists Are Functions

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 1-11.

Figure 1-11

Each item on the menu has only one price, so the price is a function of the item.

b. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 1-12.

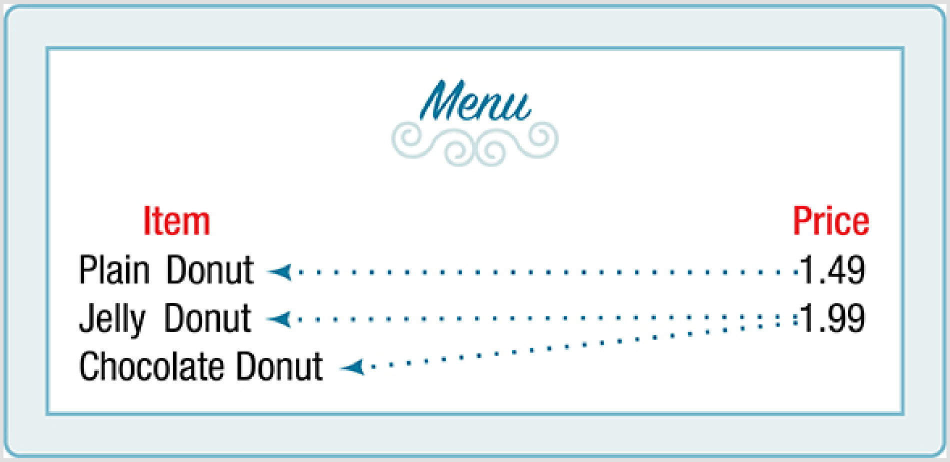


Figure 1-12

Therefore, the item is a not a function of price.

Example 2: Determining if Class Grade Rules are Functions

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Example 3: Determining if a Function is Present

a. Yes.
b. Yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

1.3 Solutions

Example 1: Function Notation for Days in a Month

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

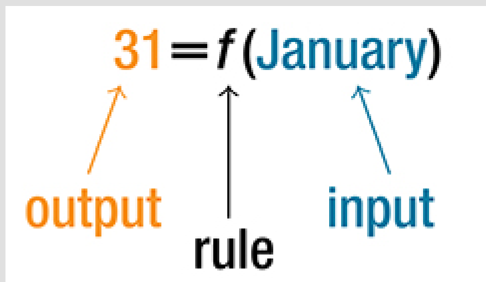


Figure 1-13

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Example 2: Interpreting Function Notation

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Example 3: Using Function Notation

$w = f(d)$

1.4 Solutions

Example 1: Identifying Tables that Represent Functions

Table 1-6 and Table 1-7 define functions. In both, each input value corresponds to exactly one output value. Table 1-8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 1-6 can be represented by writing

$$f(2) = 1, f(5) = 3, \text{and } f(8) = 6$$

Similarly, the statements

$$g(-3) = 5, g(0) = 1, \text{and } g(4) = 5$$

represent the function in Table 1-7.

Table 1-8 cannot be expressed in a similar way because it does not represent a function.

Example 2: Identifying Functions

Yes

1.5 Solutions

Exercise 1: Evaluating Functions at Specific Values

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f\left(2\right) &= 2^2 + 3\left(2\right) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f\left(a\right) = a^2 + 3a - 4$$

c. With an input value of $a + h,$ we must use the distributive property.

$$\begin{aligned} f\left(a+h\right) &= \left(a+h\right)^2 + 3\left(a+h\right) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f\left(a+h\right) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f\left(a\right) = a^2 + 3a - 4$$

Now we combine the results and simplify.

\begin{array}{l}\begin{array}{l}\hfill \\ \frac{f\left(a+h\right)-f\left(a\
right)}{h}=\frac{\left({a}^{2}+2ah+{h}^{2}+3a+3h-4\right)-\left({a}^{2}+3a-4\right)}{h}\hfill
\end{array}\hfill \\ \begin{array}{ll}\text{ }=\frac{2ah+{h}^{2}+3h}{h}\hfill & \hfill \\
\text{ }=\frac{h\left(2a+h+3\right)}{h}\hfill &
\begin{array}{cc}\begin{array}{cc}& \end{array}& \end{array}\text{Factor out }h.\hfill \\
\text{ }=2a+h+3\hfill & \begin{array}{cc}\begin{array}{cc}& \end{array}&
\end{array}\text{Simplify}.\hfill \end{array}\hfill \end{array}

Example 2: Evaluating Functions

a. To evaluate $h\left(4\right)$, we substitute the value 4 for the input variable p in the given function.

\begin{array}{l}\text{ }h\left(p\right)={p}^{2}+2p\hfill \\ \text{ }h\left(4\right)={\left(4\right)}^{2}+2\
left(4\right)\hfill \\ \text{ }=16+8\hfill \\ \text{ }=24\hfill \end{array}

Therefore, for an input of 4, we have an output of 24.

b. $g\left(5\right) = 1$

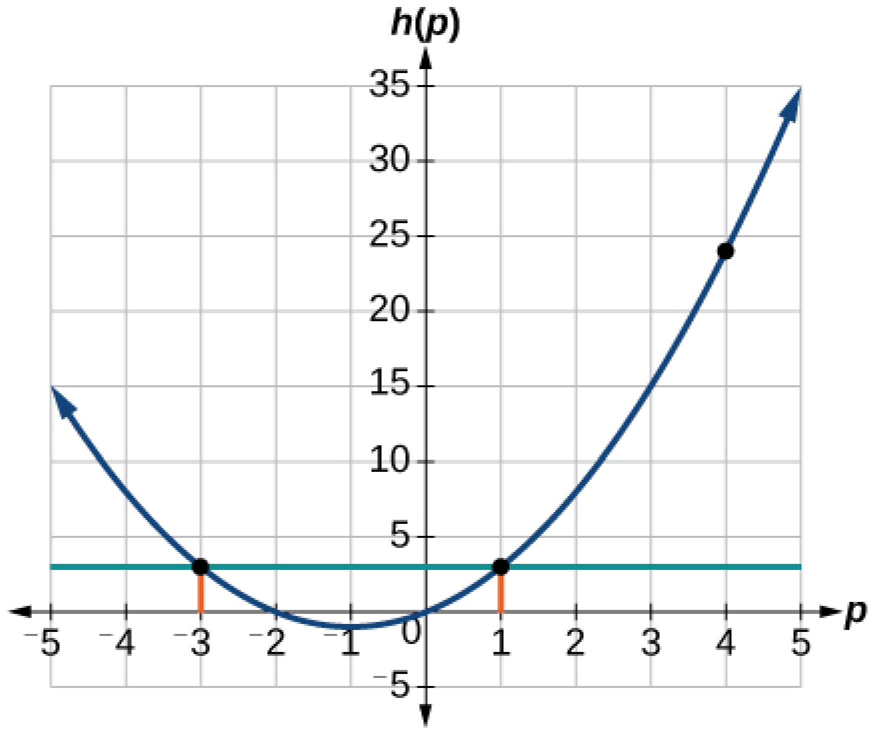
Example 3: Solving Functions

a. \begin{array}{llll}\text{ }h\left(p\right)=3\hfill & \hfill & \hfill & \hfill \\ \text{ }{p}^{2}+2p=3\hfill & \hfill & \hfill & \text{Substitute the original function }h\left(p\right)={p}^{2}+2p.\hfill \\ \text{ }{p}^{2}+2p-3=0\hfill & \hfill & \hfill & \text{Subtract 3 from each side}.\hfill \\ \text{ }\left(p+3\text{)(}p-1\right)=0\hfill & \hfill & \hfill & \text{Factor}.\hfill \end{array}

If $\left(p+3\right)\left(p-1\right)=0$, either $\left(p+3\right)=0$ or $\left(p-1\right)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3)=0, \quad p=-3$$
$$(p-1)=0, \quad p=1$$

This gives us two solutions. The output $h\left(p\right)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in Figure 1-14. The graph verifies that $h\left(1\right)=h\left(-3\right)=3$ and $h\left(4\right)=24$.



p	$^-3$	$^-2$	0	1	4
$h(p)$	3	0	0	3	24

Figure 1-14

b. $m = 8$

Example 4: Finding an Equation of a Function

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p = [\text{expression involving } n]$.

$$2n + 6p = 12$$

$$6p = 12 - 2n \qquad \text{Subtract } 2n \text{ from both sides.}$$

$$p = \frac{12-2n}{6} \qquad \text{Divide both sides by 6 and simplify.}$$

$$p = \frac{12}{6} - \frac{2n}{6}$$

$$p = 2 - \frac{1}{3}n$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n$$

Example 5: Expressing the Equation of a Circle as a Function

a. First we subtract x^2 from both sides.

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation.

\begin{array}{l}y=±\sqrt{1-{x}^{2}}\hfill \\ \text{ }=+\sqrt{1-{x}^{2}}\text{ and }-\sqrt{1-{x}^{2}}\hfill \end{array}

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

b. $y = f(x) = \dfrac{\sqrt[3]{x}}{2}$

Example 6: Evaluating and Solving a Tabular Function

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.

b. Solving $g(n) = 6$ means identifying the input values, n, that produce an output value of 6. Table 1-15 shows two solutions: 2 and 4.

Table 1-15

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

c. $g(1) = 8$

Example 7: Reading Function Values from a Graph

a. To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the *y*-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 1-15.

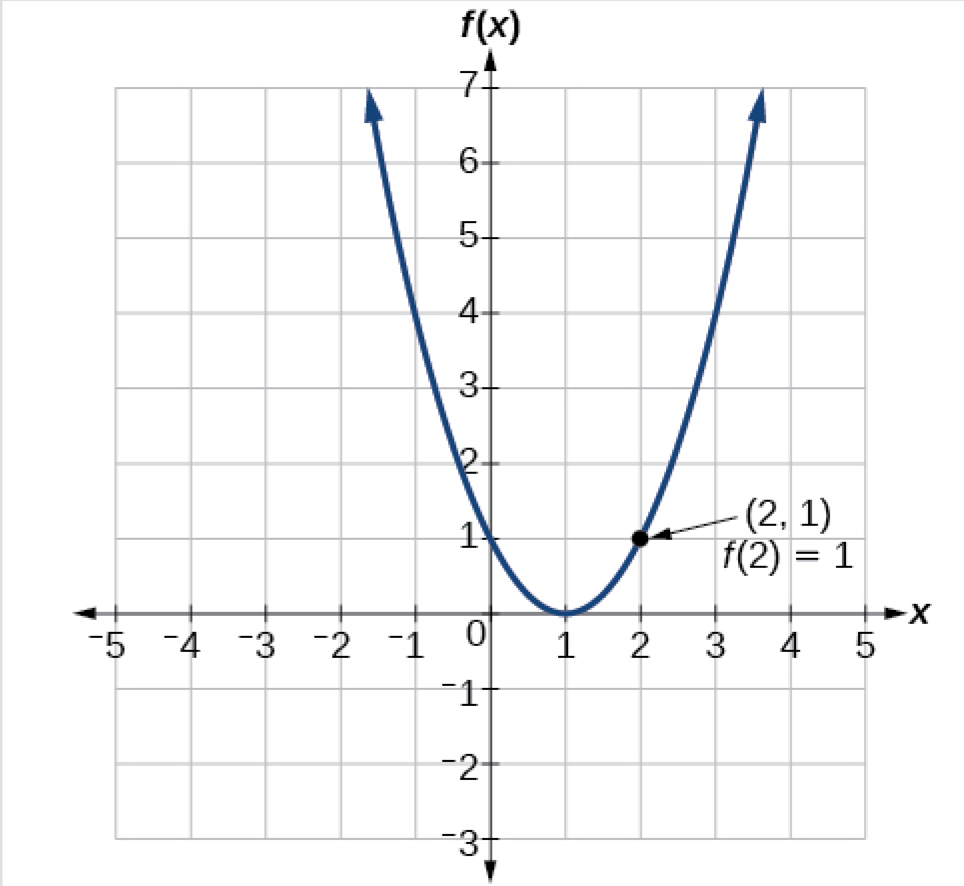


Figure 1-15

b. To solve $f\left(x\right)=4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y=4$, we locate two points of the curve with output value $4:\left(-1,4\right)$ and $\left(3,4\right)$. These points represent the two solutions to $f\left(x\right)=4:-1$ or 3. This means $f\left(-1\right)=4$ and $f\left(3\right)=4$ or when the input is -1 or $3,$ the output is 4. See Figure 1-16.

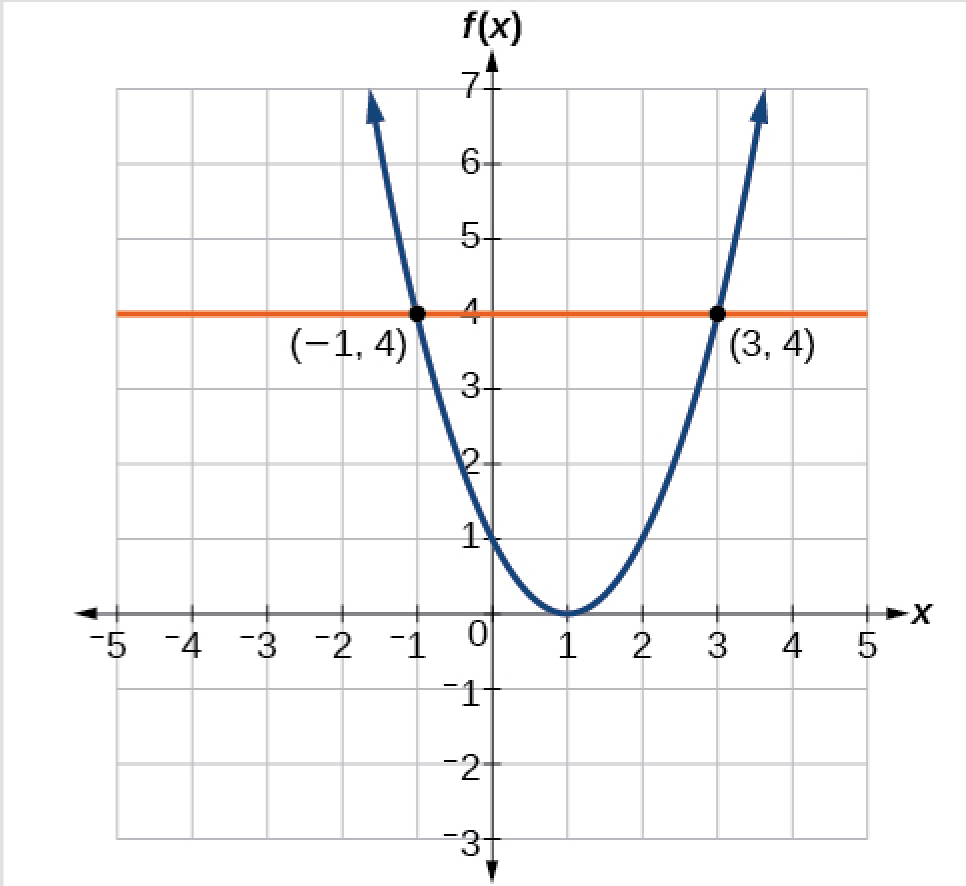


Figure 1-16

c. $x = 0$ or $x = 2$

1.6 Solutions

Example 1: Determining Whether a Relationship is a One-to-One Function

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Example 2: Determining Whether a Relationship is a One-to-One Function

1. Yes, because each bank account has a single balance at any given time;
2. No, because several bank account numbers may have the same balance;
3. No, because the same output may correspond to more than one input.

Example 3: Determining Whether a Relationship is a One-to-One Function

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

Example 4: Applying the Vertical Line Test

a. If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of [1.6 Figure 1-8](). From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most *x*-values, a vertical line would intersect the graph at more than one point, as shown in Figure 1-17.

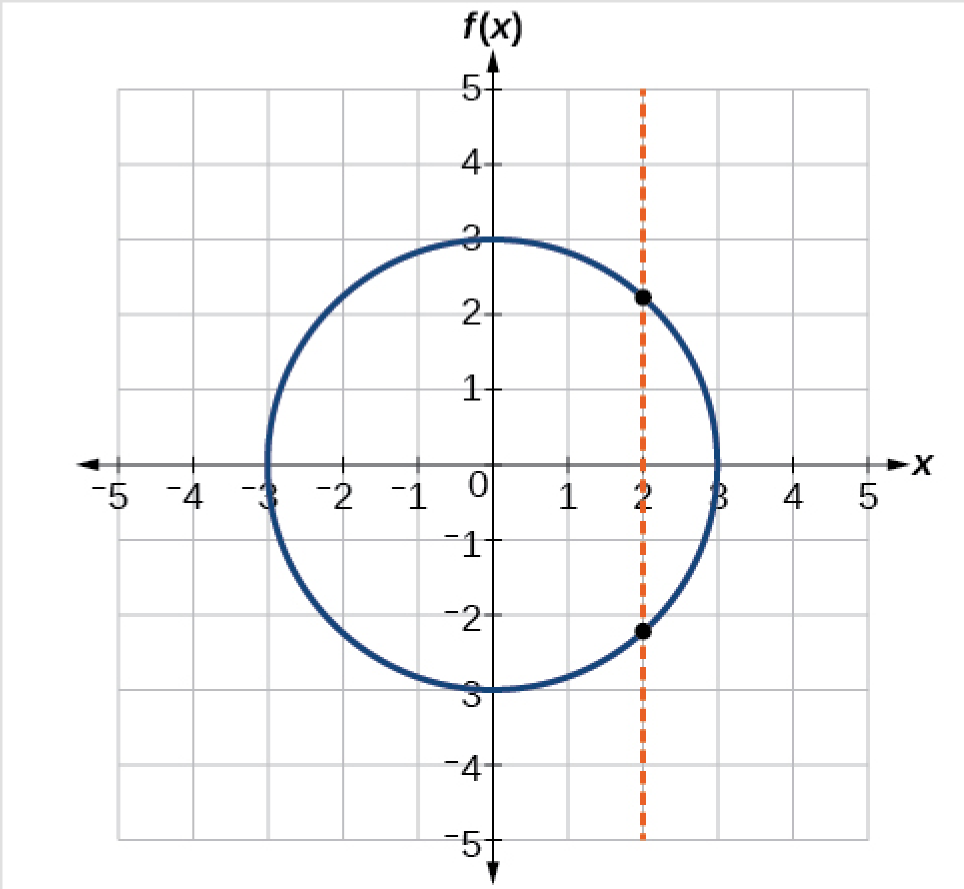


Figure 1-17

b. Yes

Example 5: Applying the Horizontal Line Test

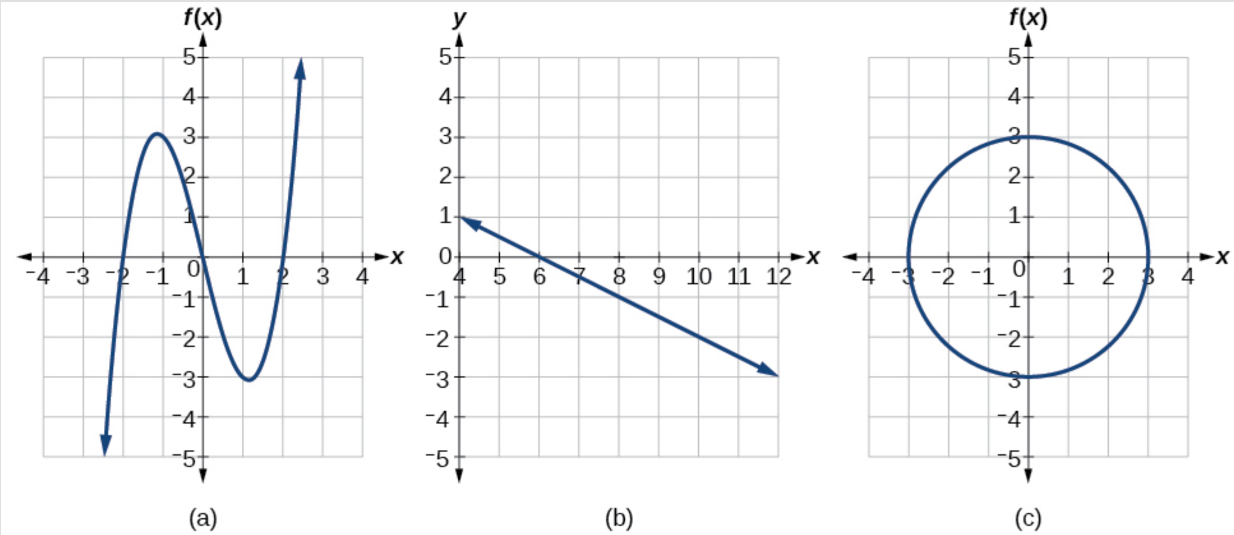


Figure 1-18

a. The function in Figure 1-18 **(a)** is not one-to-one. The horizontal line shown in Figure 1-19 intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

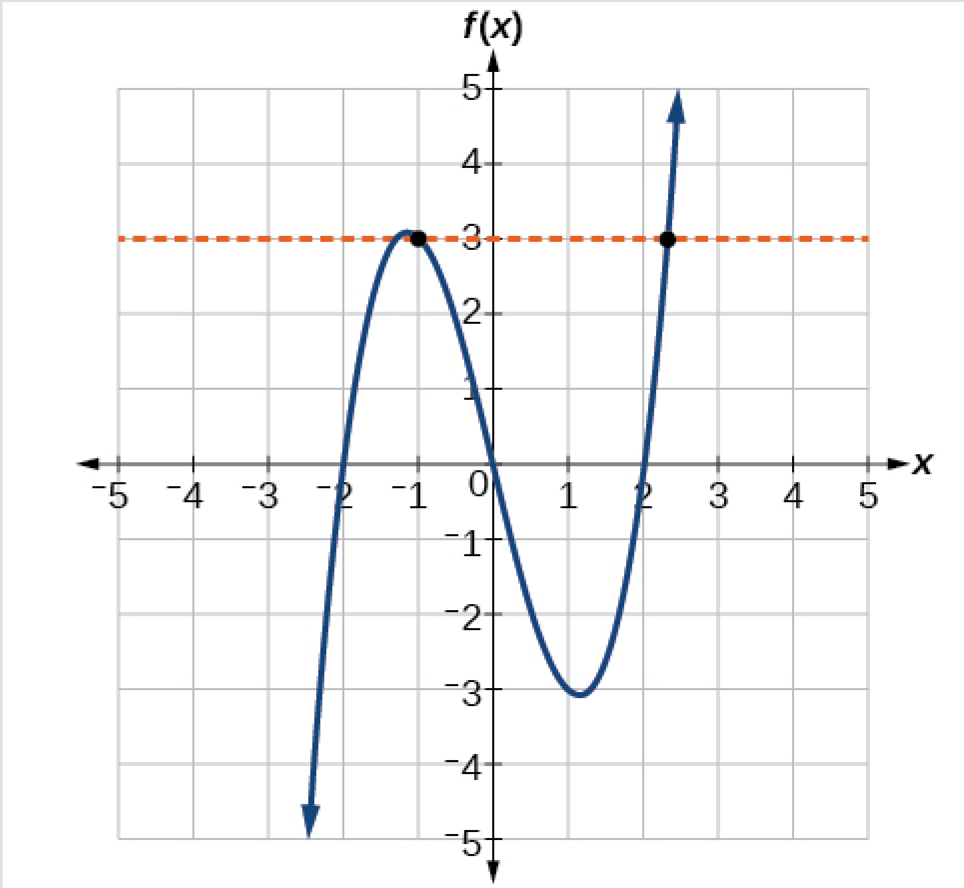


Figure 1-19

The function in Figure 1-18 **(b)** is one-to-one. Any horizontal line will intersect a diagonal line at most once.

b. No, because it does not pass the horizontal line test.

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1.11 Practice Question Solutions

Verbal Question Solutions

1. A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

3. When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

5. When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

Algebraic Question Solutions

7. function

9. function

11. function

13. function

15. function

17. function

19. function

21. function

23. function

25. not a function.

27. $f(-3) = -11;$

$f(2) = -1;$

$f(-a) = -2a - 5;$

$-f(a) = -2a + 5;;$

$f(a+h) = 2a + 2h - 5$

29. $f(-3) = \sqrt{5} + 5;$

$f(2) = 5;$

$f(-a) = \sqrt{2+a} + 5;$

$-f(a) = -\sqrt{2-a} - 5;$

$f(a+h) = \sqrt{2-a-h} + 5$

31. $f(-3) = 2;$

$f(2) = 1 - 3 = -2;$

$f(-a) = |-a-1| - |-a+1|;$

$-f(a) = -|a-1| + |a+1|;$

$f(a+h) = |a+h-1| - |a+h+1|$

33. $\dfrac{g(x) - g(a)}{x - a} = x + a + 2, x \neq a$

35. a. $f(-2) = 14;$

b. $x = 3$

37. a. $f(5) = 10;$

b. $x = -1$ or $x = 4$

39. a. $f(t) = 6 - \dfrac{2}{3}t;$

b. $f(-3) = 8;$

c. $t = 6$

Graphical Question Solutions

41. not a function

43. function

45. function

47. function

49. function

51. function

53. a. $f\left(0\right)=1;$

b. $f\left(x\right)=-3, x=-2$ or $x=2$

55. not a function so it is also not a one-to-one function

57. one-to-one function

59. function, but not one-to-one

Numeric Question Solutions

61. function

63. function

65. not a function

67. $f\left(x\right)=1, x=2$

69. $f\left(-2\right)=14;$

$f\left(-1\right)=11;$

$f\left(0\right)=8;$

$f\left(1\right)=5;$

$f\left(2\right)=2$

71. $f\left(-2\right)=4;$

$f\left(-1\right)=4.414;$

$f\left(0\right)=4.732;$

$f\left(1\right)=4.5;$

$f\left(2\right)=5.236$

73. $f\left(-2\right)=\frac{1}{9};$

$f\left(-1\right)=\frac{1}{3};$

$f\left(0\right)=1;$

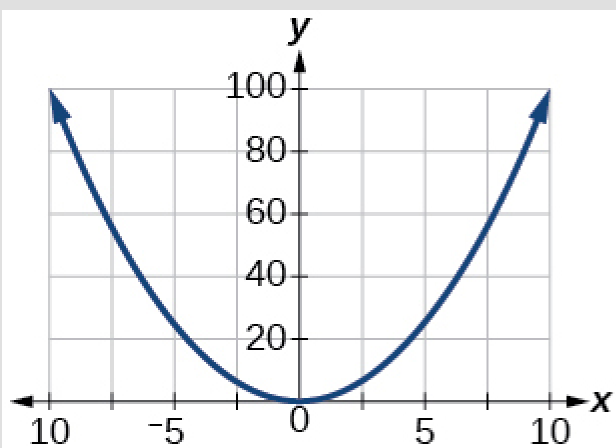
$f\left(1\right)=3;$

$f\left(2\right)=9$

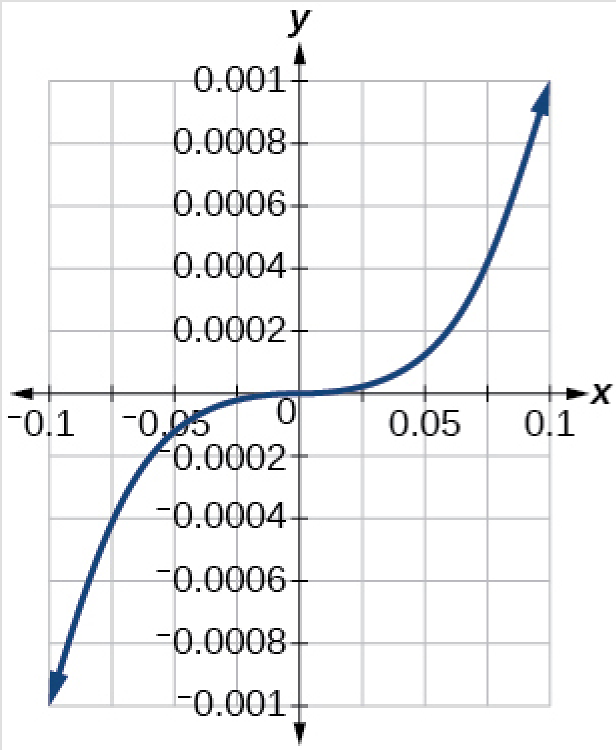
75. 20

Technology Question Solutions

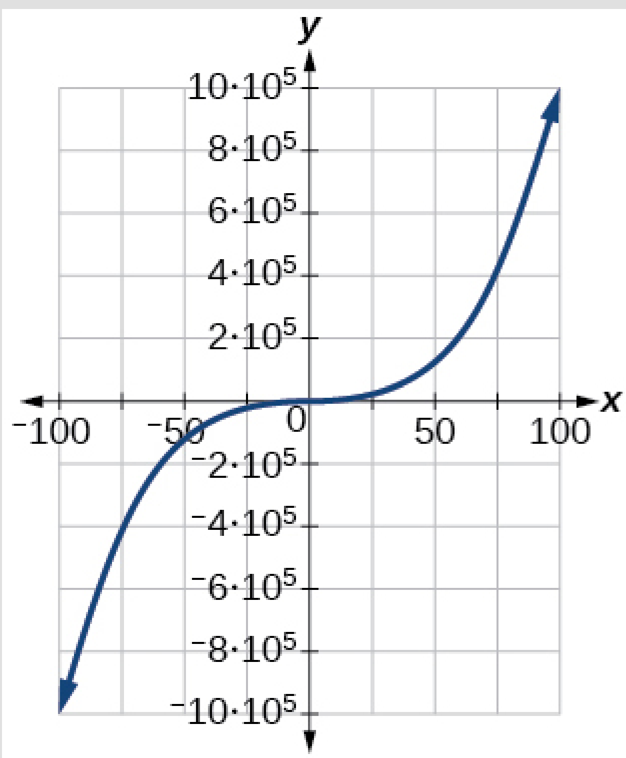
77. $\left[0,\ 100\right]$



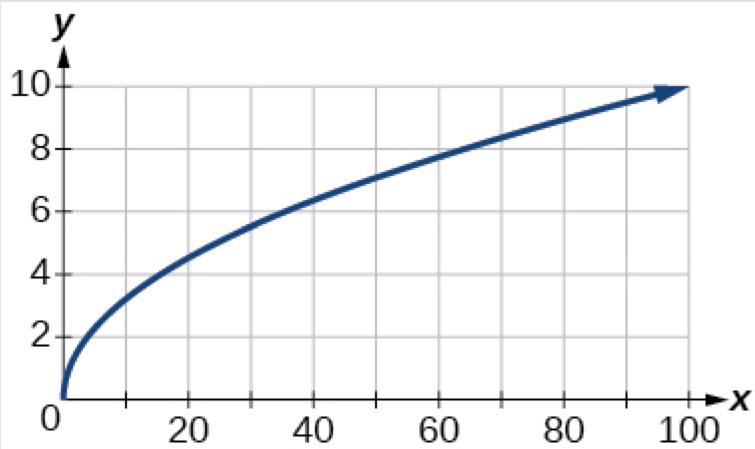
79. $\left[-0.001,\ 0.001\right]$



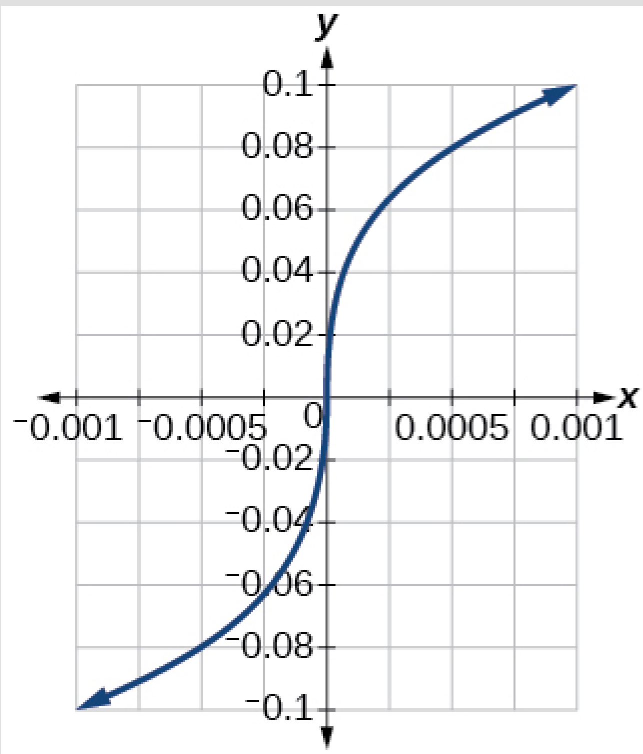
81. $[-1,000,000,\ 1{,}000{,}000]$



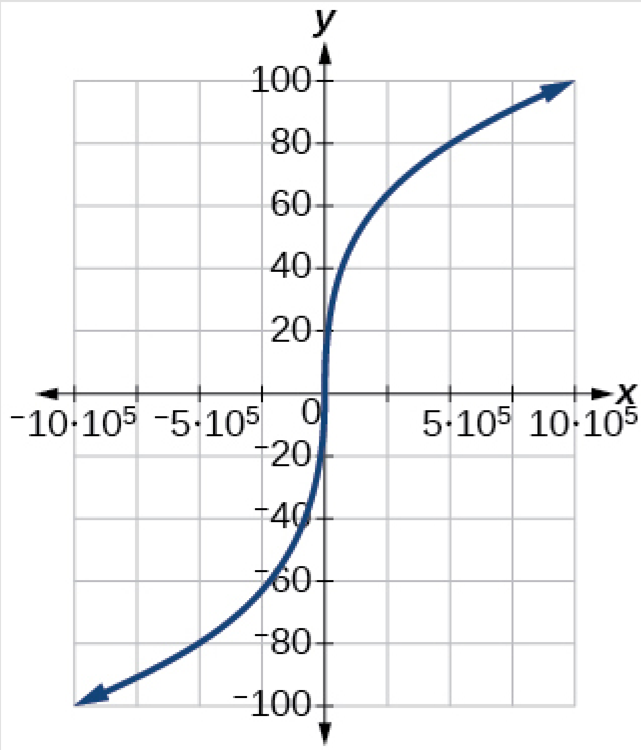
83. $[0,\ 10]$



85. $[-0.1, 0.1]$



87. $[-100,\ 100]$



Real-World Application Question Solutions

89. a. $g\left(5000\right)=50;$

b. The number of cubic yards of dirt required for a garden of 100 square feet is 1.

91. a. The height of a rocket above ground after 1 second is 200 ft.

b. the height of a rocket above ground after 2 seconds is 350 ft.

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CHAPTER 2: DOMAIN AND RANGE

2.1 Introduction to Domain and Range

Learning Objectives

In this section, you will:

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—*I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. Figure 2-1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

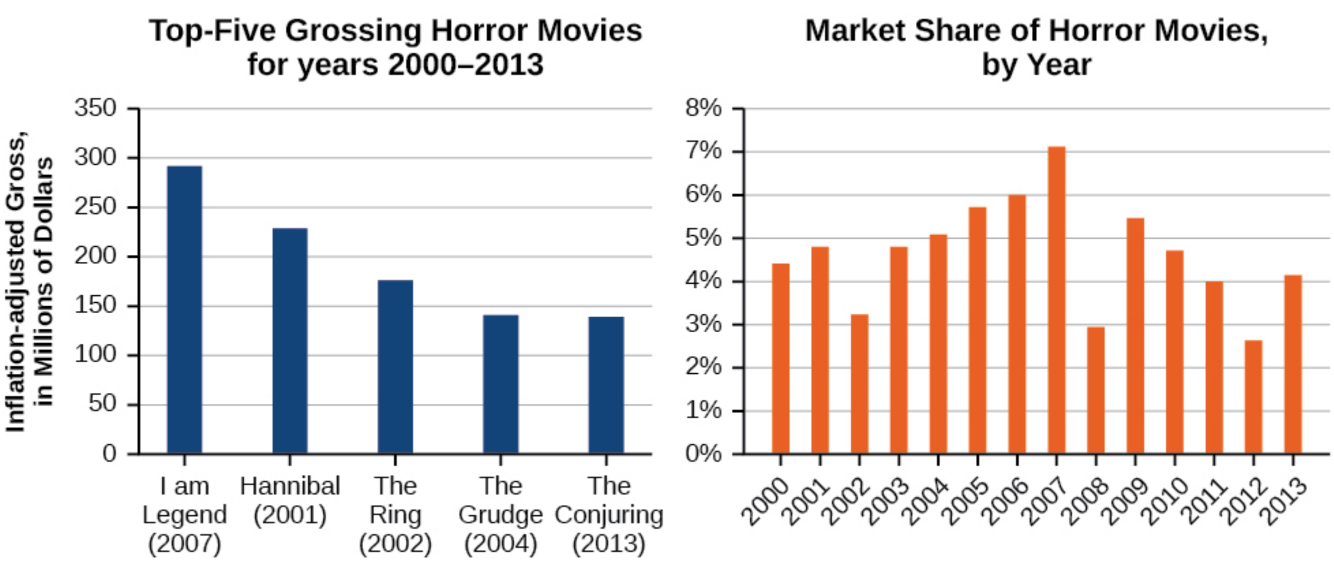

Figure 2-1: The Numbers: Where Data and the Movie Business Meet. "Box Office History for Horror Movies." http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

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2.2 Finding the Domain of a Function Defined by an Equation

In 1.3 Using Function Notation, we were introduced to the concepts of domain and range. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a “holding area” that contains “raw materials” for a “function machine” and the range as another “holding area” for the machine’s products. See Figure 2-2.

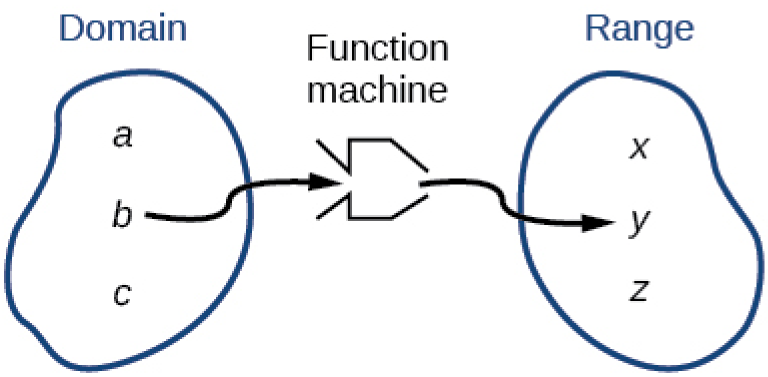

Figure 2-2

We can write the domain and range in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$ We will discuss interval notation in greater detail later.

Let’s turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function’s equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See Figure 2-3 for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		*x* is greater than *a*
$x < a$	$(-\infty, a)$		*x* is less than *a*
$x \geq a$	$[a, \infty)$		*x* is greater than or equal to *a*
$x \leq a$	$(-\infty, a]$		*x* is less than or equal to *a*
$a < x < b$	(a, b)		*x* is strictly between *a* and *b*
$a \leq x < b$	$[a, b)$		*x* is between *a* and *b*, to include *a*
$a < x \leq b$	$(a, b]$		*x* is between *a* and *b*, to include *b*
$a \leq x \leq b$	$[a, b]$		*x* is between *a* and *b*, to include *a* and *b*

Figure 2-3

Example 1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$

Solution

Example 2: Find the Domain of a Function

Find the domain of the function: $\{(-5,4),(0,0),(5,-4),(10,-8),(15,-12)\}$

Solution

How To

Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 3: Finding the Domain of a Function

1. Find the domain of the function $f(x) = x^2 - 1$
2. Find the domain of the function: $f(x) = 5 - x + x^3$

Solution

How To

Given a function written in an equation form that includes a fraction, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x. If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 4: Finding the Domain of a Function Involving a Denominator

1. Find the domain of the function $f(x)=\dfrac{x+1}{2-x}$
2. Find the domain of the function: $f(x)=\dfrac{1+4x}{2x-1}$

Solution

How To

Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the

radicand. Set the radicand greater than or equal to zero and solve for x.

3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 5: Finding the Domain of a Function with an Even Root

1. Find the domain of the function $f(x) = \sqrt{7 - x}$
2. Find the domain of the function $f(x) = \sqrt{5 + 2x}$

Solution

Question & Answer

Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x) = -\dfrac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

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2.3 Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in **set-builder notation**. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of *x*-values such that 10 is less than or equal to x and x is less than 30."

Figure 2-4 compares inequality notation, set-builder notation, and interval notation.

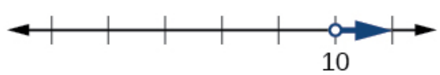
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 2-4

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\bigcup$ to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3 = (-\infty, -3] \cup [3, \infty)\}$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x| \ statement \ about \ x\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

How To

Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (●) or (for each excluded end value (○).
3. At the right end of each interval, use] with each end value to be included in the set (●) or) for each excluded end value (○).
4. Use the union symbol, ∪, to combine all intervals into one set.

Example 1: Describing Sets on the Real-Number Line

1. Describe the intervals of values shown in Figure 2-4 using inequality notation, set-builder notation, and interval notation.

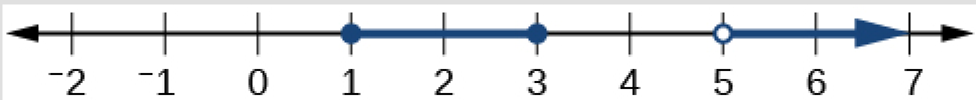


Figure 2-4

2. Given Figure 2-5, specify the graphed set in

   a. words
   b. set-builder notation
   c. interval notation

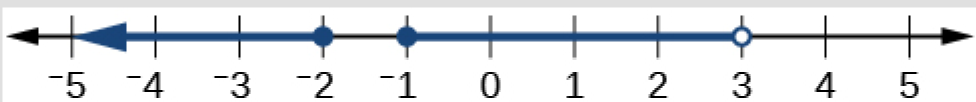


Figure 2-5

[Solution](#)

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2.4 Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the *x*-axis. The range is the set of possible output values, which are shown on the *y*-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 2-6.

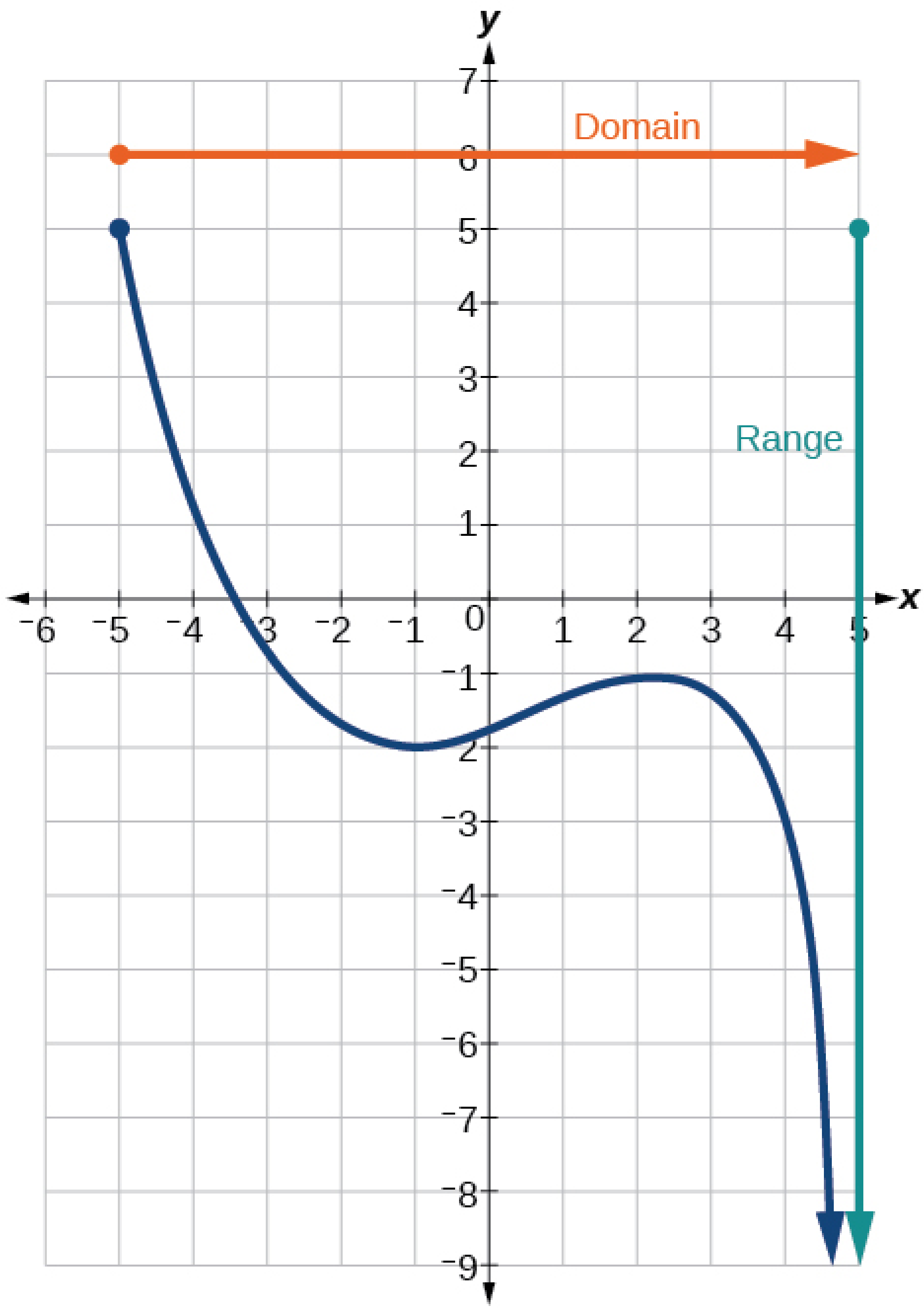

Figure 2-6

We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 1: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 2-7.

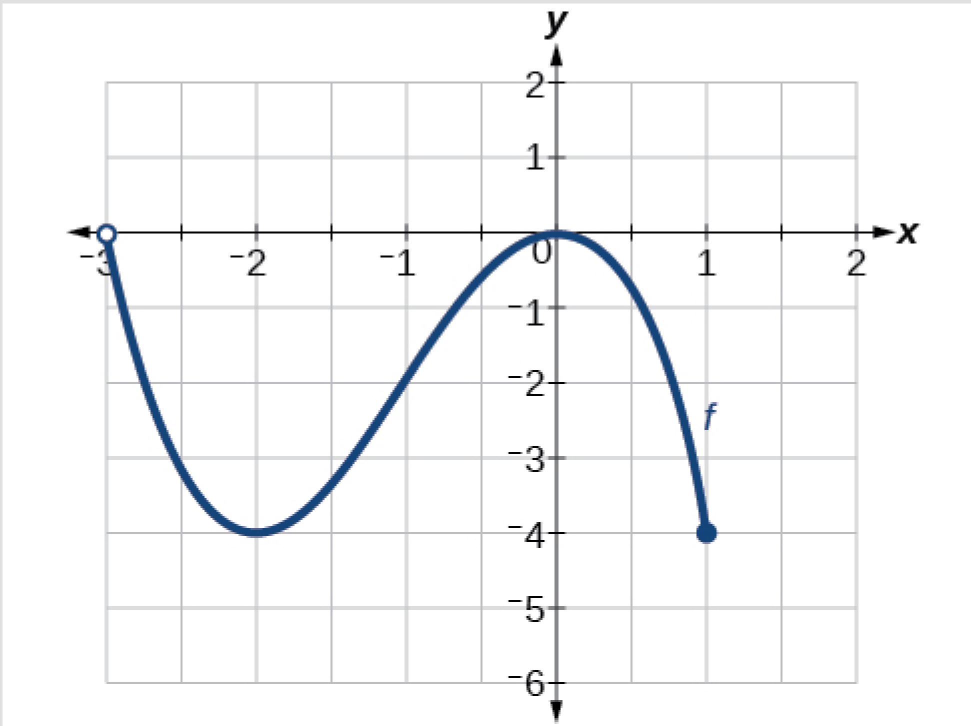

Figure 2-7

Solution

Example 2: Finding the Domain and Range from a Graph

1. Find the domain and range of the function f whose graph is shown in Figure 2-8.[1]

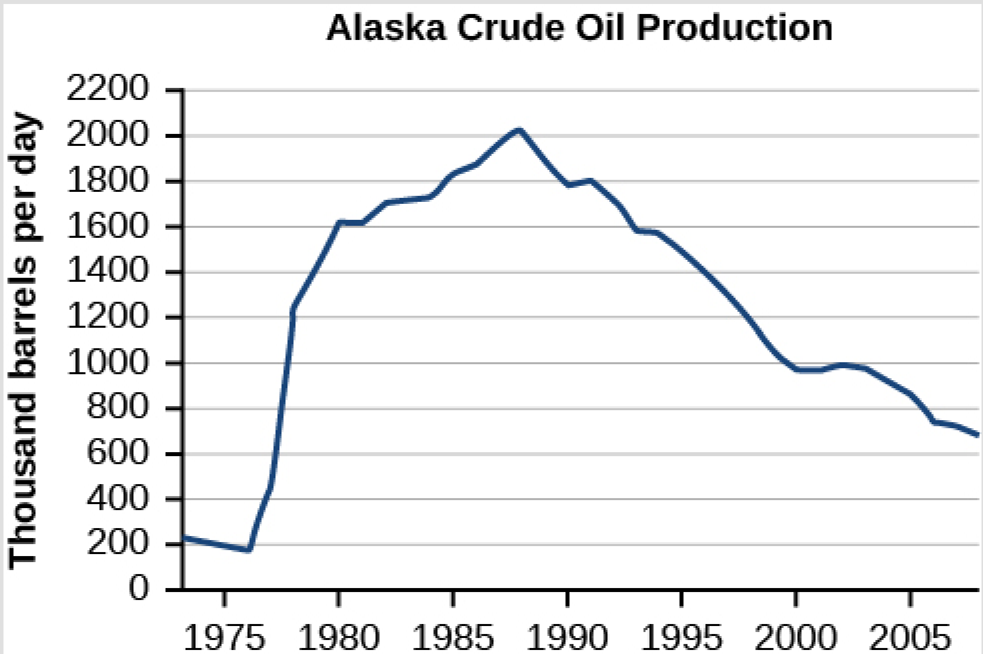

Figure 2-8: (credit: modification of work by the U.S. Energy Information Administration)

2. Given Figure 2-9, identify the domain and range using interval notation.

1. http://www.eia.gov/dnav/pet/hist/LeafHandler.ashx?n=PET&s=MCRFPAK2&f=A.

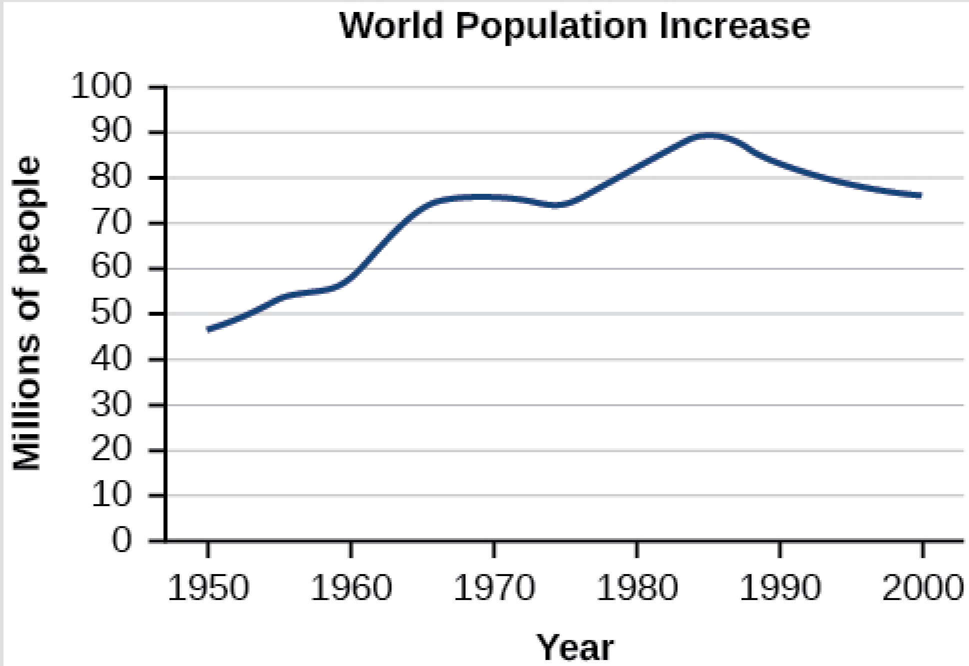

Figure 2-9

Solution

Question & Answer

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

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2.5 Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

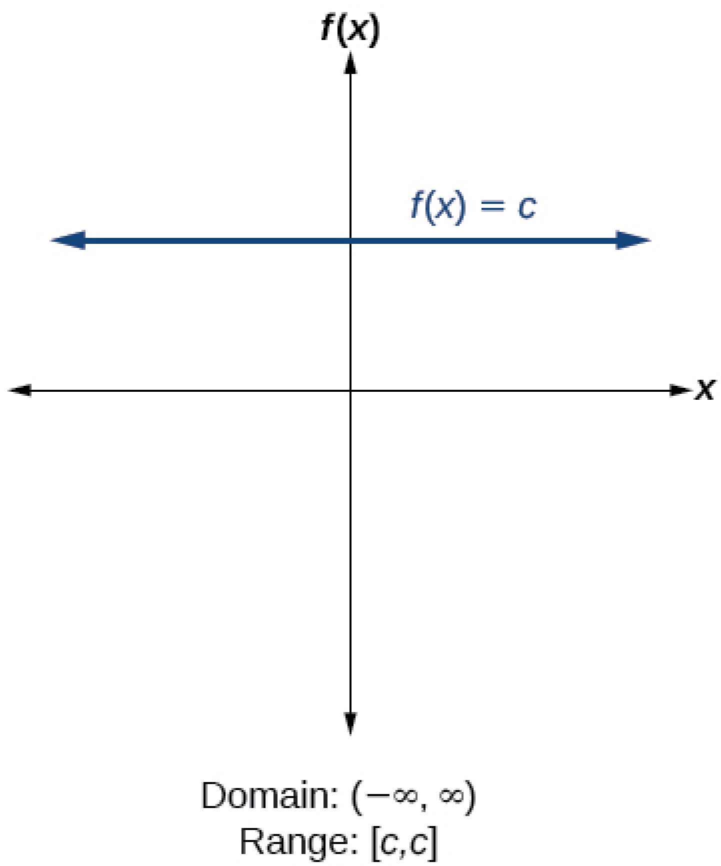

Figure 2-10: For the ***constant function*** *[latex]\text{}f\left(x\right)=c,\text{}[/latex] the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant [latex]\text{}c,\text{}[/latex] so the range is the set [latex]\text{}\left\{c\ right\}\text{}[/latex] that contains this single element. In interval notation, this is written as [latex]\text{}\left[c,c\right],\text{}[/latex] the interval that both begins and ends with [latex]\text{}c[/latex].*

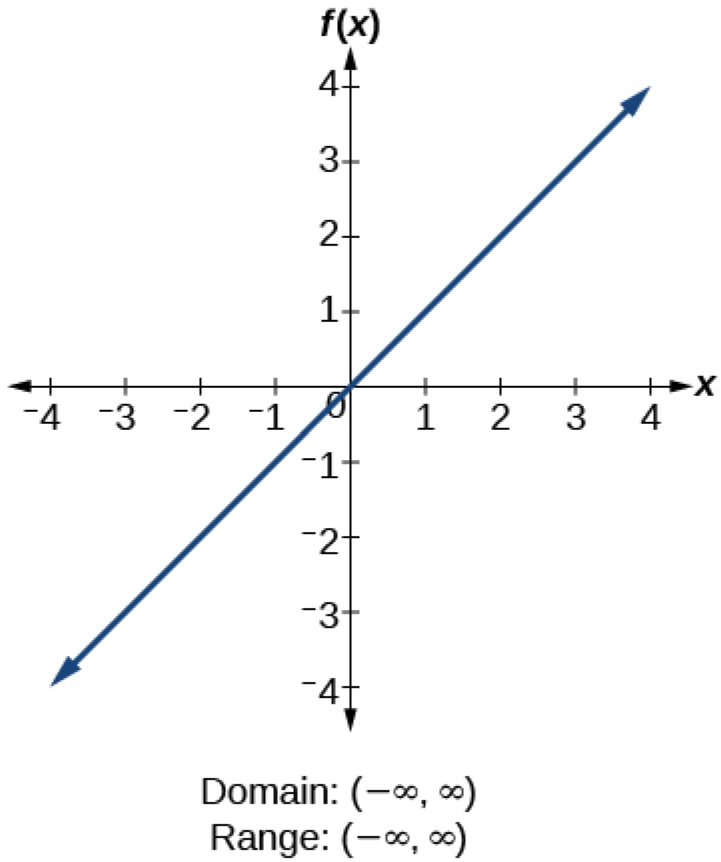

*Figure 2-11: For the **identity function** [latex]\text{}f\left(x\right)=x,\text{}[/latex] there is no restriction on [latex]\text{}x\text{}[/latex]. Both the domain and range are the set of all real numbers.*

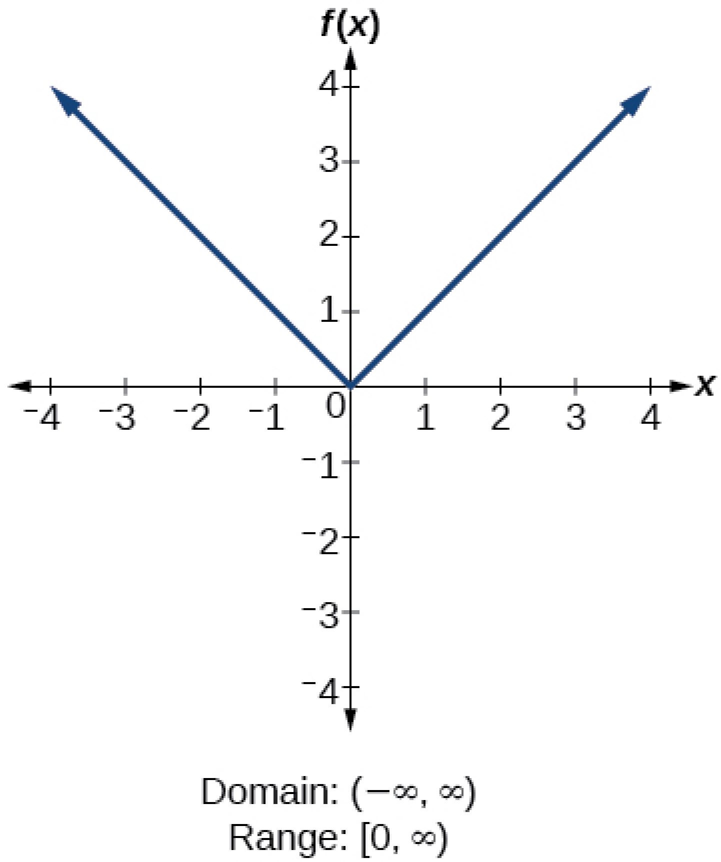

*Figure 2-12: For the **absolute value** function [latex]\text{}f\left(x\right)=|x|,\text{}[/latex]there is no restriction on [latex]\text{}x\text{}[/latex]. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.*

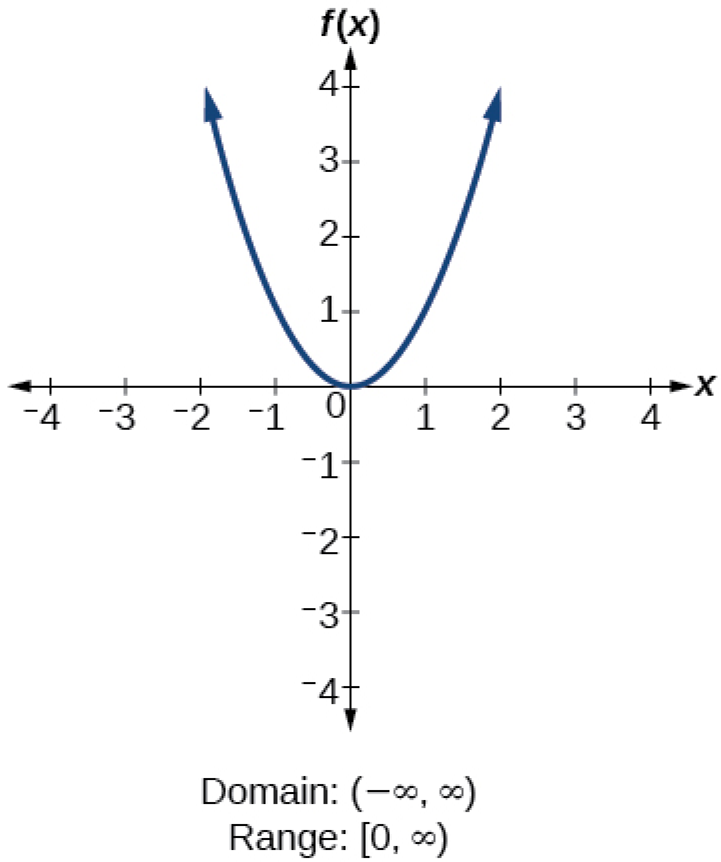

Figure 2-13: For the ***quadratic function*** *[latex]\text{}f\left(x\right)={x}^{2},\text{}[/latex] the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.*

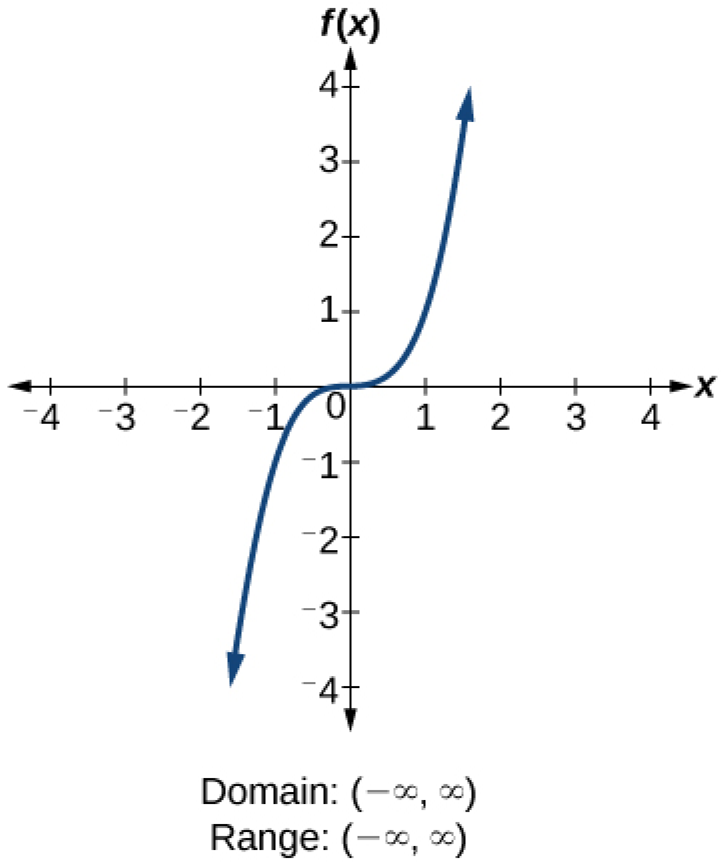

Figure 2-14: For the ***cubic function*** *[latex]\text{}f\left(x\right)={x}^{3}\text{}[/latex], the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.*

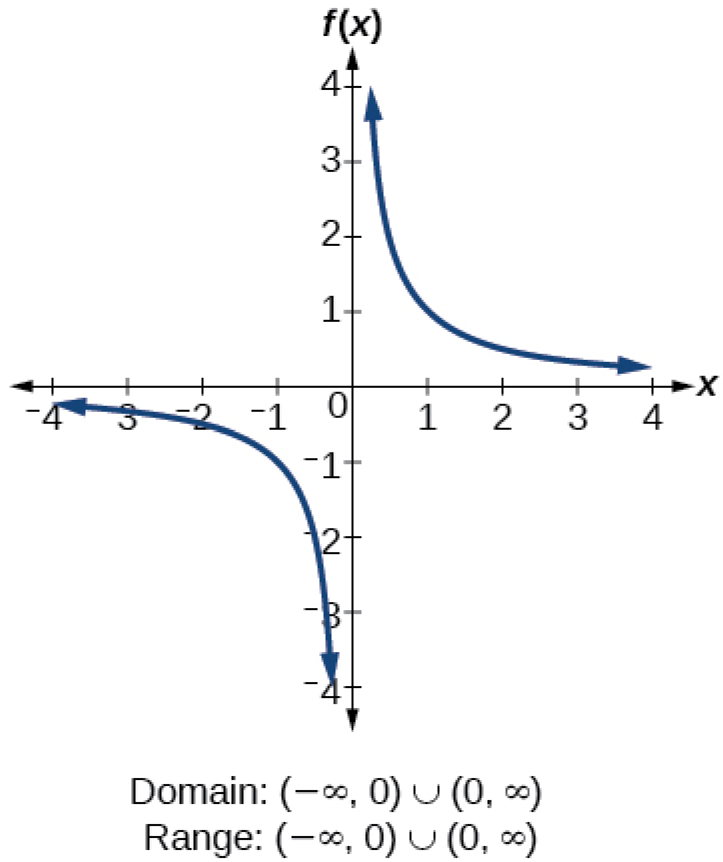

Figure 2-15: For the ***reciprocal function*** *[latex]\text{}f\left(x\right)=\frac{1}{x}\text{}[/latex], we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write [latex]\left\{x|\text{ }x\ne 0\right\}[/latex], the set of all real numbers that are not zero.*

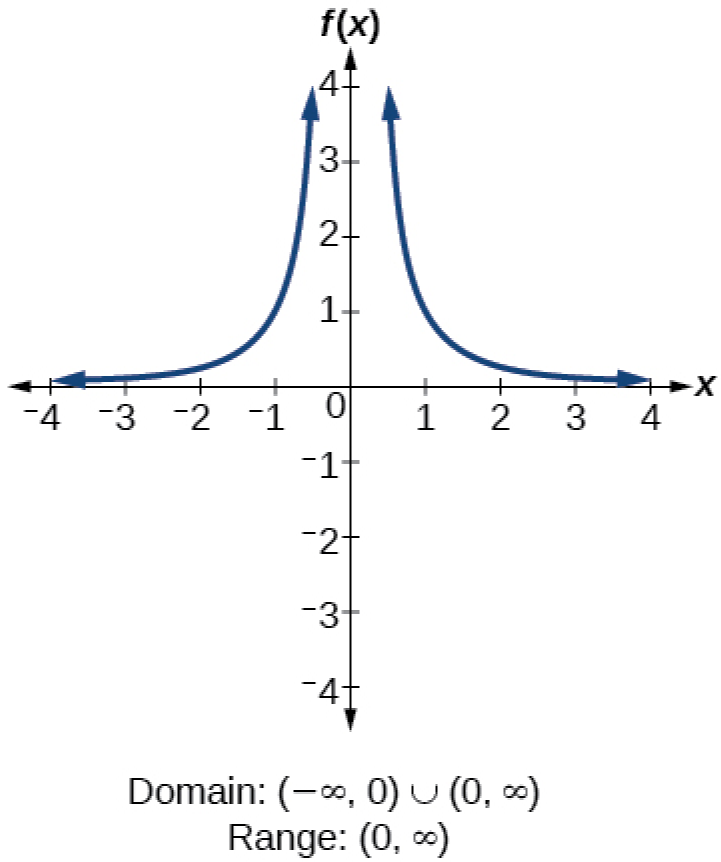

Figure 2-16: For the ***reciprocal squared function*** *[latex]\text{}f\left(x\ right)=\frac{1}{{x}^{2}}[/latex], we cannot divide by [latex]0[/latex], so we must exclude [latex]0[/latex] from the domain. There is also no [latex]x[/latex] that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.*

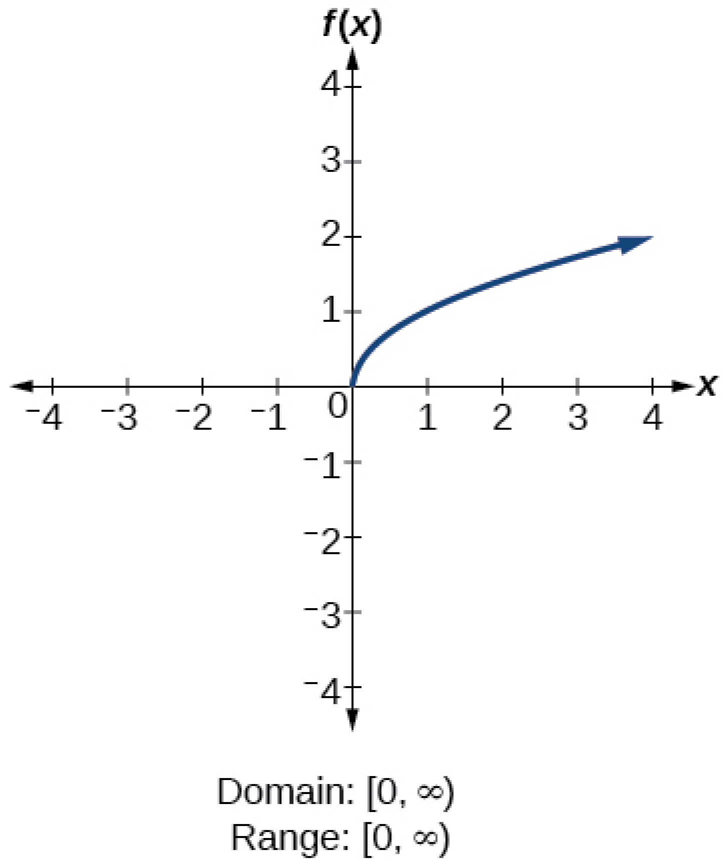

Figure 2-17: For the ***square root function*** *[latex]\text{}f\left(x\right)=\sqrt[]{x}\text{}[/latex], we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number [latex]\text{}x\text{}[/latex] is defined to be positive, even though the square of the negative number [latex]\text{}-\sqrt{x}\text{}[/latex] also gives us [latex]\text{}x[/latex].*

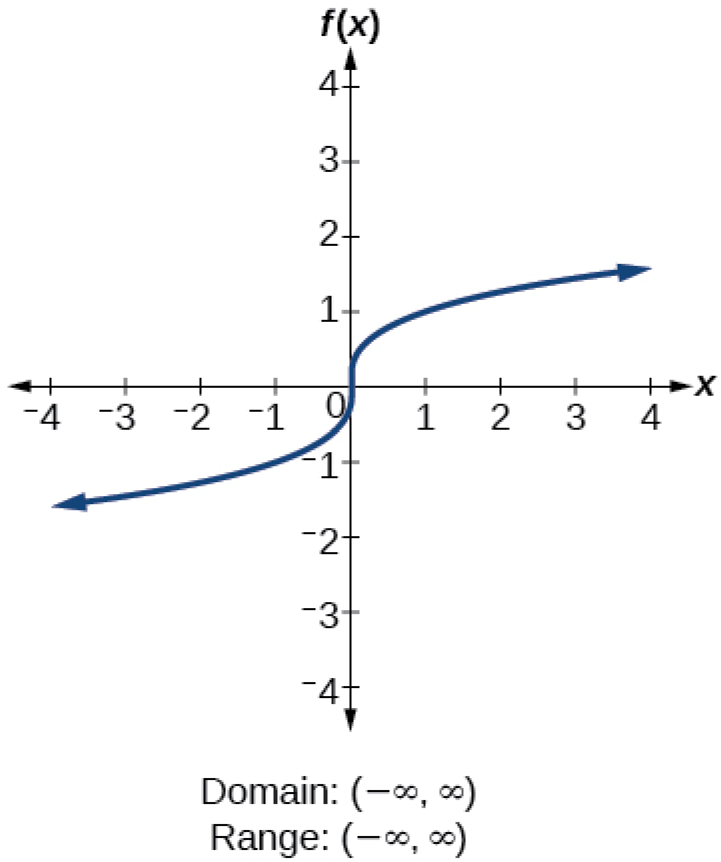

Figure 2-18: For the ***cube root function*** *[latex]\text{}f\left(x\right)=\sqrt[3]{x}\text{}[/latex], the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).*

How To

Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have non-real (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behaviour.

Example 1: Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f\left(x\right)=2x^3-x$.

Solution

Example 2: Finding Domain and Range

Find the domain and range of $f\left(x\right)=\dfrac{2}{x+1}$.

Solution

Example 3: Finding the Domain and Range

Find the domain and range of $f\left(x\right)=2\sqrt{x+4}$.

Analysis

Figure 2-19 represents the function f.

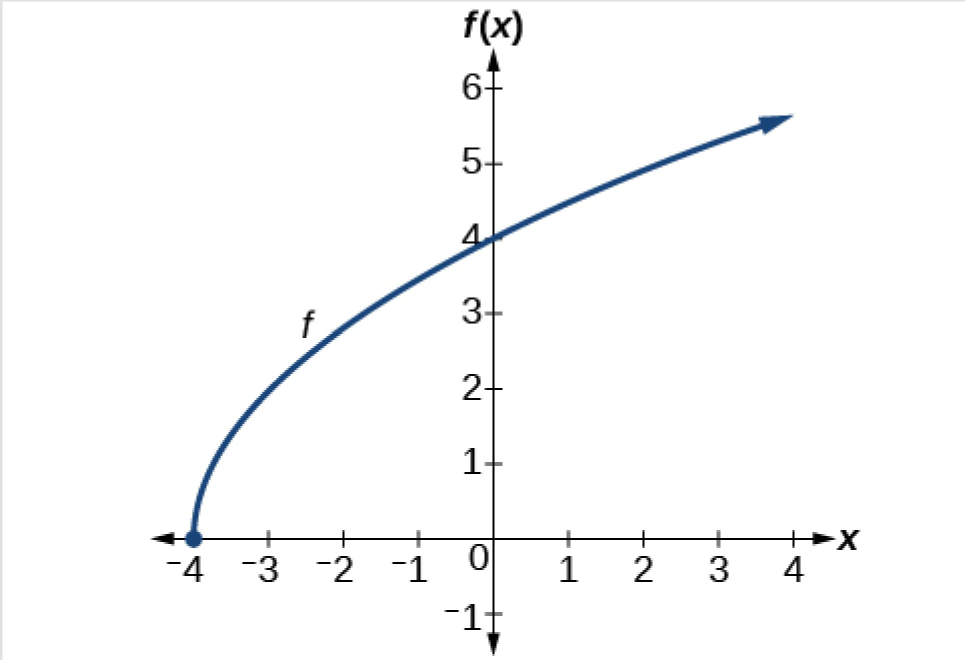

Figure 2-19

Solution

Example 4: Finding the Domain and Range

Find the domain and range of $f\left(x\right)=-\sqrt{2-x}$.

Solution

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2.6 Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f\left(x\right)=\left|x\right|$. With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the magnitude, or modulus, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f\left(x\right)=x\text{ if }x\ge 0$$

If we input a negative value, the output is the opposite of the input.

$$f\left(x\right)=-x\text{ if }x<0$$

Because this requires two different processes or pieces, the absolute value function is an example of a piecewise function. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S\le \$10{,}000$ and $\$1000+0.2\left(S-\$10{,}000\right)$ if $S>\$10{,}000$.

Piecewise Function

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f\left(x\right)\begin{cases} formula\ 1 & if\ x\ is\ in\ domain\ 1\\ formula\ 2 & if\ x\ is\ in\ domain\ 2\\ formula\ 3 & if\ x\ is\ in\ domain\ 3\end{cases}$$

In piecewise notation, the absolute value function is

$$\left|x\right|=\begin{cases} x\ if\ x & \ge 0\\ -x\ if\ x & <0\end{cases}$$

How To

Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example 1: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a function relating the number of people, n, to the cost, C.

Analysis

The function is represented in Figure 2-20. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10,$ but not all piecewise functions have this property.

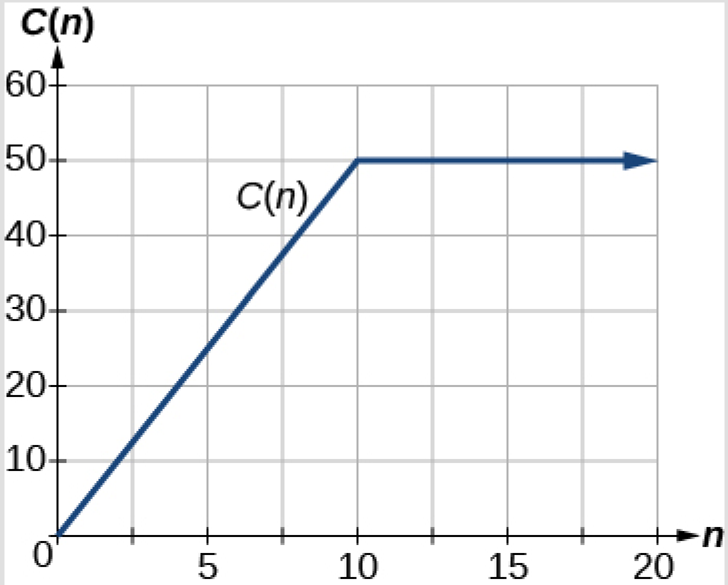

Figure 2-20

Solution

Example 2: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } \; 0 < g < 2 \\ 25 + 10(g-2) & \text{if } \; g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Analysis

The function is represented in Figure 2-21. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

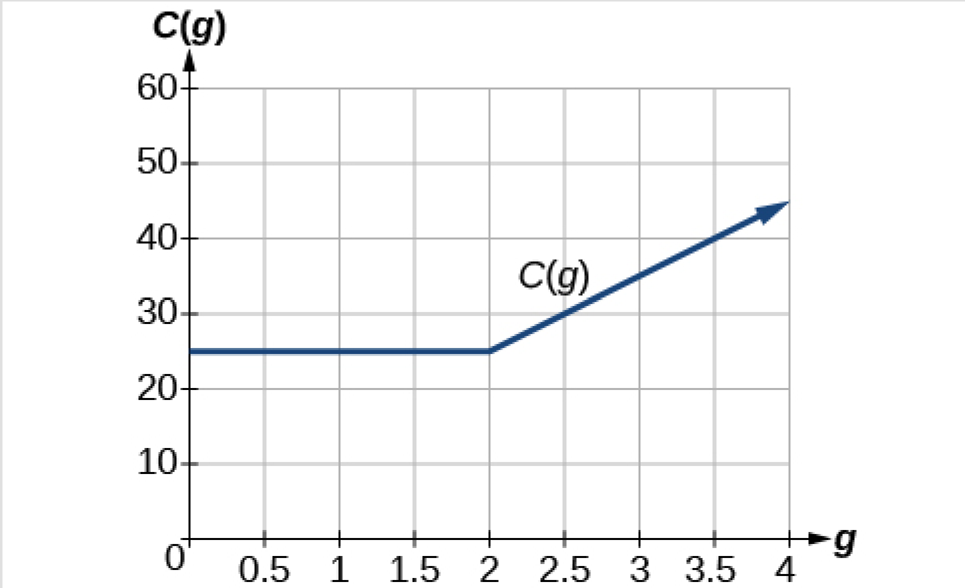

Figure 2-21

[Solution](#)

How To

Given a piecewise function, sketch a graph.

1. Indicate on the *x*-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example 3: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 1 \\ 3 & \text{if} \quad 1 < x \leq 2 \\ x & \text{if} \quad x > 2 \end{cases}$$

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

Solution

Example 4: Graphing a Piecewise Function

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if} \quad x < -1 \\ -2 & \text{if} \quad -1 < x < 4 \\ \sqrt{x} & \text{if} \quad x > 4 \end{cases}$$

Solution

Question & Answer

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

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2.7 Review and Summary

Additional Resources

Access these online video resources for additional instruction and practice with domain and range.

- Domain and Range of Square Root Functions
- Determining Domain and Range
- Find Domain and Range Given the Graph
- Find Domain and Range Given a Table
- Find Domain and Range Given Points on a Coordinate Plane

Key Terms

Interval Notation – a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

Piecewise Function – a function in which more than one formula is used to define the output

Set-builder Notation – a method of describing a set by a rule that all of its members obey; it takes the form $\{x|statement\ about\}\ x$

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs. See 2.2 Finding the Domain of a Function Defined by an Equation.
- The domain of a function can also be determined by identifying the input values of a function written as an equation. See 2.2 Finding the Domain of a Function Defined by an Equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation. See 2.3 Using Notations to Specify Domain and Range.

- For many functions, the domain and range can be determined from a graph. See 2.4 Finding Domain and Range from Graphs.
- An understanding of toolkit functions can be used to find the domain and range of related functions. See 2.5 Finding Domains and Ranges of the Toolkit Functions.
- A piecewise function is described by more than one formula. See 2.6 Graphing Piecewise-Defined Functions.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain. See 2.6 Graphing Piecewise-Defined Functions.

2.8 Practice Questions

Verbal Questions

1. Why does the domain differ for different functions?
2. How do we determine the domain of a function defined by an equation?
3. Explain why the domain of $f\left(x\right)=\sqrt[3]{x}$ is different from the domain of $f\left(x\right)=\sqrt{x}$.
4. When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?
5. How do you graph a piecewise function?

Odd Number Verbal Solutions

Algebraic Questions

For the following exercises, find the domain of each function using interval notation.

6. $f\left(x\right)=-2x\left(x-1\right)\left(x-2\right)$
7. $f\left(x\right)=5-2x^2$
8. $f\left(x\right)=3\sqrt{x-2}$
9. $f\left(x\right)=3-\sqrt{6-2x}$
10. $f\left(x\right)=\sqrt{4-3x}$
11. $f\left(x\right)=\sqrt{x^2+4}$
12. $f\left(x\right)=\sqrt[3]{1-2x}$
13. $f\left(x\right)=\sqrt[3]{x-1}$
14. $f\left(x\right)=\dfrac{9}{x-6}$
15. $f\left(x\right)=\dfrac{3x+1}{4x+2}$

16. $f(x) = \dfrac{\sqrt{x+4}}{x-4}$

17. $f(x) = \dfrac{x-3}{x^2+9x-22}$

18. $f(x) = \dfrac{1}{x^2-x-6}$

19. $f(x) = \dfrac{2x^3-250}{x^2-2x-15}$

20. $\dfrac{5}{\sqrt{x-3}}$

21. $\dfrac{2x+1}{\sqrt{5-x}}$

22. $f(x) = \dfrac{\sqrt{x-4}}{\sqrt{x-6}}$

23. $f(x) = \dfrac{\sqrt{x-6}}{\sqrt{x-4}}$

24. $f(x) = \dfrac{x}{x}$

25. $f(x) = \dfrac{x^2-9x}{x^2-81}$

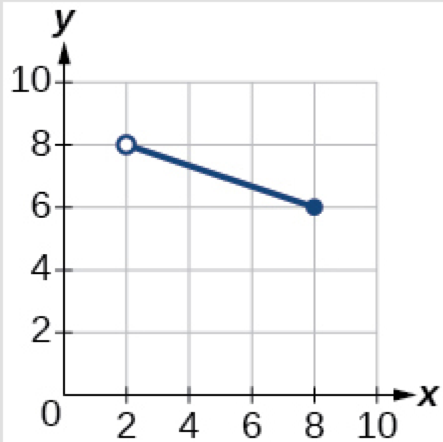
26. Find the domain of the function $f(x) = \sqrt{2x^3-50x}$ by:

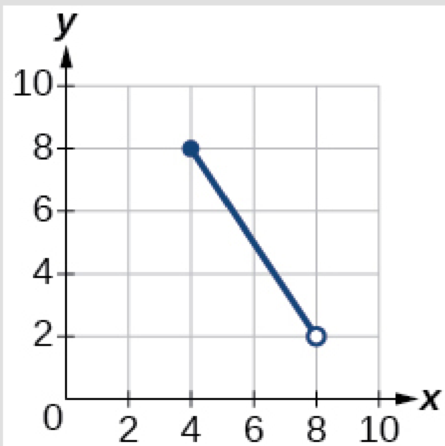
   a. using algebra.
   b. graphing the function in the radicand and determining intervals on the *x*-axis for which the radicand is nonnegative.

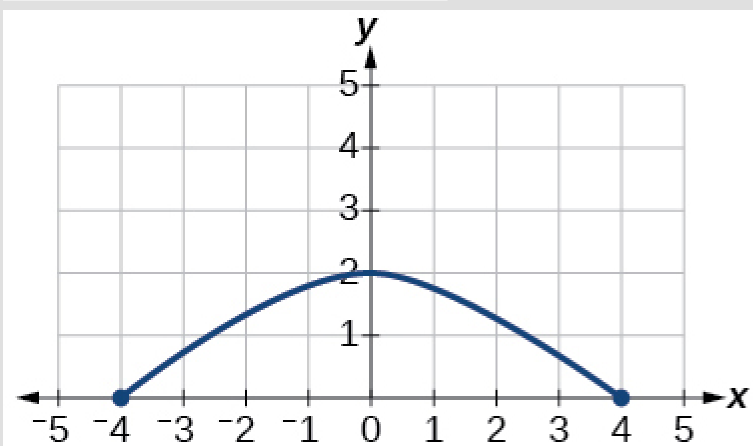
Odd Number Algebraic Solutions

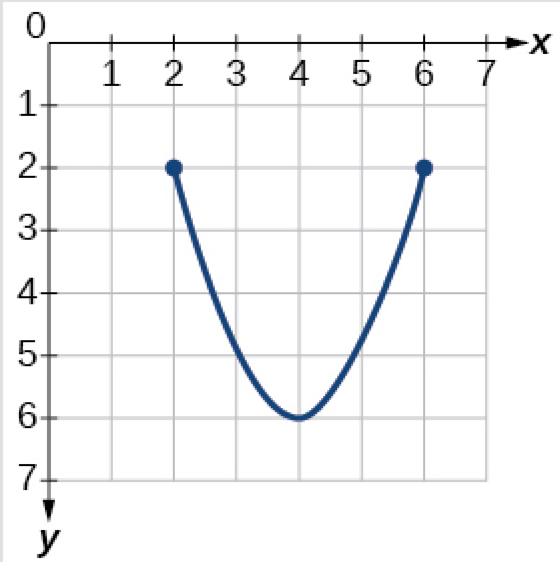
Graphical Questions

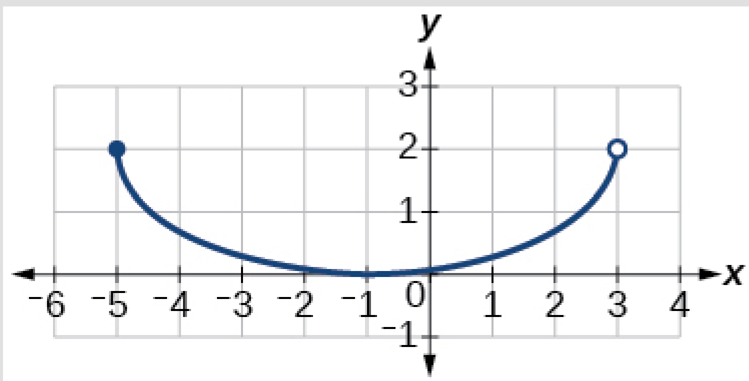
For the following exercises, write the domain and range of each function using interval notation.

27. 

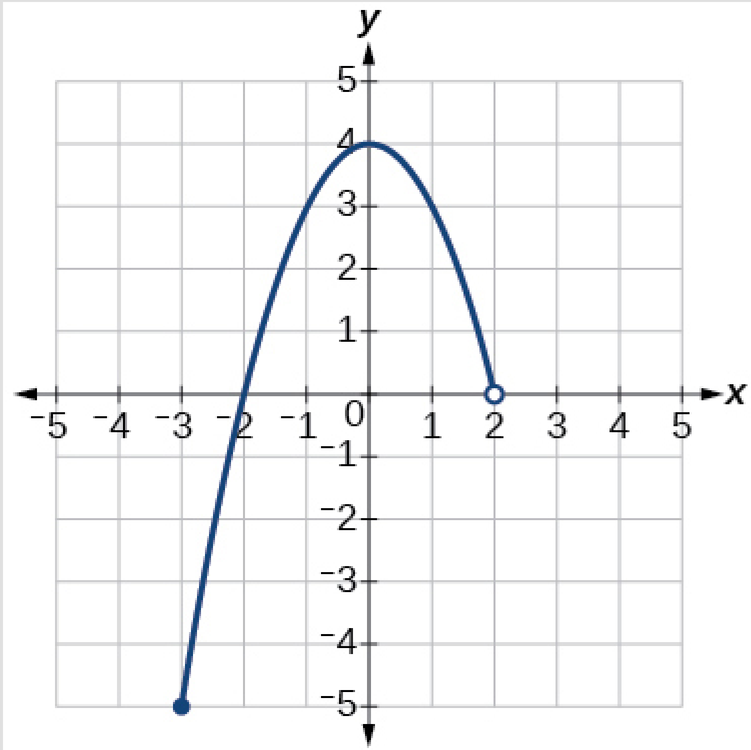
28. 

29. 

30. 

31. 

32.



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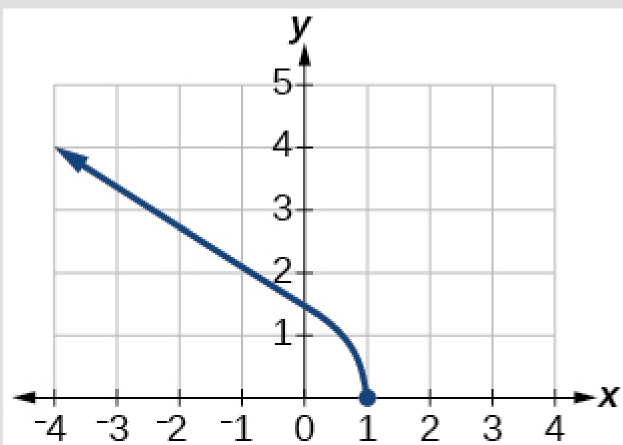
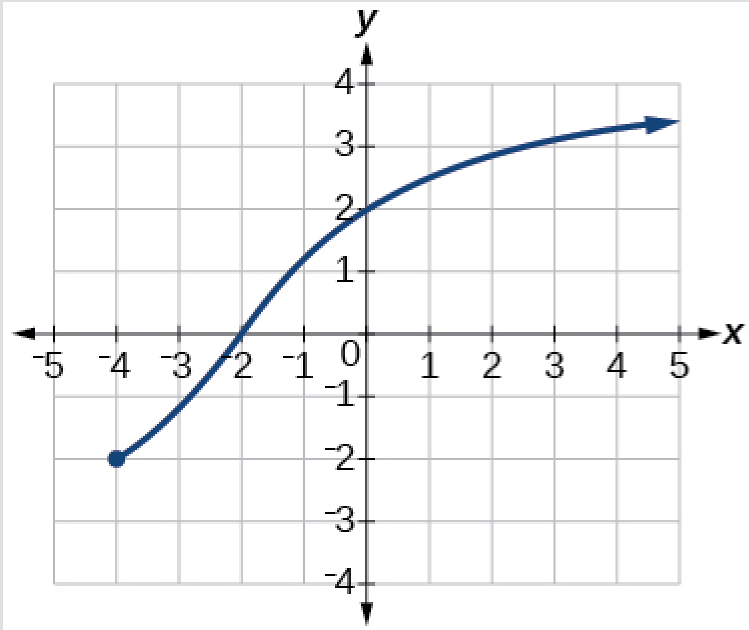


34. 
y
x
4
3
2
1
0
-1
-2
-3
-4
-5
-4
-3
-2
-1
1
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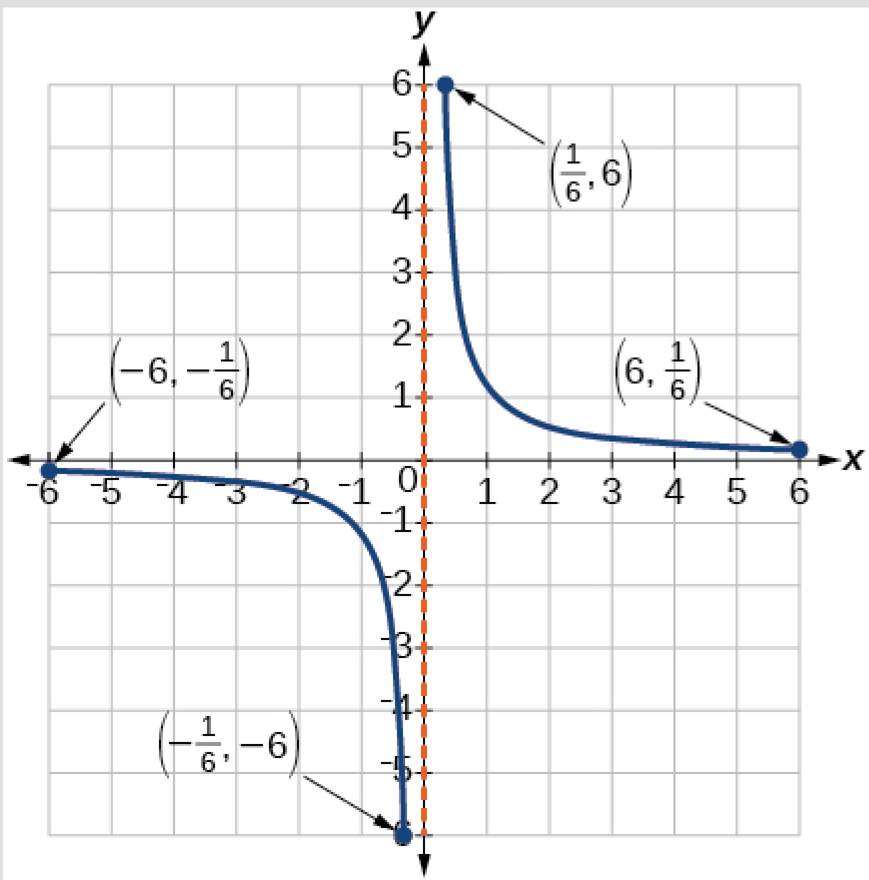
4
5

35. 
y
x
$\left(\frac{1}{6}, 6\right)$
$\left(-6, -\frac{1}{6}\right)$
$\left(6, \frac{1}{6}\right)$
$\left(-\frac{1}{6}, -6\right)$
6
5
4
3
2
1
0
-1
-2
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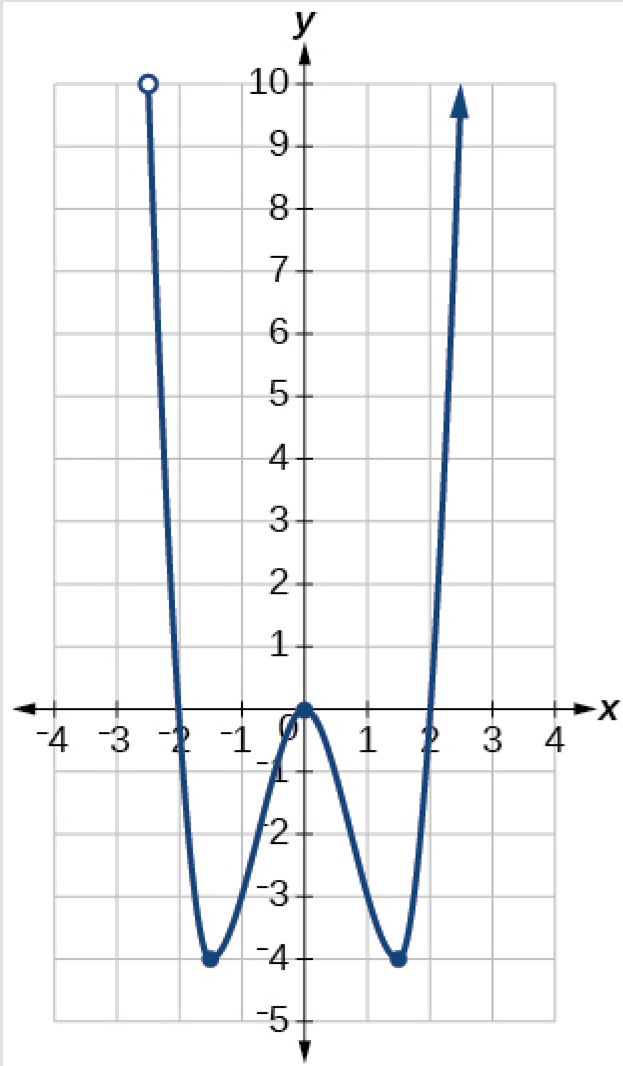
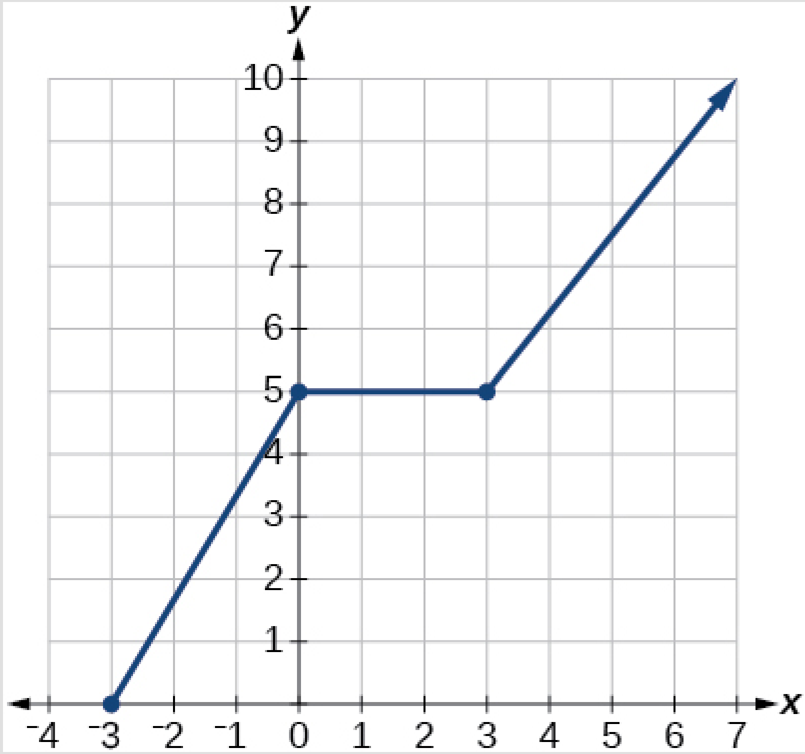
36. 

37. 

For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

38. $f\left(x\right)=\begin{cases}x+1\ if\ x\ <-2\\ -2x-3\ if\ x\geq-2\end{cases}$

39. $f\left(x\right)=\begin{cases}2x-1\ if\ x\ <1\\ 1+x\ if\ x\geq1\end{cases}$

40. $f\left(x\right)=\begin{cases}x+1\ if\ x\ <0\\ x-1\ if\ x>0\end{cases}$

41. $f\left(x\right)=\begin{cases}3\ if\ x\ <0\\ \sqrt{x}\ if\ x\geq0\end{cases}$

42. $f\left(x\right)=\begin{cases}x^2\ if\ x<0\\ 1-x\ if\ x>0\end{cases}$

43. $f\left(x\right)=\begin{cases}x^2\ if\ x\ <0\\ x+2\ if\ x\geq0\end{cases}$

44. $f\left(x\right)=\begin{cases}x+1\ if\ x\ <1\\ x^3\ if\ x\geq1\end{cases}$

45. $f\left(x\right)=\begin{cases}\left|x\right|\ if\ x<2\\ 1\ if\ x\geq2\end{cases}$

Odd Number Graphical Solutions

Numeric Questions

For the following exercises, given each function f, evaluate $f(-3), f(-2), f(-1)$, and $f(0)$.

46. $f(x)=\begin{cases} x+1 \; if \; x < -2 \\ -2x-3 \; if \; x \geq -2 \end{cases}$

47. $f(x)=\begin{cases} 1 \; if \; x \leq -3 \\ 0 \; if \; x > -3 \end{cases}$

48. $f(x)=\begin{cases} -2x^2+3 \; if \; x \leq -1 \\ 5x-7 \; if \; x > -1 \end{cases}$

For the following exercises, given each function f, evaluate $f(-1), f(0), f(2)$, and $f(4)$.

49. $f(x)=\begin{cases} 7x+3 \; if \; x < 0 \\ 7x+6 \; if \; x \geq 0 \end{cases}$

50. $f(x)=\begin{cases} x^2-2 \; if \; x < 2 \\ 4+|x-5| \; if \; x \geq 2 \end{cases}$

51. $f(x)=\begin{cases} 5x \; if \; x < 0 \\ 3 \; if \; 0 \leq x \geq 3 \\ x^2 \; if \; x > 3 \end{cases}$

For the following exercises, write the domain for the piecewise function in interval notation.

52. $f(x)=\begin{cases} x+1 \; if \; x < -2 \\ -2x-3 \; if \; x \geq -2 \end{cases}$

53. $f(x)=\begin{cases} x^2-2 \; if \; x < 1 \\ -x^2+2 \; if \; x > 1 \end{cases}$

54. $f(x)=\begin{cases} 2x-3 \; if \; x < 0 \\ -3x^2 \; if \; x \geq 2 \end{cases}$

Odd Number Numeric Solutions

Technology Questions

55. Graph $y = \frac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1,\ 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.
56. Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1,\ 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Odd Number Technology Solutions

Extension Questions

57. Suppose the range of a function f is $[-5,\ 8]$. What is the range of $|f(x)|$?
58. Create a function in which the range is all nonnegative real numbers.
59. Create a function in which the domain is $x > 2$.

Odd Number Extension Solutions

Real-World Applications Questions

60. The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?
61. The cost in dollars of making x items is given by the function $C(x) = 10x + 500$.

a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.
b. What is the cost of making 25 items?
c. Suppose the maximum cost allowed is $1500. What are the domain and range of the cost function, $C(x)$?

Even Number Real-World Applications Solutions

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2.9 Chapter 2 Example Solutions

2.2 Example Solutions

Example 1: Finding the Domain of a Function as a Set of Ordered Pairs

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

Example 2: Find the domain of the function

$$\{-5, 0, 5, 10, 15\}$$

Example 3: Finding the Domain of a Function

1. The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

   In interval form, the domain of $f\ is\ (-\infty,\ \infty)$.

2. $(\ -\infty\ ,\ \infty\ )$

Example 4: Finding the Domain of a Function Involving a Denominator

1. When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$
\begin{aligned}
2 - x &= 0 \\
-x &= -2 \\
x &= 2
\end{aligned}
$$

Now, we will exclude 2 from the domain. The answers are all real numbers where < mi > x < /mi >< mo ><< /mo >< mn > 2 < /mn > or < mi > x < /mi >< mo >>< /mo >< mn > 2. < /mn >

We can use a symbol known as the union, $\bigcup$ to combine the two sets. In interval notation, we write the solution: $(-\infty, 2) \cup (2, \infty)$.

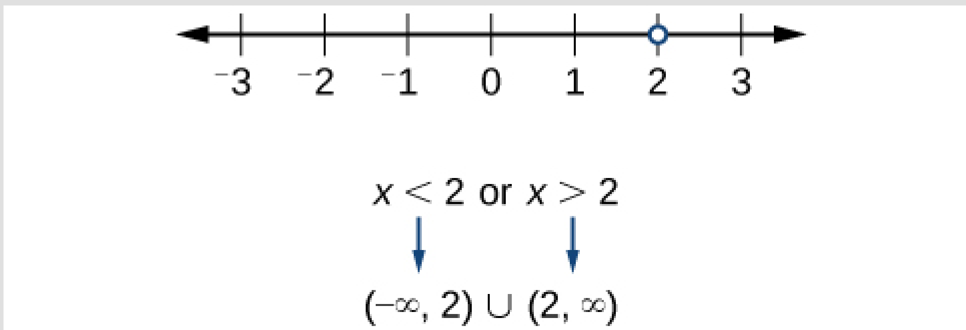


Figure 2-22

In interval form, the domain of $f\ is\ (-\infty, 2) \cup (2, \infty)$

2. $\left(-\infty, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$

Example 5: Finding the Domain of a Function with an Even Root

1. When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

   Set the radicand greater than or equal to zero and solve for x.

   $$
   \begin{aligned}
   &7 - x \geq 0 \\
   &-x \geq -7 \\
   &x \leq 7
   \end{aligned}
   $$

   Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to $7\ or\ (-\infty, 7]$.

2. $\left[-\dfrac{5}{2}, \infty\right)$

2.3 Example Solutions

Example 1: Describing Sets on the Real-Number Line

1. To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality	$1 \leq x \leq 3 \text{ or } x > 5$
Set-builder notation	$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$
Interval notation	$[1, 3] \cup (5, \infty)$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

a. values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;
b. $\{x \mid x \leq -2 \text{ or } -1 \leq x < 3\}$;
c. $(-\infty, -2] \cup [-1, 3)$

2.4 Example Solutions

Example 1: Finding Domain and Range from a Graph

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0]$. See Figure 2-23.

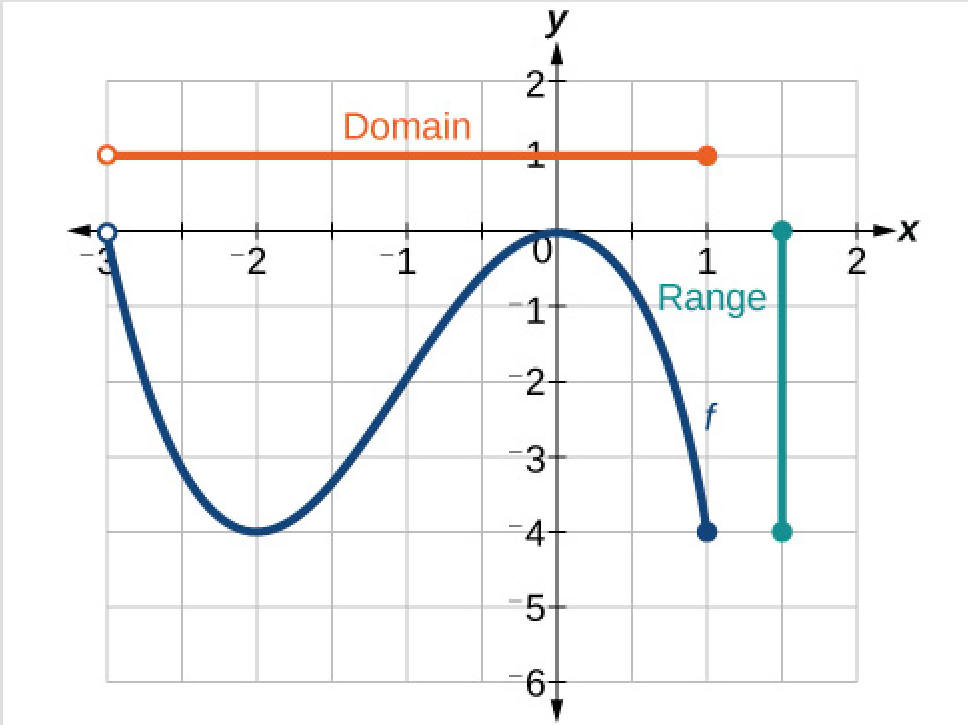


Figure 2-23

Example 2: Finding Domain and Range from a Graph

1. The input quantity along the horizontal axis is "years," which we represent with the variable t for time. The output quantity is "thousands of barrels of oil per day," which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \le t \le 2008$ and the range as approximately $180 \le b \le 2010$

   In interval notation, the domain is [1973, 2008], and the range is about [180, 2010]. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.
2. domain =[1950, 2002] range = [47,000,000, 89,000,000]

2.5 Example Solutions

Example 1: Finding the Domain and Range Using Toolkit Functions

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 2: Finding the Domain and Range

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 3: Finding the Domain and Range

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$$x + 4 \geq 0 \text{ when } x \geq -4$$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Example 4: Finding the Domain and Range

domain: $(-\infty, 2]$; range: $(-\infty, 0]$

2.6 Example Solutions

Example 1: Writing a Piecewise Function

Two different formulas will be needed. For *n*-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } 0 < n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Example 2: Working with a Piecewise Function

To find the cost of using 1.5 gigabytes of data, $C\left(1.5\right)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C\left(1.5\right) = \$25$$

To find the cost of using 4 gigabytes of data, $C\left(4\right)$, we see that our input of 4 is greater than 2, so we use the second formula.

$$C\left(4\right) = 25 + 10\left(4 - 2\right) = \$45$$

Example 3: Graphing a Piecewise Function

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 2-24 shows the three components of the piecewise function graphed on separate coordinate systems.

(a) \text{}f\left(x\right)={x}^{2}\text{ if }x\le 1;\text{} (b) $f\left(x\right) = 3 \text{ if } 1 < x \leq 2;$ (c) \text{}f\left(x\right)=x\text{ if }x>2

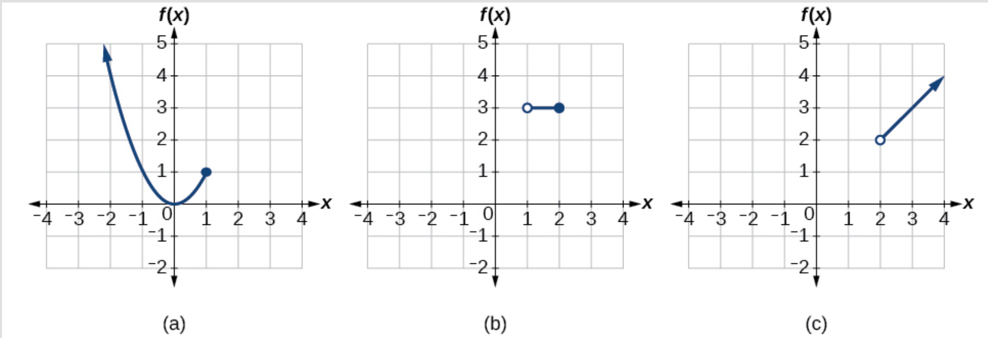


Figure 2-24

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 2-25.

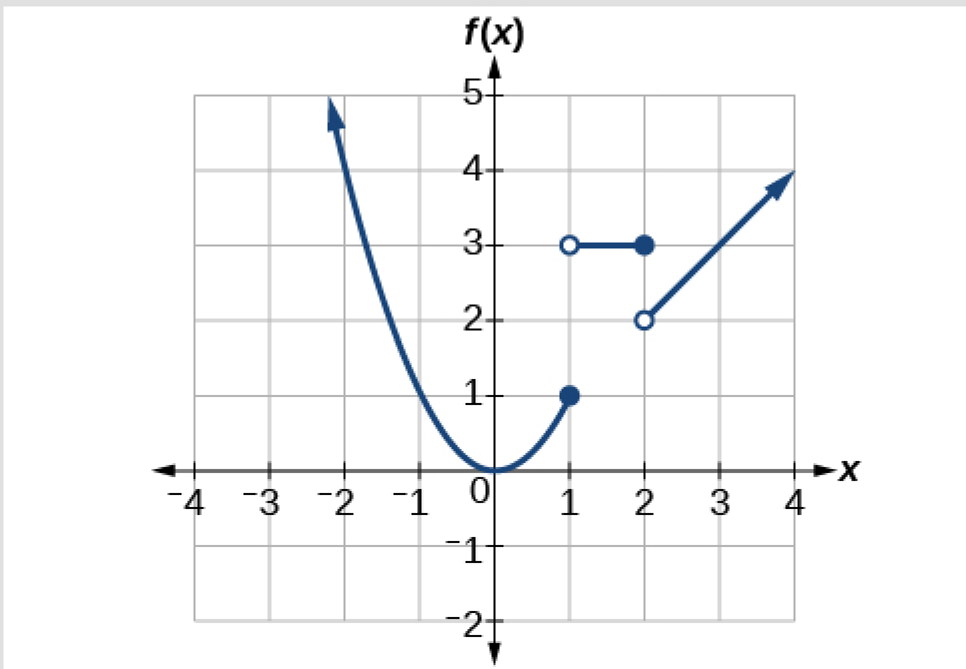
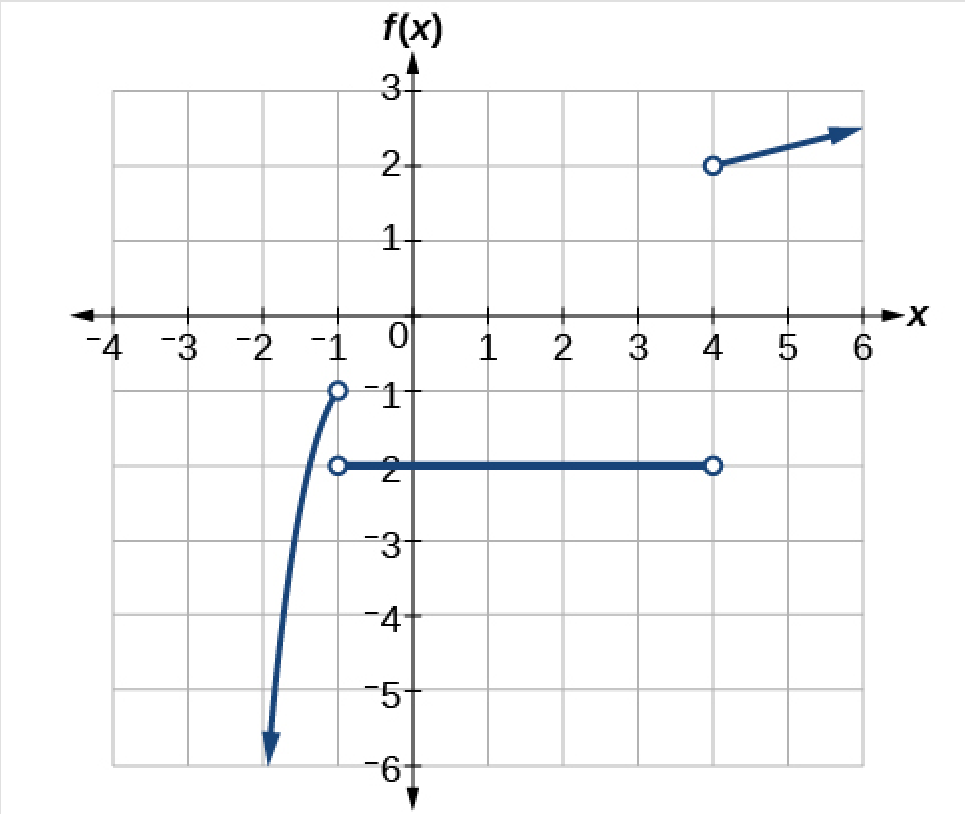


Figure 2-25

Example 4: Graphing a Piecewise Function



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2.10 Practice Question Solutions

Verbal Question Solutions

1. The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

3. There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$.

5. Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function.

Algebraic Question Solutions

7. $(-\infty, \infty)$

9. $(-\infty, 3]$

11. $(-\infty, \infty)$

13. $(-\infty, \infty)$

15. $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$

17. $(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$

19. $(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

21. $(-\infty, 5)$

23. $[6, \infty)$

25. $(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$

Graphical Solution Answers

27. domain: $(2, 8]$; range: $[6, 8)$

29. domain: $[-4, 4]$; range: $[0, 2]$

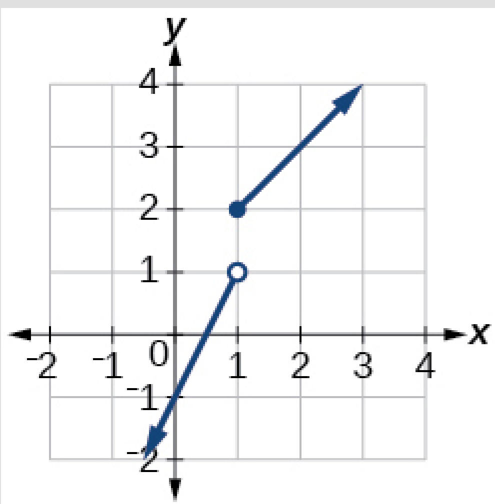
31. domain: $[-5,\ 3)$; range: $[0, 2]$

33. domain: $(-\infty, 1]$; range:$[0, \infty)$

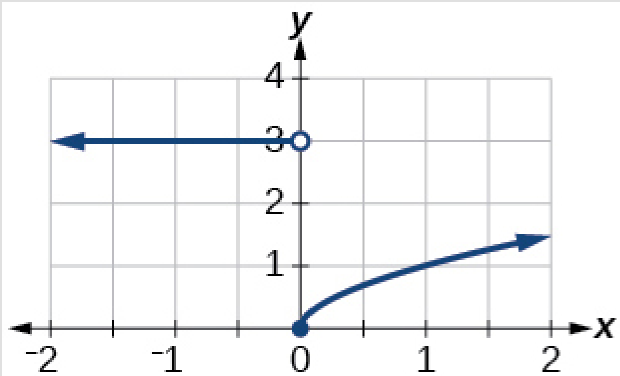
35. domain: $\left[-6, -\frac{1}{6}\right] \cup \left[\frac{1}{6}, 6\right]$; range: $\left[-6, -\frac{1}{6}\right] \cup \left[\frac{1}{6}, 6\right]$

37. domain: $[-3,\ \infty)$; range: $[0, \infty)$

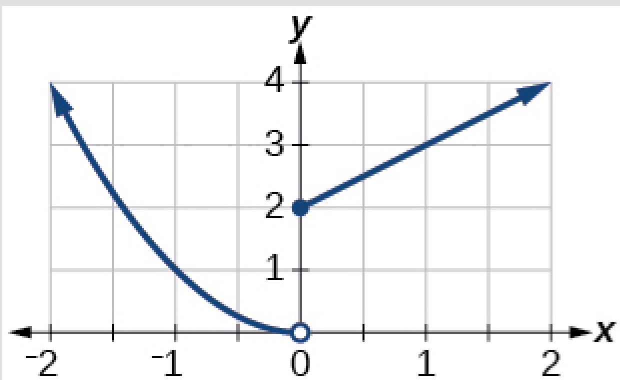
39. domain: $(-\infty, \infty)$



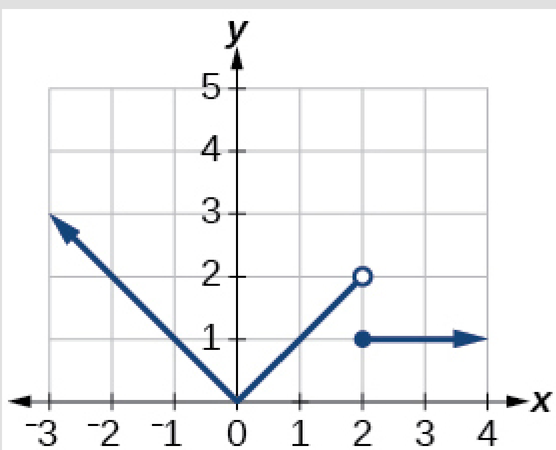
41. domain: $(-\infty, \infty)$



43. domain: $(-\infty, \infty)$



45. domain: $(-\infty, \infty)$



Numeric Question Solutions

47. $f(-3) = 1; \quad f(-2) = 0; \quad f(-1) = 0; \quad f(0) = 0$

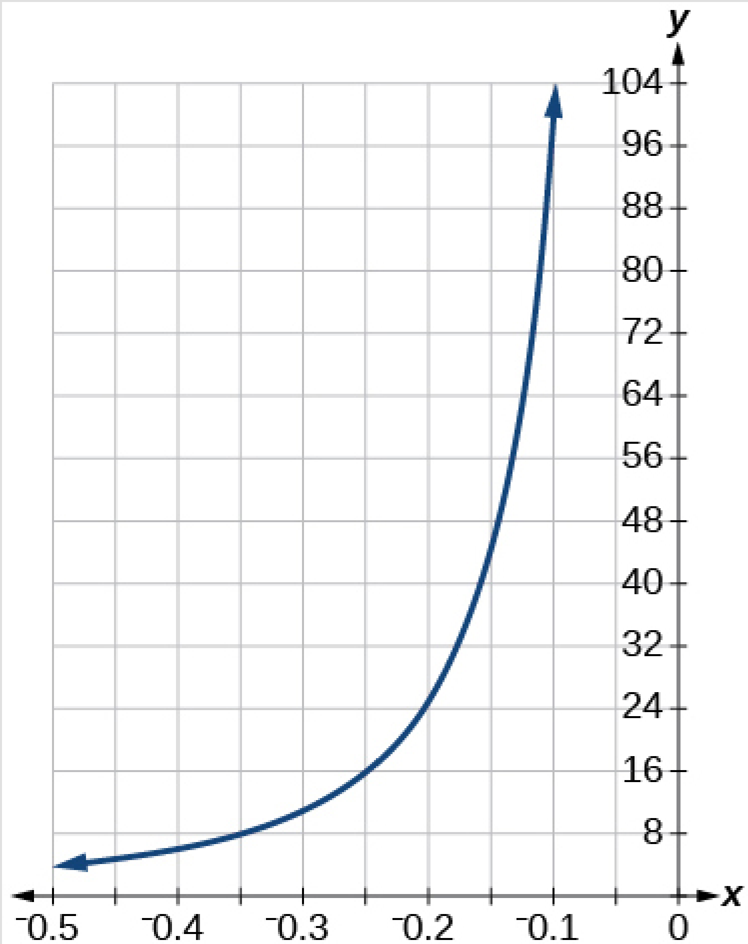
49. $f(-1) = -4; \quad f(0) = 6; \quad f(2) = 20; \quad f(4) = 34$

51. $f(-1) = -5; \quad f(0) = 3; \quad f(2) = 3; \quad f(4) = 16$

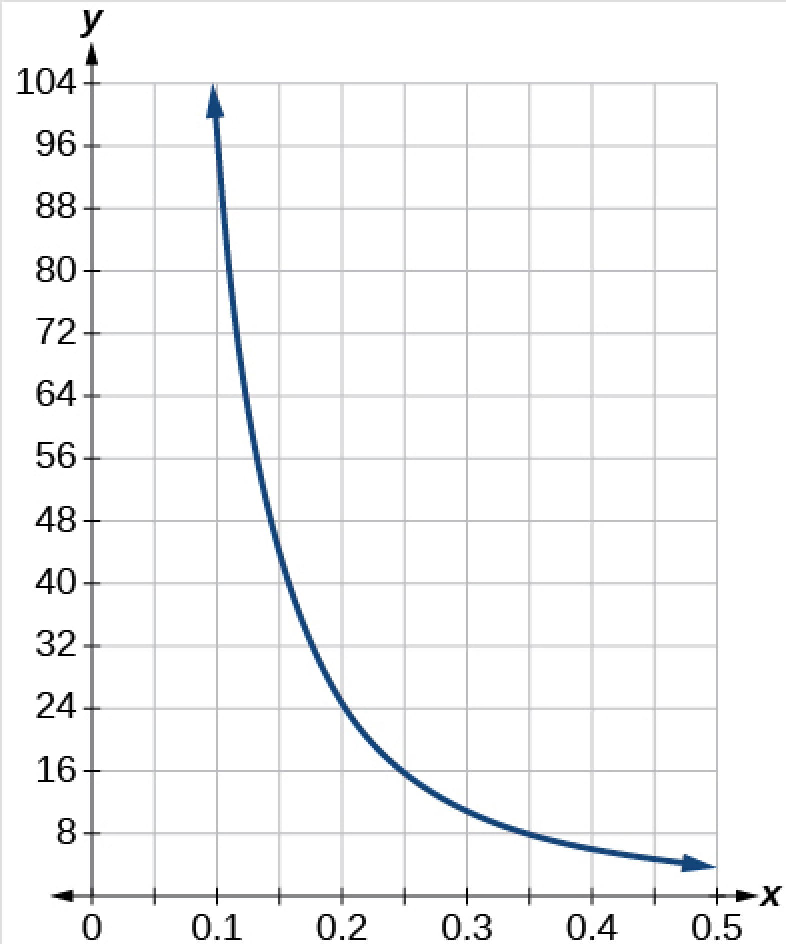
53. domain: $(-\infty, 1) \cup (1, \infty)$

Technology Question Solutions

55.



window: $[-0.5, -0.1]$; range: $[4,\ 100]$



window: $[0.1,\ 0.5]$; range: $[4,\ 100]$

Extension Question Solutions

57. $[0,\ 8]$

59. Many answers. One function is $f\left(x\right)=\dfrac{1}{\sqrt{x-2}}$.

Real-World Applications Solutions

60. The domain is $[0, 6]$; it takes 6 seconds for the projectile to leave the ground and return to the ground.

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CHAPTER 3: TRANSFORMATION OF FUNCTIONS

3.1 Transformation of Functions

Learning Objectives

In this section, you will:

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

Figure 3-1: (credit: "Misko"/Flickr)

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

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3.2 Graphing Functions Using Vertical and Horizontal Shifts

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g\left(x\right)=f\left(x\right)+k,$ the function $f\left(x\right)$ is shifted vertically k units. See Figure 3-2 for an example.

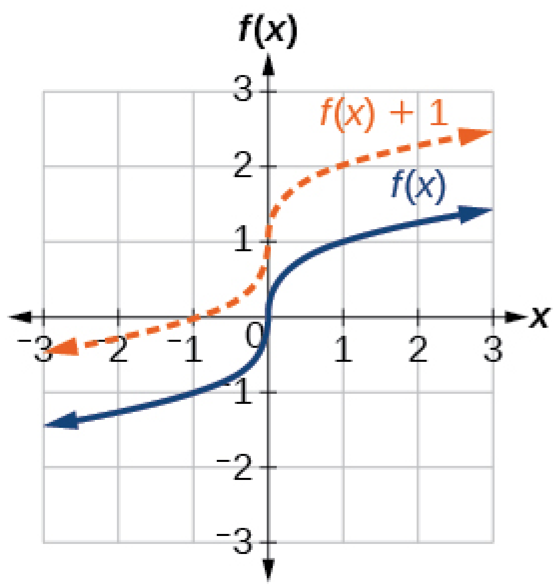


Figure 3-2: Vertical shift by [latex]\text{}k=1\text{}[/latex] of the cube root function [latex]\text{}f\left(x\right)=\sqrt[3]{x}[/latex].

To help you visualize the concept of a vertical shift, consider that $y=f\left(x\right)$. Therefore, $f\left(x\right)+k$ is equivalent to $y+k$. Every unit of y is replaced by $y+k,$ so the y- value increases or decreases depending on the value of k. The result is a shift upward or downward.

Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a vertical shift of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 3-3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

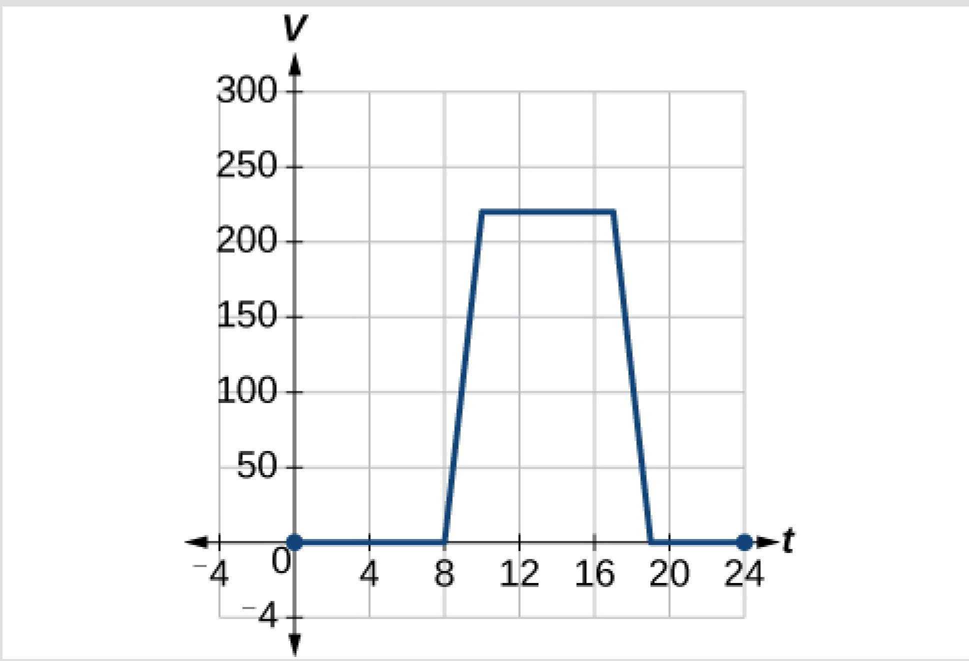


Figure 3-3

Solution

How To

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 1. Create a table for the function $g(x) = f(x) - 3$.

Table 1

x	2	4	6	8
$f(x)$	1	3	7	11

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Solution

Example 3: Vertical Shift

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Solution

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 3-4.

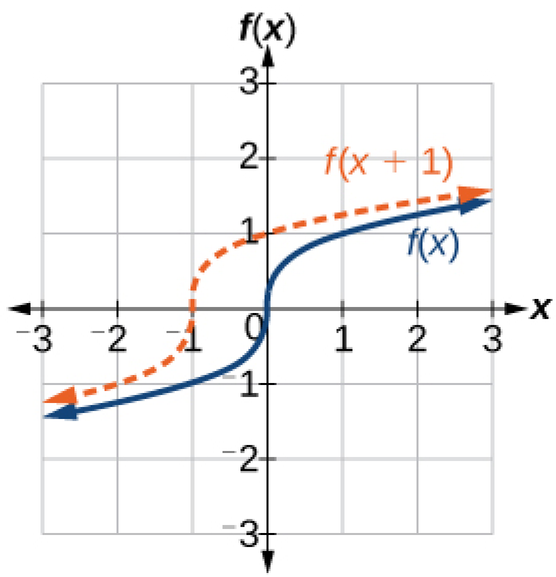


Figure 3-4: Horizontal shift of the function [latex]\text{}f\left(x\right)=\sqrt[3]{x}\text{}[/latex]. Note that [latex]\text{}h=+1\text{}[/latex] shifts the graph to the left, that is, towards negative values of [latex]\text{}x[/latex].

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Horizontal Shift

Given a function f, a new function $g(x) = f(x - h)$, where h is a constant, is a horizontal shift of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 4: Adding a Constant to an Input

Returning to our building airflow example from Example 1 (Figure 3-5 below), suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

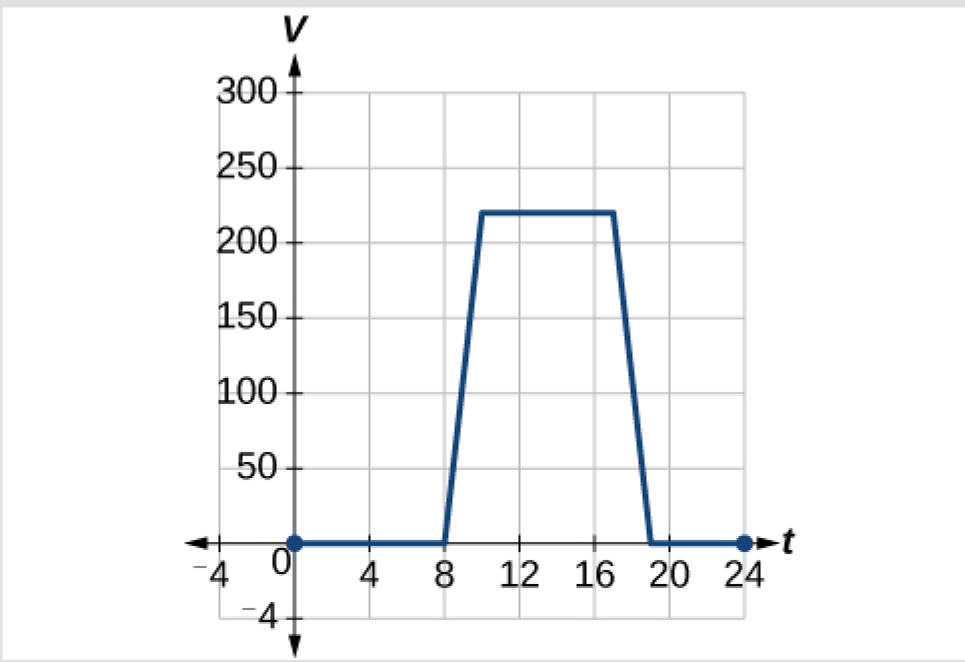


Figure 3-5

Analysis

Note that $V\left(t+2\right)$ has the effect of shifting the graph to the *left*.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F\left(t\right)$ uses the same outputs as $V\left(t\right)$, but matches those outputs to inputs 2 hours earlier than those of $V\left(t\right)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F: F\left(t\right)=V\left(t+2\right)$.

Solution

How To

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 5: Shifting a Tabular Function Horizontally

A function $f\left(x\right)$ is given in Table 2. Create a table for the function $g\left(x\right)=f\left(x-3\right)$.

Table 2

x	2	4	6	8
$f\left(x\right)$	1	3	7	11

Analysis

Figure 3-6 represents both of the functions. We can see the horizontal shift in each point.

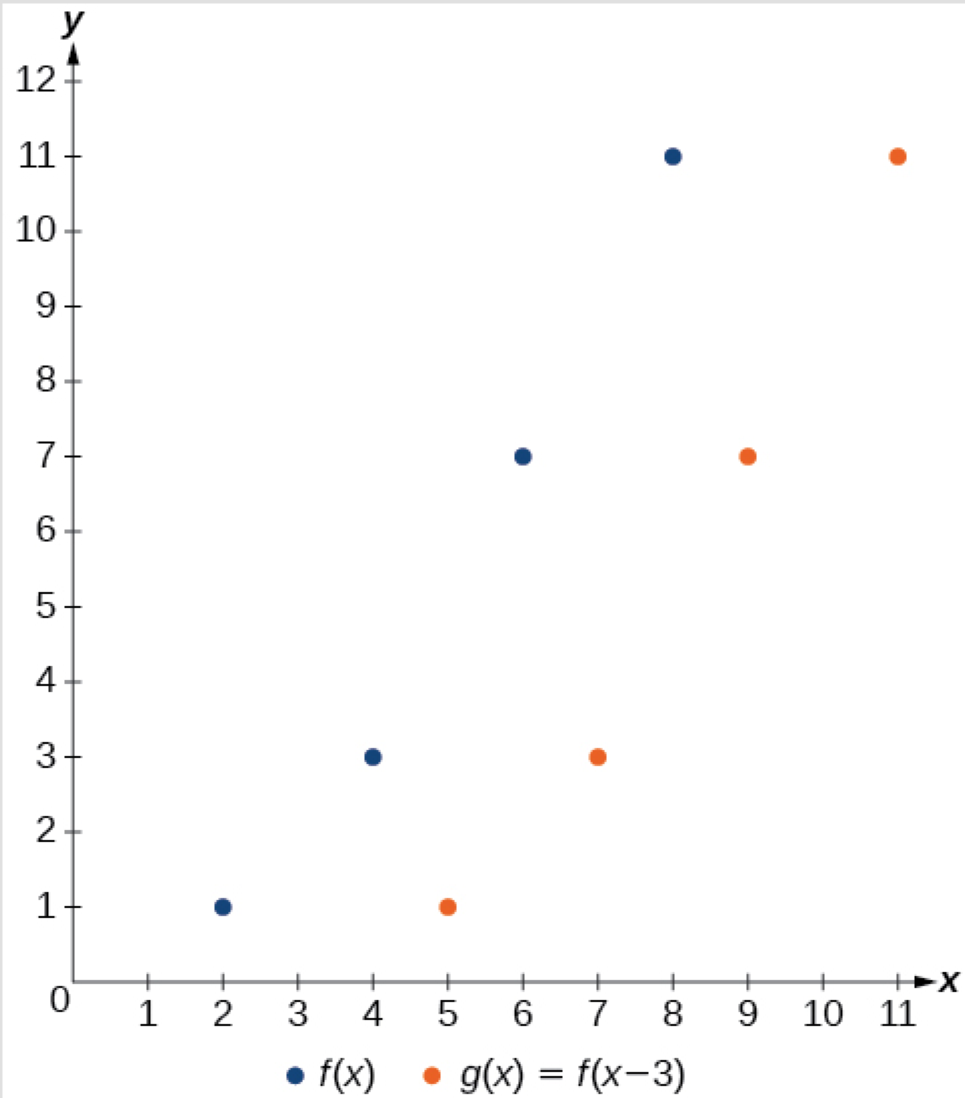


Figure 3-6

[Solution](#)

Example 6: Identifying a Horizontal Shift of a Toolkit Function

Figure 3-7 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

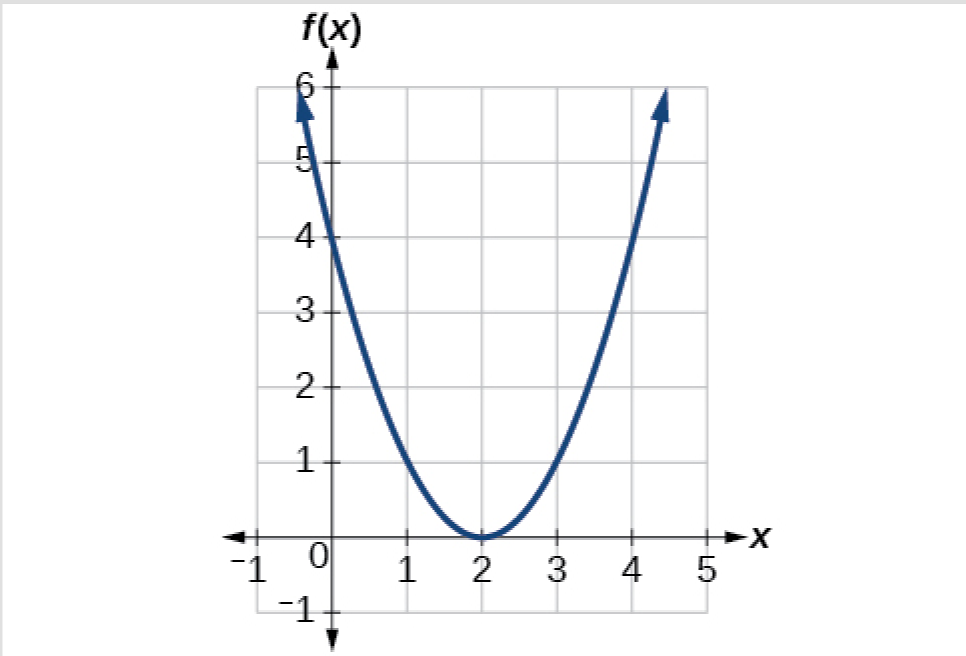


Figure 3-7

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x-2) = f(0) = 0$. For this to work, we will need to *subtract* 2 units from our input values.

Solution

Example 7: Interpreting Horizontal versus Vertical Shifts

1. The function $G\left(m\right)$ gives the number of gallons of gas required to drive m miles. Interpret $G\left(m\right)+10$ and $G\left(m+10\right)$.
2. Given the function $f\left(x\right)=\sqrt{x}$, graph the original function $f\left(x\right)$ and the transformation $g\left(x\right)=f\left(x+2\right)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Solution

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $\left(y\text{-}\right)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $\left(x\text{-}\right)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* right or left.

How To

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Exercise 8: Graphing Combined Vertical and Horizontal Shifts

1. Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3$.
2. Given $f(x) = |x|$, sketch a graph of $h(x) = f(x-2) + 4$.

Solution

Example 9: Identifying Combined Vertical and Horizontal Shifts

1. Write a formula for the graph shown in Figure 3-8, which is a transformation of the toolkit square root function.

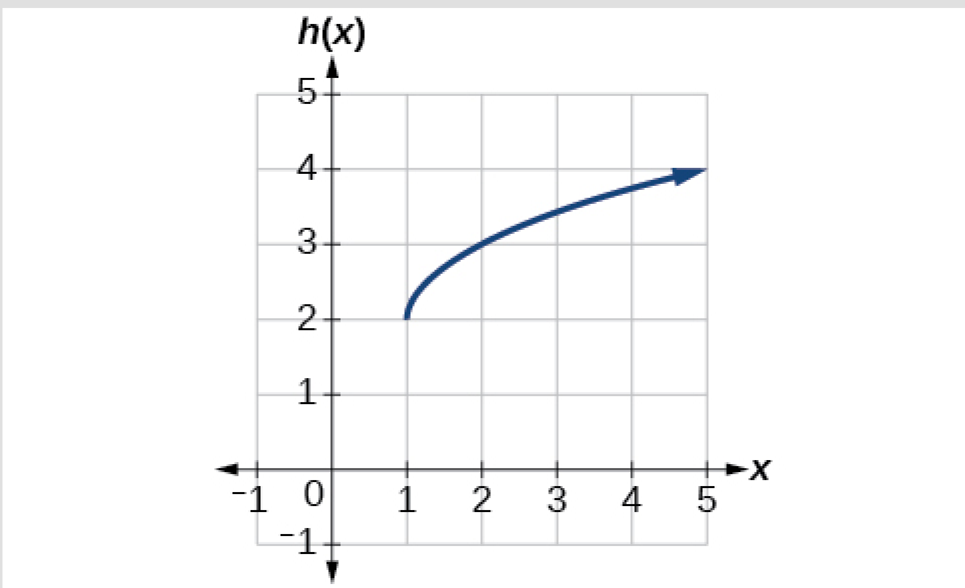


Figure 3-8

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

2. Write a formula for a transformation of the toolkit reciprocal function $f\left(x\right) = \dfrac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Solution

Access for free at https://openstax.org/books/precalculus/pages/1-introduction-to-functions

3.3 Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the *x*– or *y*-axis. A **vertical reflection** reflects a graph vertically across the *x*-axis, while a **horizontal reflection** reflects a graph horizontally across the *y*-axis. The reflections are shown in Figure 3-9.

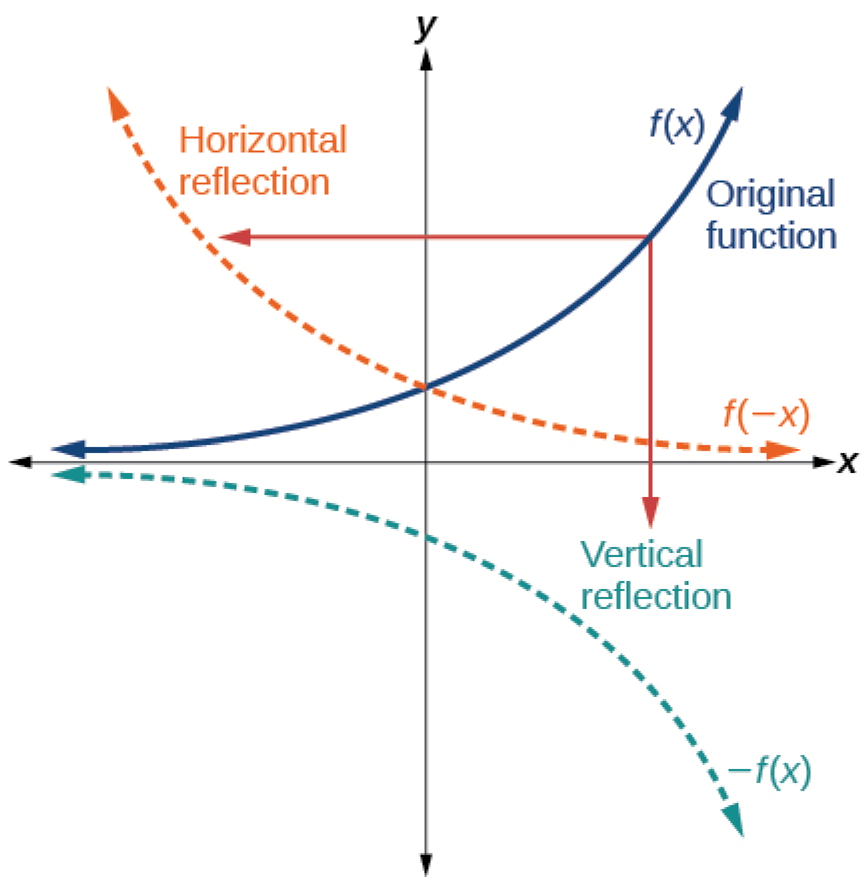


Figure 3-9: Vertical and horizontal reflections of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the *x*-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the *y*-axis.

Reflections

Given a function $f\left(x\right)$, a new function $g\left(x\right)=-f\left(x\right)$ is a vertical reflection of the function $f\left(x\right)$, sometimes called a reflection about (or over, or through) the *x*-axis.

Given a function $f\left(x\right)$, a new function $g\left(x\right)=f\left(-x\right)$ is a horizontal reflection of the function $f\left(x\right)$, sometimes called a reflection about the *y*-axis.

How To

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the *x*-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the *y*-axis.

Example 1: Reflecting a Graph Horizontally and Vertically

1. Reflect the graph of $s\left(t\right)=\sqrt{t}$ (a) vertically and (b) horizontally.
2. Reflect the graph of $f\left(x\right)=\left|x-1\right|$ (a) vertically and (b) horizontally.

Solution

Example 2: Reflecting a Tabular Function Horizontally and Vertically

1. A function $f(x)$ is given as Table 3. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 3

x	2	4	6	8
$f(x)$	1	3	7	11

2. A function $f(x)$ is given as Table 4. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 4

x	–2	0	2	4
$f(x)$	5	10	15	20

Solution

Example 3: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1,$ where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 3-10. Sketch a graph of $k(t)$.

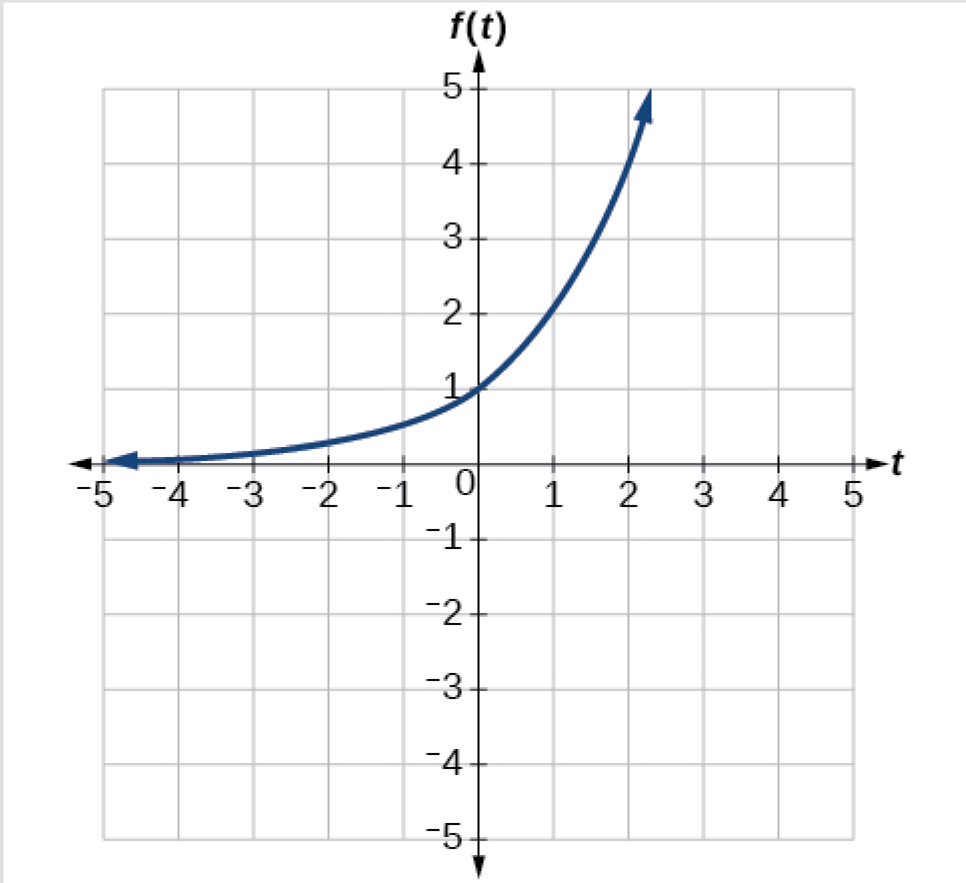


Figure 3-10

Analysis

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

Solution

Example 4: Graphing Functions

Given the toolkit function $f\left(x\right)=x^2$, graph $g\left(x\right)=-f\left(x\right)$ and $h\left(x\right)=f\left(-x\right)$. Take note of any surprising behaviour for these functions.

Solution

Access for free at https://openstax.org/books/precalculus/pages/1-introduction-to-functions

3.4 Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f\left(x\right)=x^2$ or $f\left(x\right)=\left|x\right|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f\left(x\right)=x^3$ or $f\left(x\right)=\dfrac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in Figure 3-11.

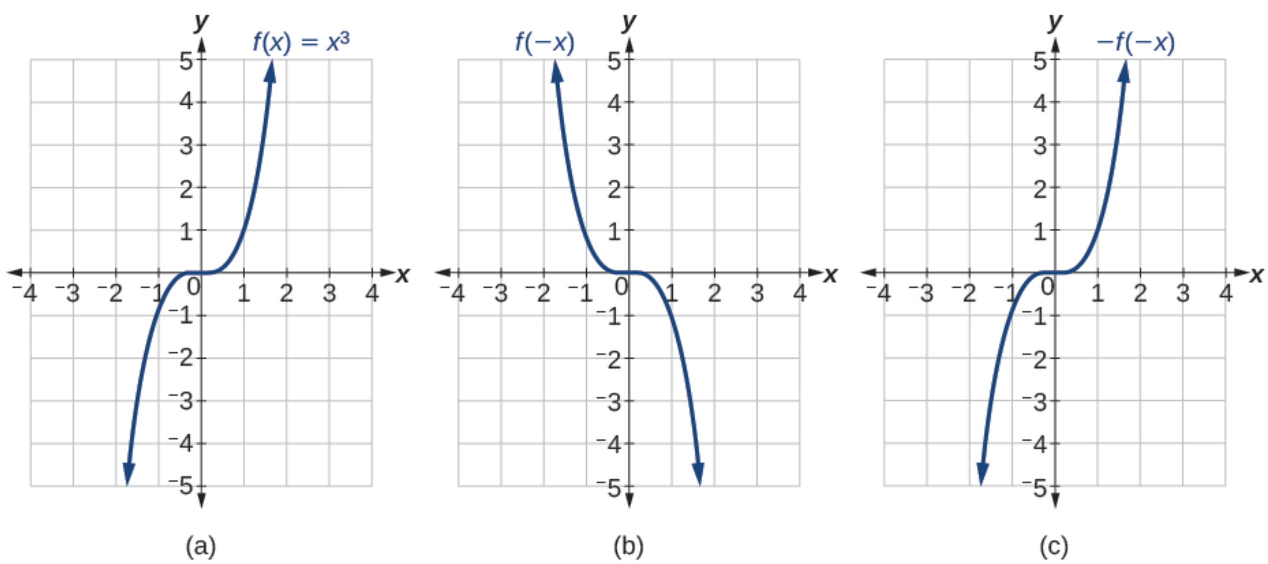


Figure 3-11: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f\left(x\right)=2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f\left(x\right)=0$.

Even and Odd Functions

A function is called an **even function** if for every input x

$$f\left(x\right)=f\left(-x\right)$$

The graph of an even function is symmetric about the y- axis.

A function is called an **odd function** if for every input x

$$f\left(x\right)=-f\left(-x\right)$$

The graph of an odd function is symmetric about the origin.

How To

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f\left(x\right)=f\left(-x\right)$. If it does, it is even.
2. Determine whether the function satisfies $f\left(x\right)=-f\left(-x\right)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 1: Determining whether a Function is Even, Odd, or Neither

1. Is the function $f\left(x\right)={x}^{3}+2x$ even, odd, or neither?

Analysis

Consider the graph of f in Figure 3-12. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, (1, 3) is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

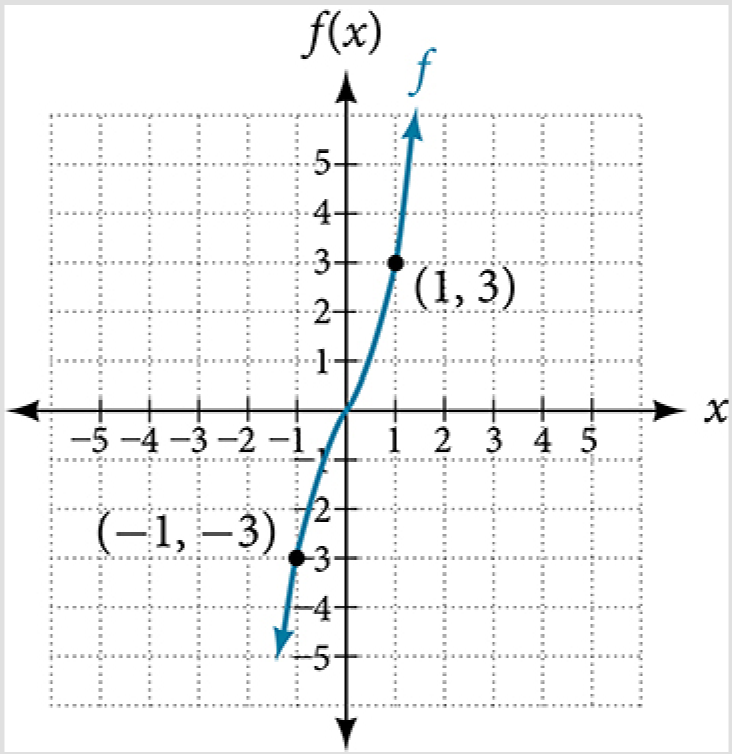


Figure 3-12

2. Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Solution

Access for free at https://openstax.org/books/precalculus/pages/1-introduction-to-functions

3.5 Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 3-13 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

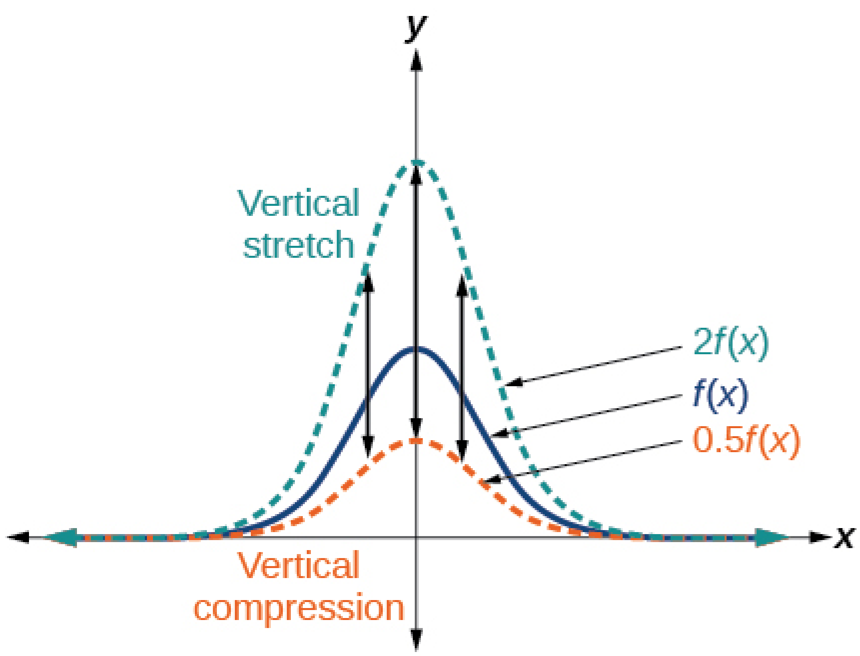


Figure 3-13: Vertical stretch and compression

Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To

Given a function, graph its vertical stretch.

1. Identify the value of a.
2. Multiply all range values by a.
3. If $a > 1$, the graph is stretched by a factor of a. If $0 < a < 1$, the graph is compressed by a factor of a. If $a < 0$, the graph is either stretched or compressed and also reflected about the *x*-axis.

Example 1: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 3-14.

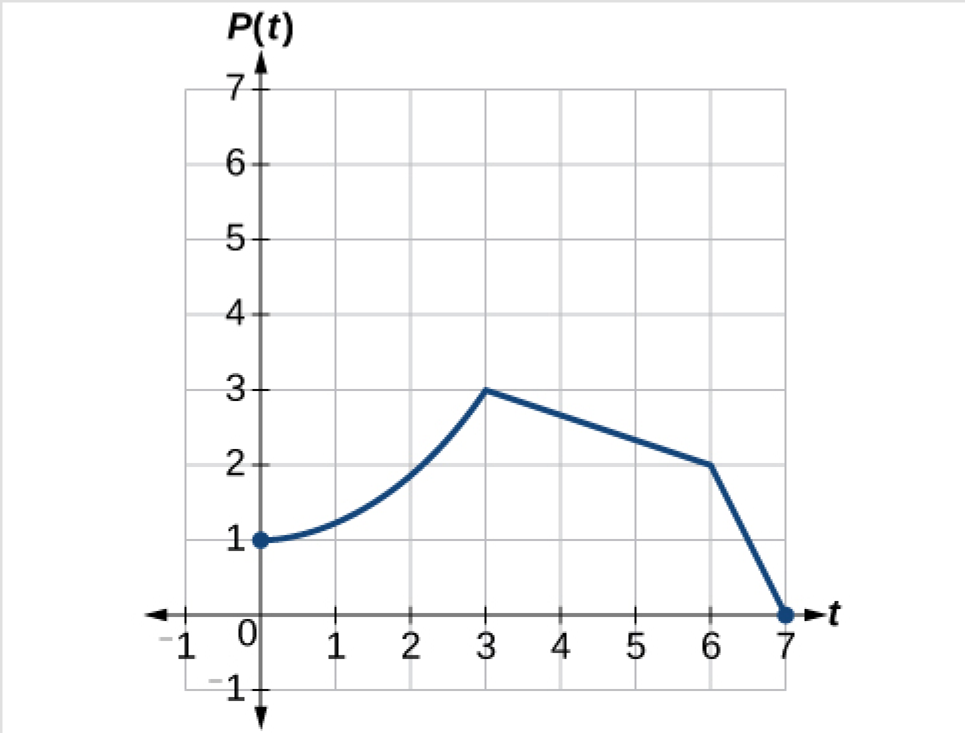


Figure 3-14

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

How To

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 2: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 5. Create a table for the function $g\left(x\right)=\frac{1}{2}f\left(x\right)$.

Table 5

x	2	4	6	8
$f\left(x\right)$	1	3	7	11

Analysis

The result is that the function $g\left(x\right)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

2. A function f is given as Table 6. Create a table for the function $g\left(x\right)=\frac{3}{4}f\left(x\right)$.

Table 6

x	2	4	6	8
$f\left(x\right)$	12	16	20	0

Solution

Example 3: Recognizing a Vertical Stretch

The graph in Figure 3-15 is a transformation of the toolkit function $f\left(x\right)=x^3$. Relate this new function $g\left(x\right)$ to $f\left(x\right)$, and then find a formula for $g\left(x\right)$.

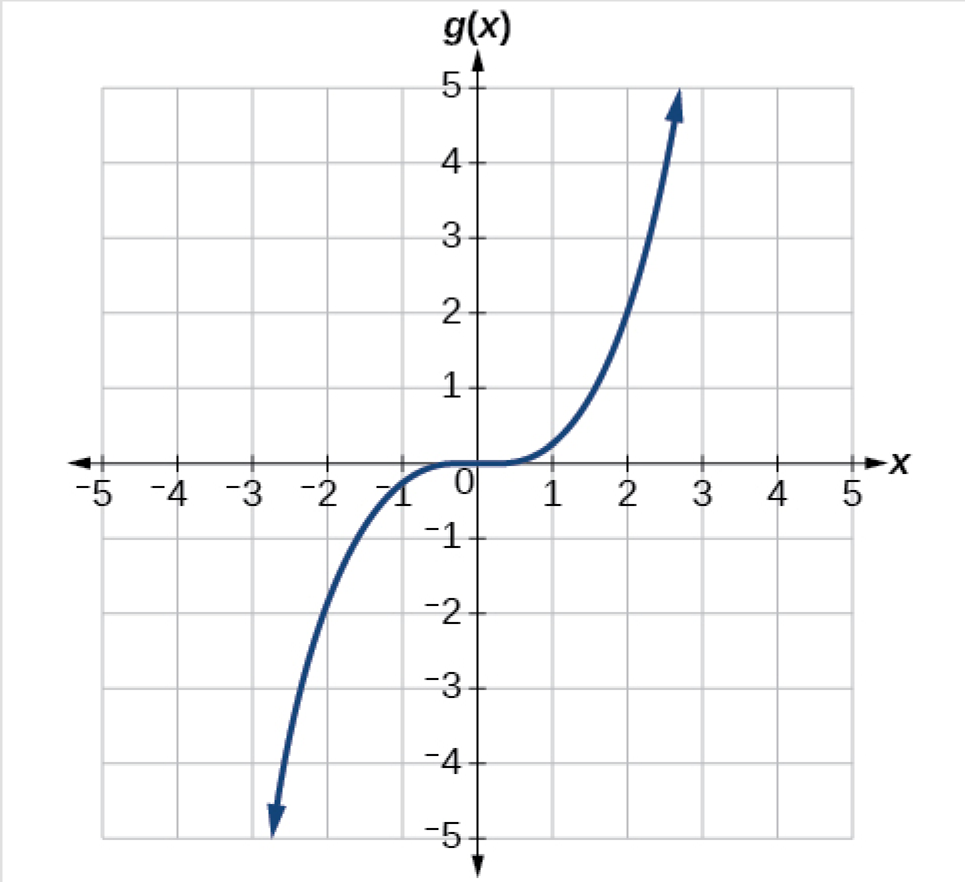


Figure 3-15

Solution

Example 4: Vertical Stretch

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Solution

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

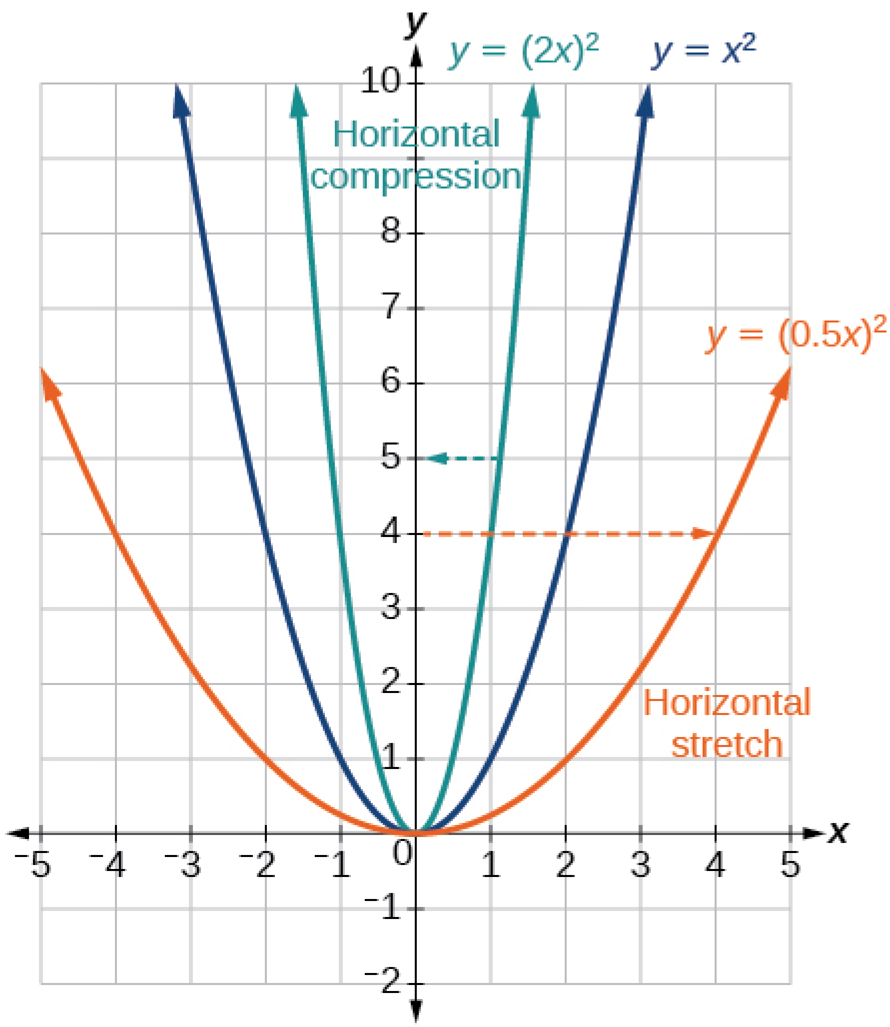


Figure 3-16

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 3-16. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of $\dfrac{1}{2}$.

Horizontal Stretches and Compressions

Given a function $f\left(x\right)$, a new function $g\left(x\right)=f\left(bx\right)$, where b is a constant, is a horizontal stretch or horizontal compression of the function $f\left(x\right)$.

- If $b>1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0<b<1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b<0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g\left(x\right)=f\left(bx\right)$ where $b>1$ for a compression or $0<b<1$ for a stretch.

Example 5: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Solution

Example 6: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 7. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$.

Table 7

x		2	4	6	8
$f(x)$		1	3	7	11

Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Solution

Example 7: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 3-17.

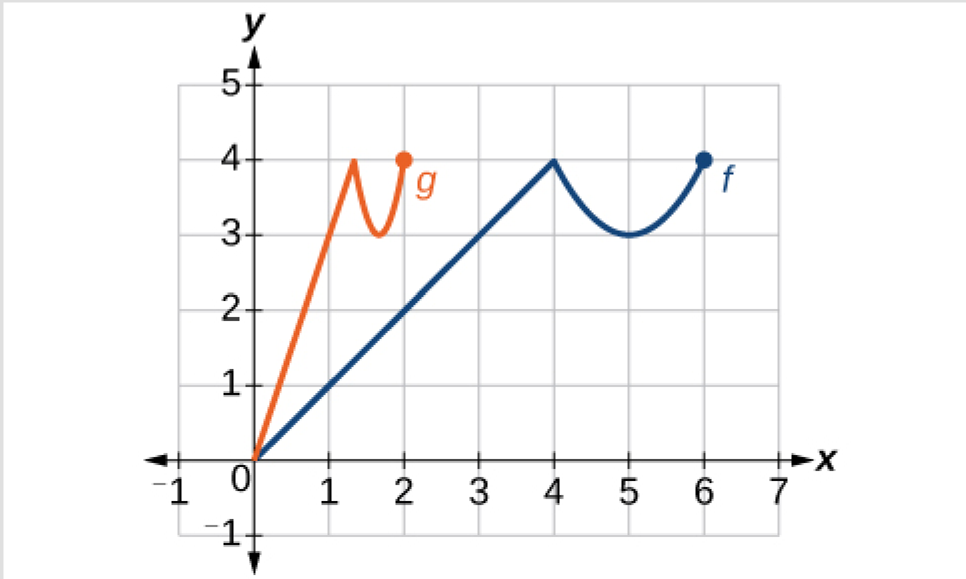


Figure 3-17

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f\left(\frac{1}{4}x\right)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Solution

Example 8: Horizontal Stretch

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Solution

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7)=12$. What input to g would produce that output? In other words, what value of x will allow $g(x)=f(2x+3)=12$? We would need $2x+3=7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f\left(b\left(x+\frac{p}{b}\right)\right)$$

Let's work through an example.

$$f(x)=(2x+4)^2$$

We can factor out a 2.

$$f(x)=(2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.

When combining horizontal transformations written in the form $f(bx-h)$, first horizontally shift by h and then horizontally stretch by $\dfrac{1}{b}$.

When combining horizontal transformations written in the form $f(b(x-h))$, first horizontally stretch by $\dfrac{1}{b}$ and then horizontally shift by h.

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 9: Finding a Triple Transformation of a Tabular Function

Given Table 8 for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x) + 1$.

Table 8

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

Example 10: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 3-18 to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.

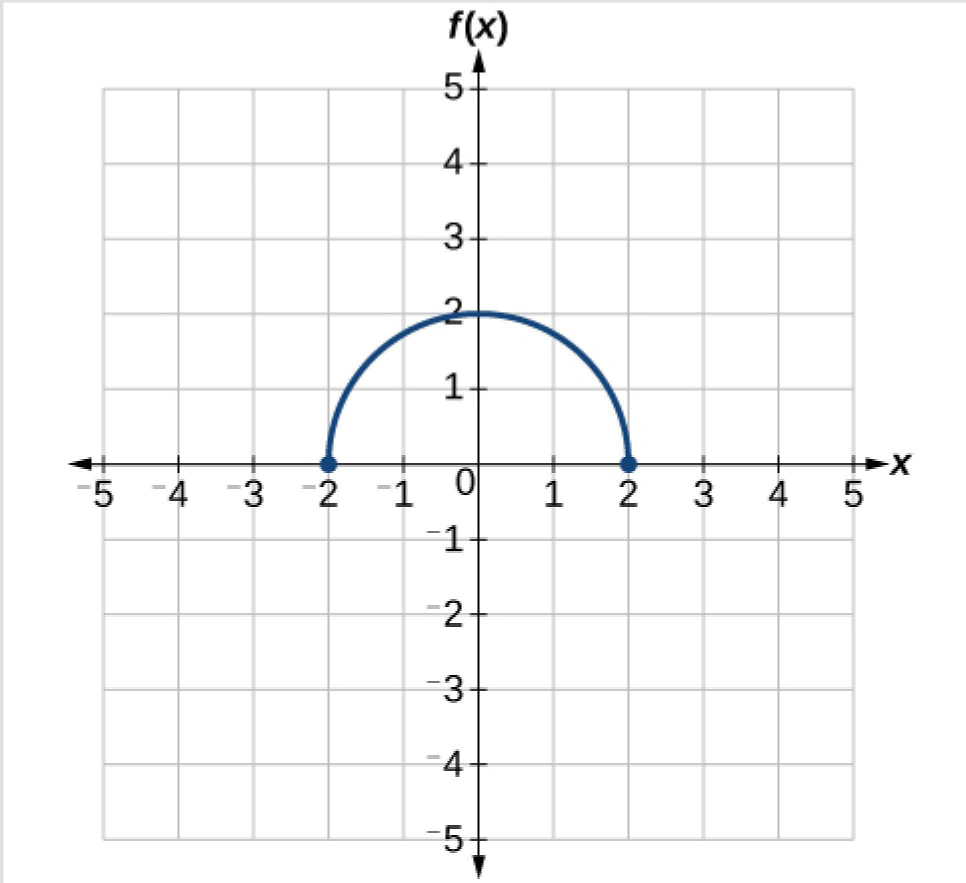


Figure 3-18

Solution

Access for free at https://openstax.org/books/precalculus/pages/1-introduction-to-functions

3.6 Review and Summary

Additional Information

Access this online video resource for additional instruction and practice with transformation of functions.

- [Function Transformations]

Key Equations

Vertical shift	$g(x) = f(x) + k$ (up for $k > 0$)
Horizontal shift	$g(x) = f(x - h)$ (right for $h > 0$)
Vertical reflection	$g(x) = -f(x)$
Horizontal reflection	$g(x) = f(-x)$
Vertical stretch	$g(x) = af(x)$ ($a > 0$)
Vertical compression	$g(x) = af(x)$ ($0 < a < 1$)
Horizontal stretch	$g(x) = f(bx)$ ($0 < b < 1$)
Horizontal compression	$g(x) = f(bx)$ ($b > 1$)

Key Terms

Even Function – a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis.

Horizontal Compression – a transformation that compresses a function's graph horizontally, by multiplying the input by a constant $b > 1$.

Horizontal Reflection – a transformation that reflects a function's graph across the y-axis by multiplying the input by -1.

Horizontal Shift – a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input.

Horizontal Stretch – a transformation that stretches a function's graph horizontally by multiplying the input by a constant $0 < b < 1$.

Odd Function – a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$

Vertical Compression – a function transformation that compresses the function's graph vertically by multiplying the output by a constant $0 < a < 1$.

Vertical Reflection – a transformation that reflects a function's graph across the x-axis by multiplying the output by -1.

Vertical Shift – a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output.

Vertical Stretch – a transformation that stretches a function's graph vertically by multiplying the output by a constant $a > 1$.

Key Concepts

- A function can be shifted vertically by adding a constant to the output. See 3.2 Graphing Functions Using Vertical and Horizontal Shifts.
- A function can be shifted horizontally by adding a constant to the input. See 3.2 Graphing Functions Using Vertical and Horizontal Shifts.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts. See 3.2 Graphing Functions Using Vertical and Horizontal Shifts.
- Vertical and horizontal shifts are often combined. See 3.2 Graphing Functions Using Vertical and Horizontal Shifts.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph. See 3.3 Graphing Functions Using Reflections about the Axes.

- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly. See 3.3 Graphing Functions Using Reflections about the Axes.
- A function presented as an equation can be reflected by applying transformations one at a time. See 3.3 Graphing Functions Using Reflections about the Axes.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f\left(x\right)=f\left(-x\right)$.
- Odd functions satisfy the condition $f\left(x\right)=-f\left(-x\right)$.
- A function can be odd, even, or neither. See 3.4 Determining Even and Odd Functions.
- A function can be compressed or stretched vertically by multiplying the output by a constant. See 3.5 Graphing Functions Using Stretches and Compressions.
- A function can be compressed or stretched horizontally by multiplying the input by a constant. See 3.5 Graphing Functions Using Stretches and Compressions.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order. See 3.5 Graphing Functions Using Stretches and Compressions.

3.7 Practice Questions

Verbal Questions

1. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?
2. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?
3. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?
4. When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the *x*-axis from a reflection with respect to the *y*-axis?
5. How can you determine whether a function is odd or even from the formula of the function?

Odd Number Verbal Solutions

Algebraic Questions

6. Write a formula for the function obtained when the graph of $f\left(x\right)=\sqrt{x}$ is shifted up 1 unit and to the left 2 units.
7. Write a formula for the function obtained when the graph of $f\left(x\right)=\left|x\right|$ is shifted down 3 units and to the right 1 unit.
8. Write a formula for the function obtained when the graph of $f\left(x\right)=\dfrac{1}{x}$ is shifted down 4 units and to the right 3 units.
9. Write a formula for the function obtained when the graph of $f\left(x\right)=\dfrac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

10. $y = f(x - 49)$
11. $y = f(x + 43)$
12. $y = f(x + 3)$
13. $y = f(x - 4)$
14. $y = f(x) + 5$
15. $y = f(x) + 8$
16. $y = f(x) - 2$
17. $y = f(x) - 7$
18. $y = f(x - 2) + 3$
19. $y = f(x + 4) - 1$

For the following exercises, determine the interval(s) on which the function is increasing and decreasing.

20. $f(x) = 4(x + 1)^2 - 5$
21. $g(x) = 5(x + 3)^2 - 2$
22. $a(x) = \sqrt{-x + 4}$
23. $k(x) = -3\sqrt{x} - 1$

Odd Number Algebraic Solutions

Graphical Questions

For the following exercises, use the graph of $f(x) = 2^x$ shown in Figure 3-19 to sketch a graph of each transformation of $f(x)$.

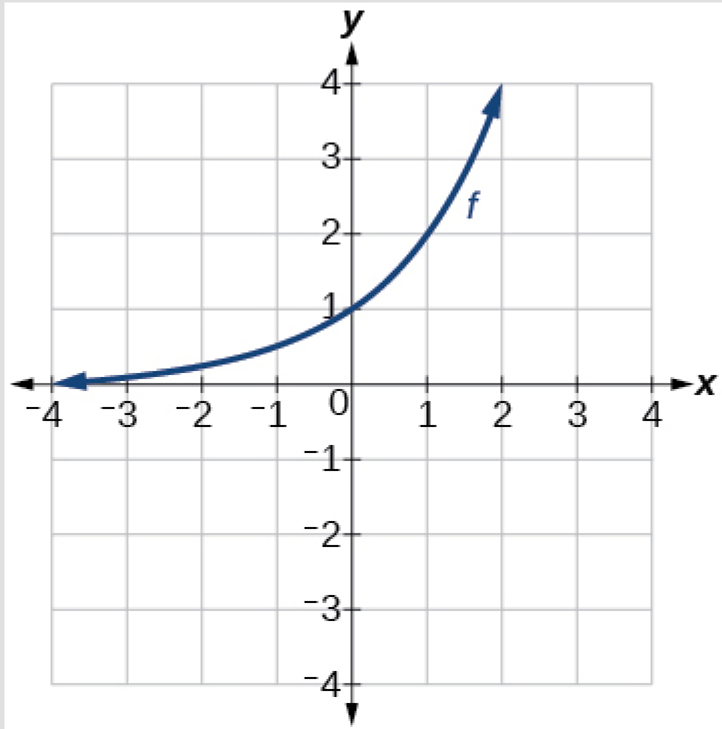


Figure 3-19

24. $g\left(x\right)=2^{x}+1$
25. $h\left(x\right)=2^{x}-3$
26. $w\left(x\right)=2^{x-1}$

For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

27. $f\left(t\right)=\left(t+1\right)^{2}-3$
28. $h\left(x\right)=\left|x-1\right|+4$
29. $k\left(x\right)=\left(x-2\right)^{3}-1$
30. $m\left(t\right)=3+\sqrt{t+2}$

Odd Number Graphical Solutions

Numeric Questions

31. Tabular representations for the functions $f, g,$ and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	–2	–1	0	1	2
$f(x)$	–2	–1	–3	1	2

x	–1	0	1	2	3
$g(x)$	–2	–1	–3	1	2

x	–2	–1	0	1	2
$h(x)$	–1	0	–2	2	3

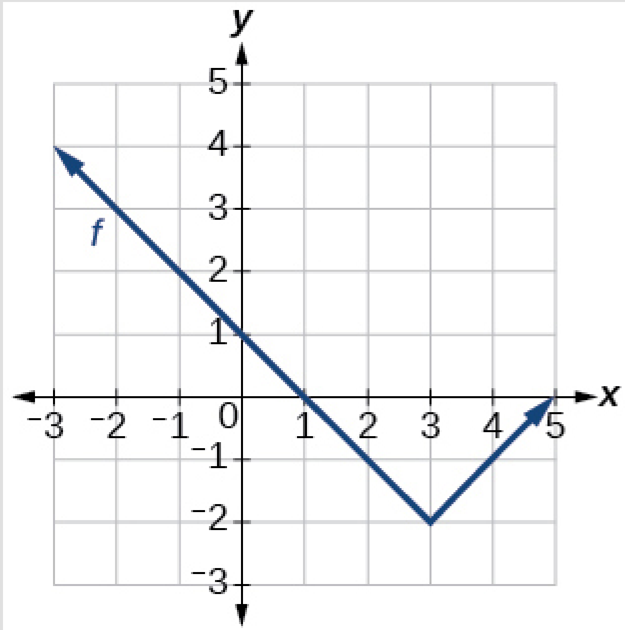
32. Tabular representations for the functions $f, g,$ and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

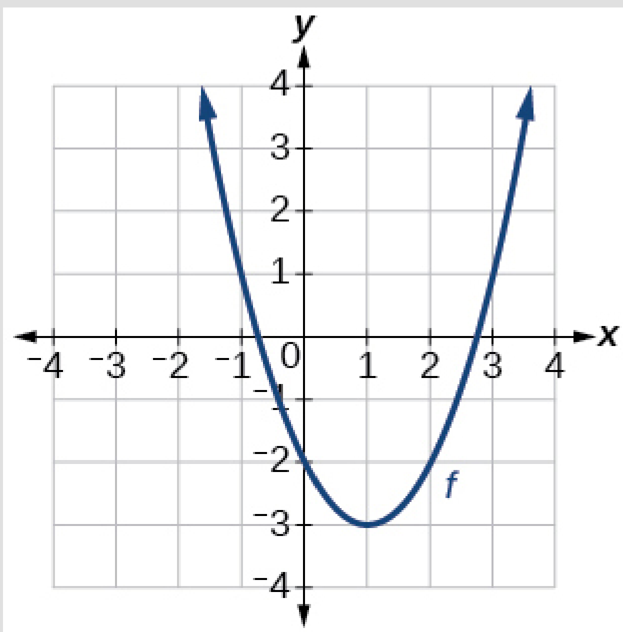
x	–2	–1	0	1	2
$f(x)$	–1	–3	4	2	1

x	–3	–2	–1	0	1
$g(x)$	–1	–3	4	2	1

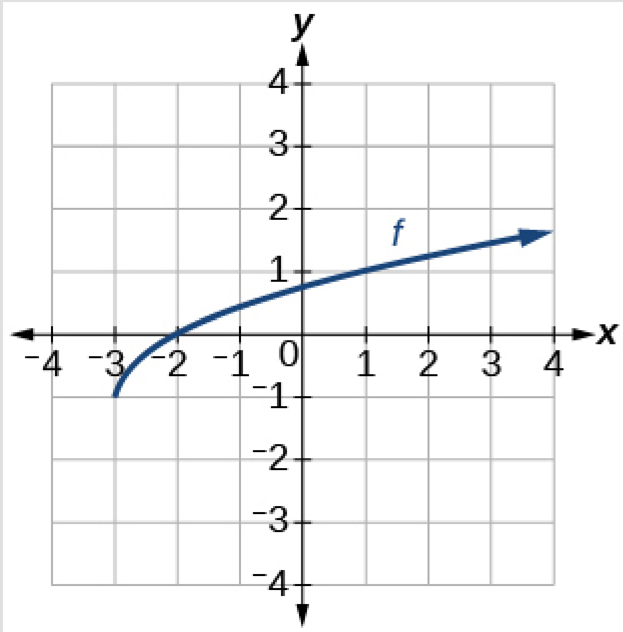
x	–2	–1	0	1	2
$h(x)$	–2	–4	3	1	0

For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

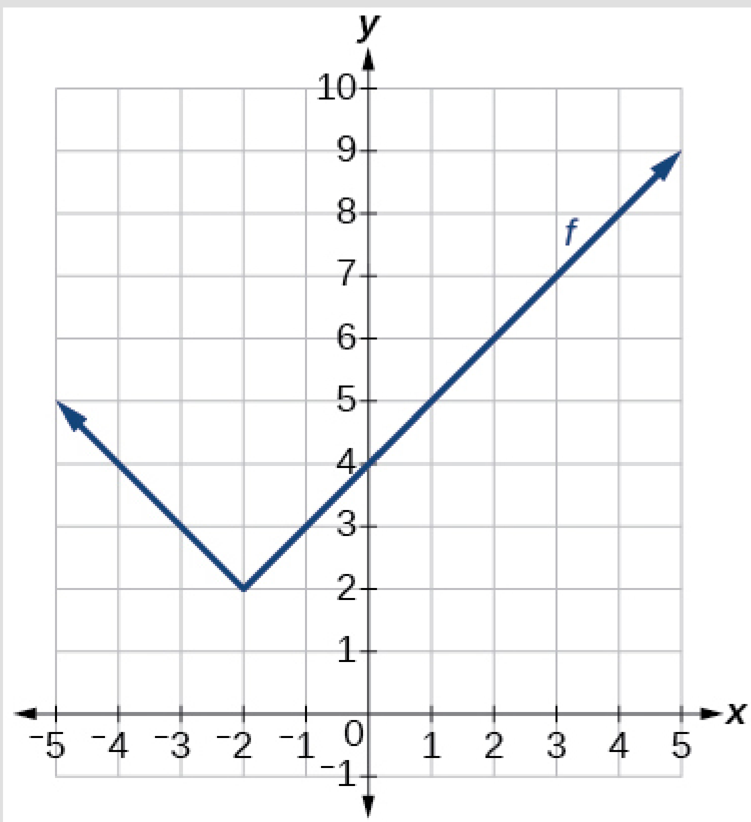
33. 

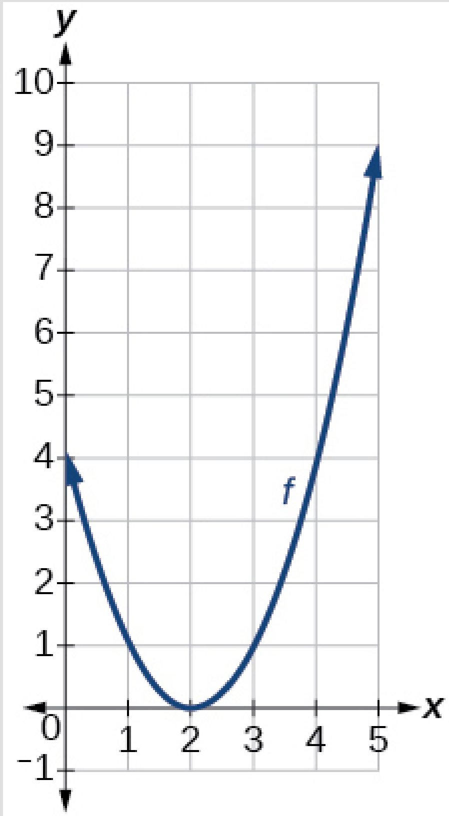
34. 

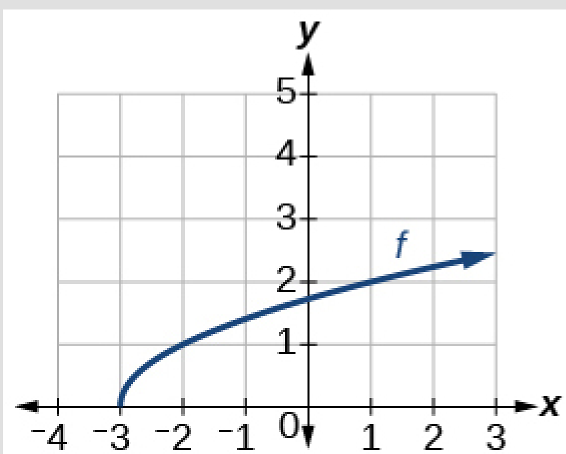
35.

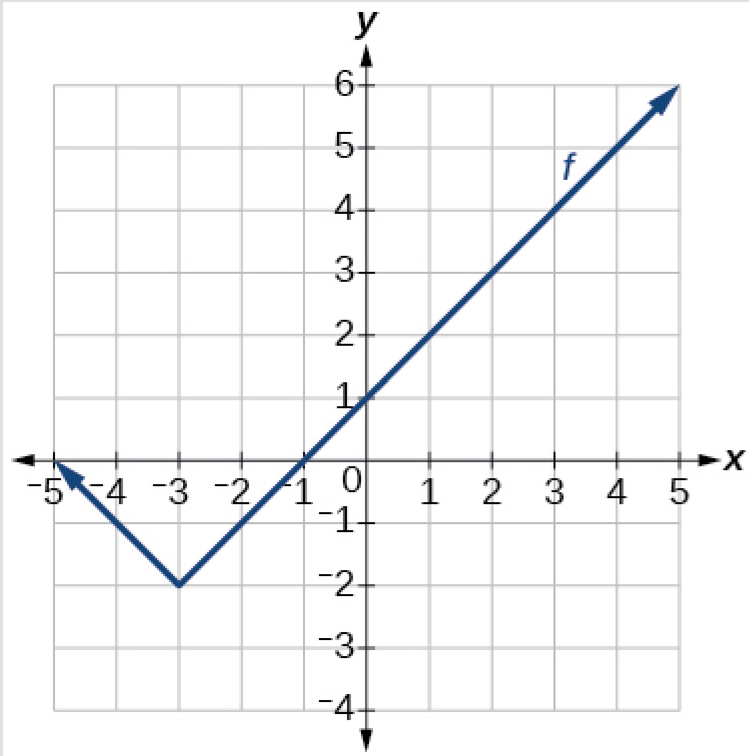


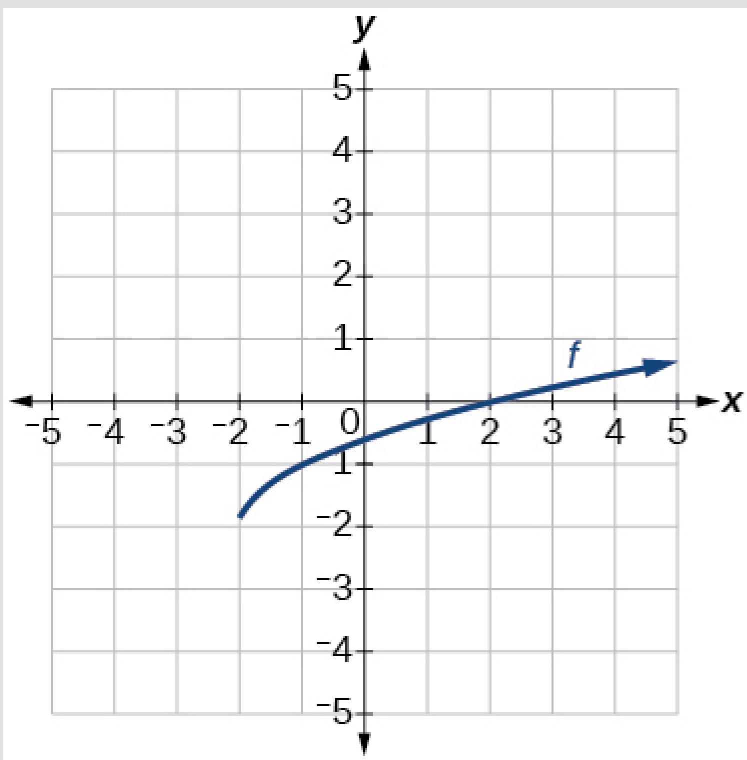
36.



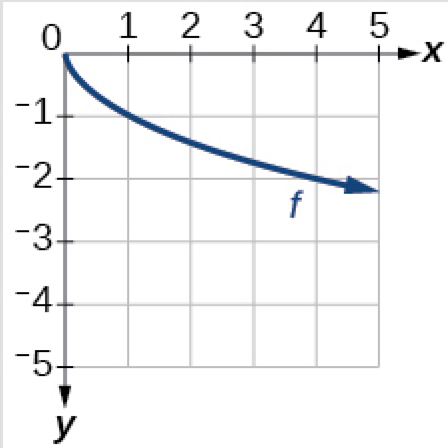
37. 

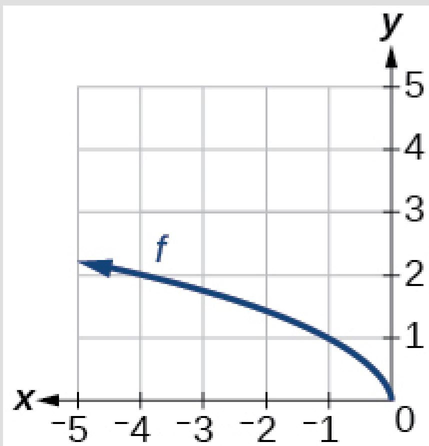
38. 

39. 

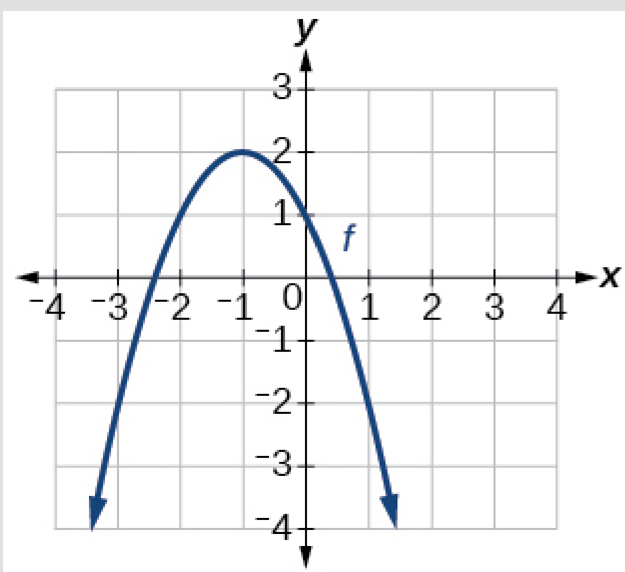
40. 

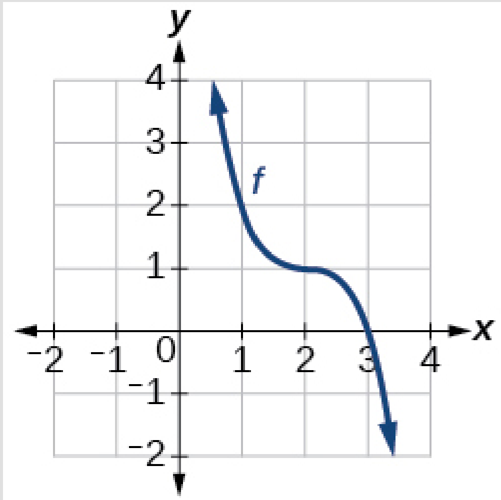
For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.

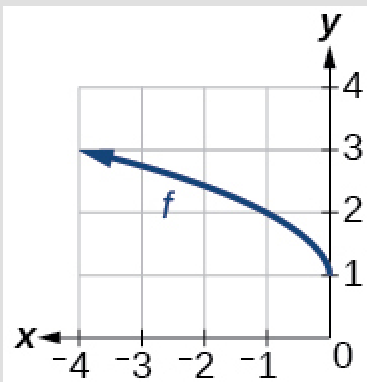
41. 

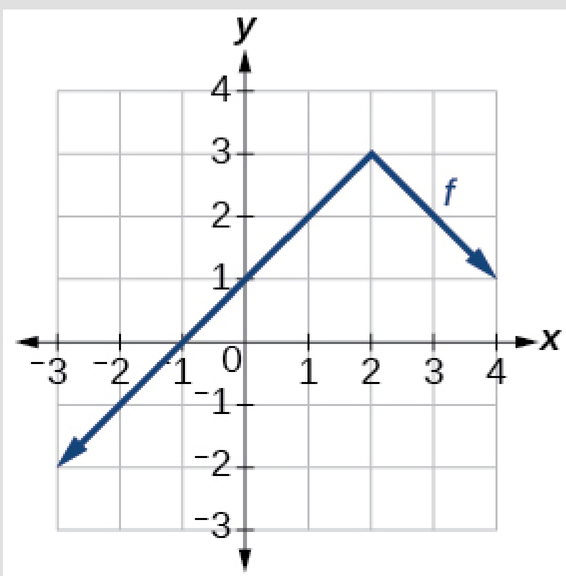
42. 

For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

43. 

44. 

45. 

46. 

For the following exercises, determine whether the function is odd, even, or neither.

47. $f\left(x\right)=3x^4$

48. $g\left(x\right)=\sqrt{x}$

49. $h(x) = \dfrac{1}{x} + 3x$

50. $f(x) = (x-2)^2$

51. $g(x) = 2x^4$

52. $h(x) = 2x - x^3$

For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

53. $g(x) = -f(x)$

54. $g(x) = f(-x)$

55. $g(x) = 4f(x)$

56. $g(x) = 6f(x)$

57. $g(x) = f(5x)$

58. $g(x) = f(2x)$

59. $g(x) = f\left(\dfrac{1}{3}x\right)$

60. $g(x) = f\left(\dfrac{1}{5}x\right)$

61. $g(x) = 3f(-x)$

62. $g(x) = -f(3x)$

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

63. The graph of $f(x) = |x|$ is reflected over the y–axis and horizontally compressed by a factor of $\dfrac{1}{4}$.

64. The graph of $f(x) = \sqrt{x}$ is reflected over the x -axis and horizontally stretched by a factor of 2.

65. The graph of $f(x) = \dfrac{1}{x^2}$ is vertically compressed by a factor of $\dfrac{1}{3}$, then shifted to the left 2 units and down 3 units.

66. The graph of $f(x) = \dfrac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

67. The graph of $f(x) = x^2$ is vertically compressed by a factor of $\dfrac{1}{2}$, then shifted to the right 5 units and up 1 unit.

68. The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4

units and down 3 units.

For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

69. $g\left(x\right)=4{\left(x+1\right)}^{2}-5$

70. $g\left(x\right)=5{\left(x+3\right)}^{2}-2$

71. $h\left(x\right)=-2\left|x-4\right|+3$

72. $k\left(x\right)=-3\sqrt{x}-1$

73. $m\left(x\right)=\frac{1}{2}x^{3}$

74. $n\left(x\right)=\frac{1}{3}\left|x-2\right|$

75. $p\left(x\right)={\left(\frac{1}{3}x\right)}^{3}-3$

76. $q\left(x\right)={\left(\frac{1}{4}x\right)}^{3}+1$

77. $a\left(x\right)=\sqrt{-x+4}$

For the following exercises, use the graph in Figure 3-20 to sketch the given transformations.

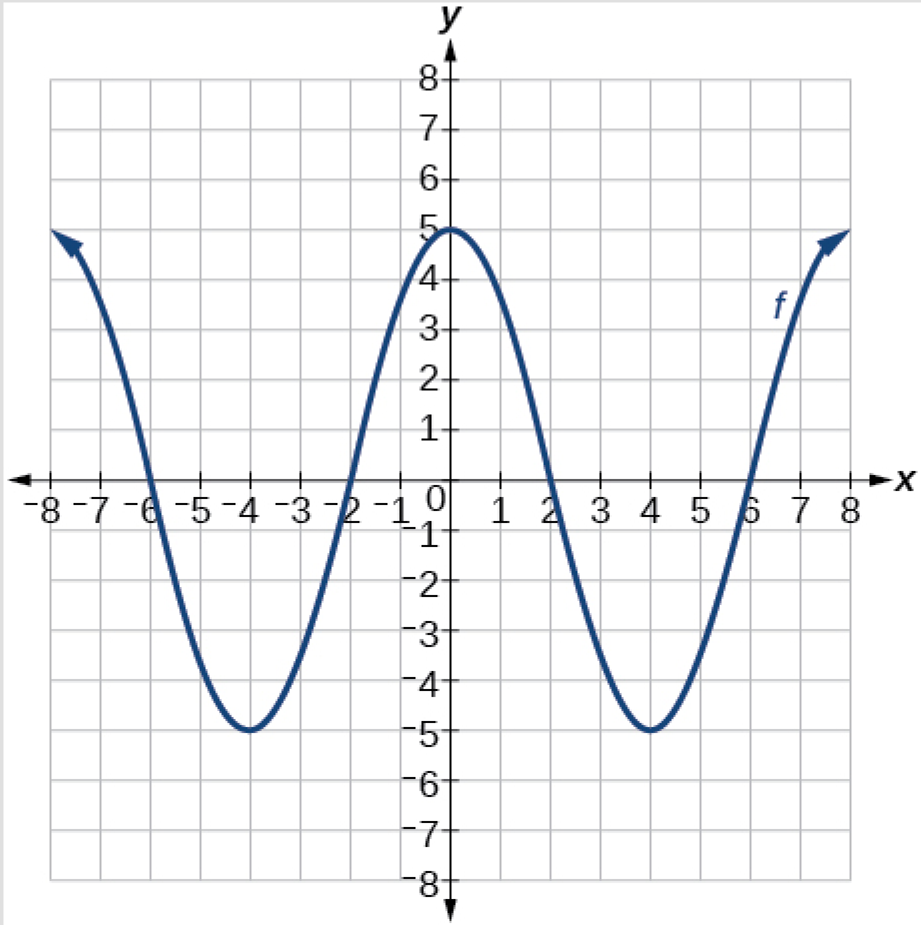


Figure 3-20

78. $g\left(x\right)=f\left(x\right)-2$
79. $g\left(x\right)=-f\left(x\right)$
80. $g\left(x\right)=f\left(x+1\right)$
81. $g\left(x\right)=f\left(x-2\right)$

[Odd Number Numeric Solutions]

Access for free at https://openstax.org/books/precalculus/pages/1-introduction-to-functions

3.8 Chapter 3 Example Solutions

3.2 Example Solutions

Example 1: Adding a Constant to a Function

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 3-21.

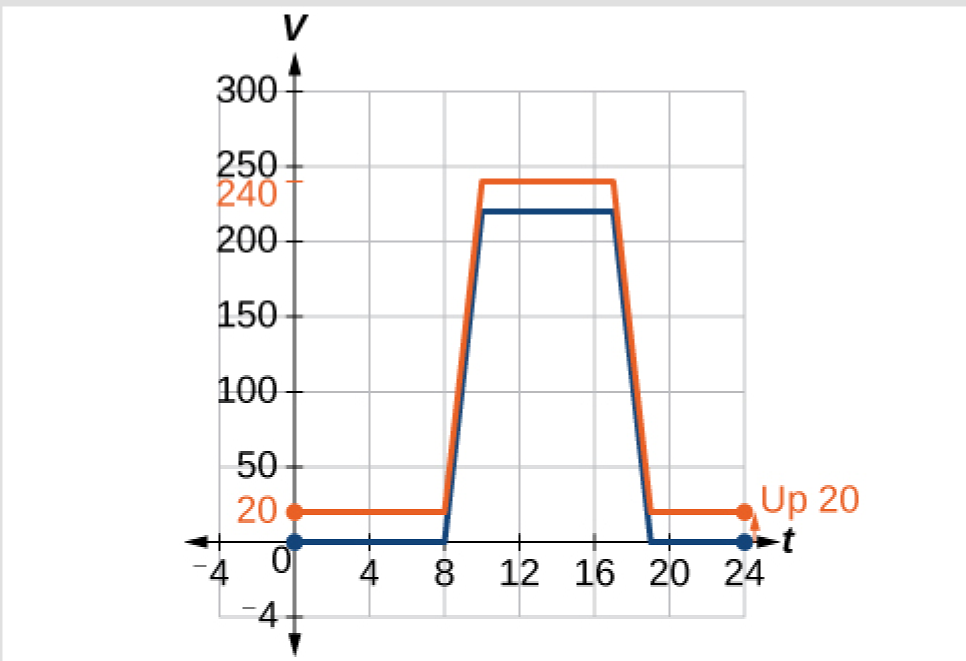


Figure 3-21

Notice that in Figure 3-21, for each input value, the output value has increased by 20, so if we call the new function $S(t)$, we could write

$$S(t) = V(t) + 20$$

This notation tells us that, for any value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 8.

Table 8

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

Example 2: Shifting a Tabular Function Vertically

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$\begin{aligned} &f(2) = 1 && \text{Given} \\ &g(x) = f(x) - 3 && \text{Given transformation} \\ &g(2) = f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 9.

Table 9

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	−2	0	4	8

Example 3: Vertical Shift

$$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$$

Example 4: Adding a Constant to an Input

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$
$$\mathrm{F}(t) = \text{starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 3-22. Notice also that the vents first opened to $220\ \mathrm{ft}^2$ at 10 a.m. under the original plan, while under the new plan the vents reach $220\ \mathrm{ft}^2$ at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t - (-2)) = V(t + 2)$.

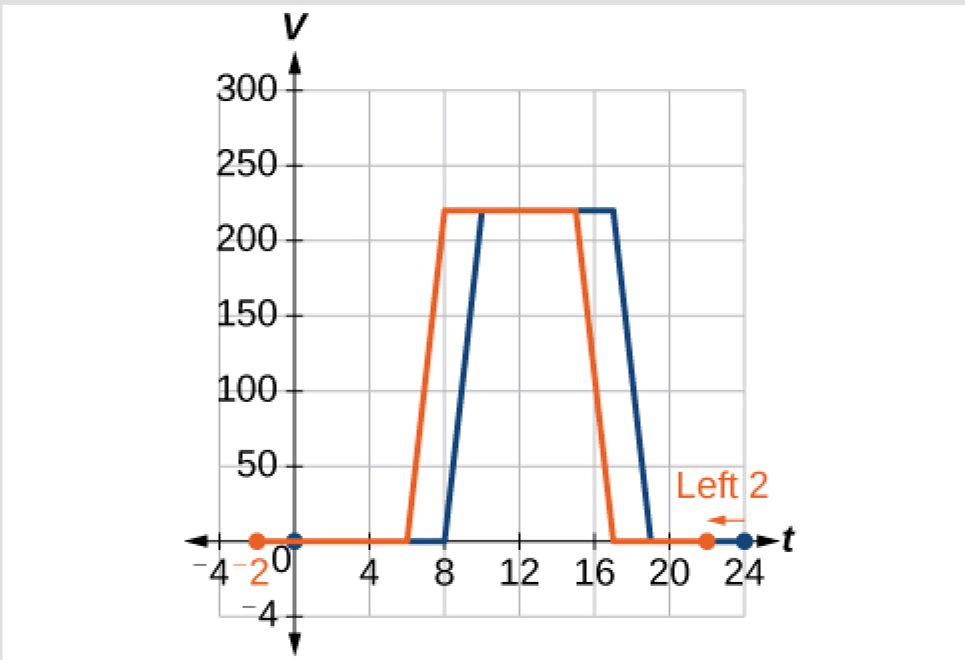


Figure 3-22

Example 5: Shifting a Tabular Function Horizontally

The formula $g(x) = f(x - 3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 *larger*. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$g(5) = f(5 - 3)$$
$$= f(2)$$
$$= 1$$

We continue with the other values to create Table 9.

Table 9

x	5	7	9	11
$x-3$	2	4	6	8
$f(x-3)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Example 6: Identifying a Horizontal Shift of a Toolkit Function

Notice that the graph is identical in shape to the $f(x)=x^2$ function, but the *x*-values are shifted to the right 2 units. The vertex used to be at (0,0), but now the vertex is at (2,0). The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x)=f(x-2)$$

Notice how we must input the value $x=2$ to get the output value $y=0$; the *x*-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

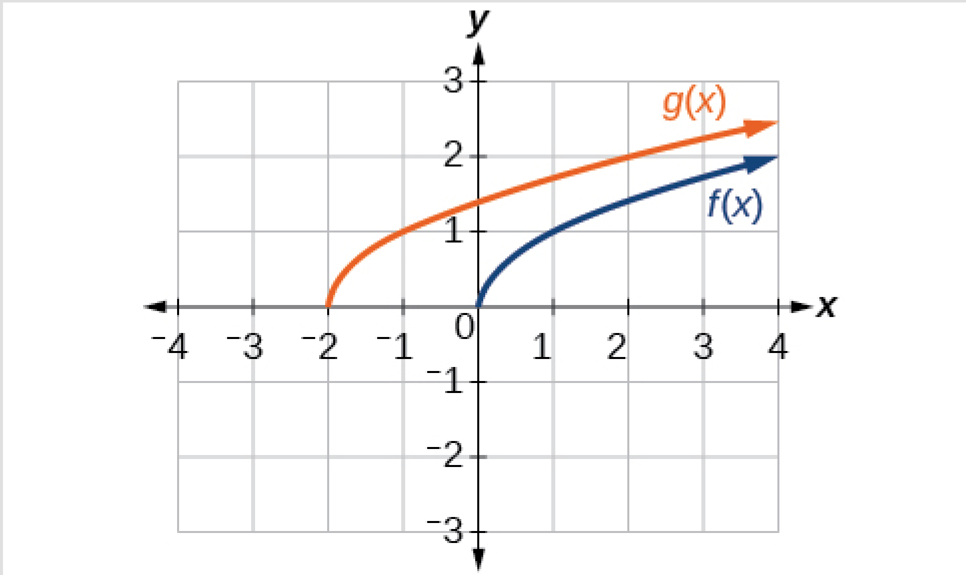
$$f(x)=x^2$$
$$g(x)=f(x-2)$$
$$g(x)=f(x-2)=(x-2)^2$$

Example 7: Interpreting Horizontal versus Vertical Shifts

1. $G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

2. The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.



Example 8: Graphing Combined Vertical and Horizontal Shifts

1. The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f\left(x+1\right)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f\left(x+1\right)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 3-23.

Let us follow one point of the graph of $f\left(x\right)=\left|x\right|$.

- The point $\left(0,0\right)$ is transformed first by shifting left 1 unit: $\left(0,0\right)\to\left(-1,0\right)$
- The point $\left(-1,0\right)$ is transformed next by shifting down 3 units: $\left(-1,0\right)\to\left(-1,-3\right)$

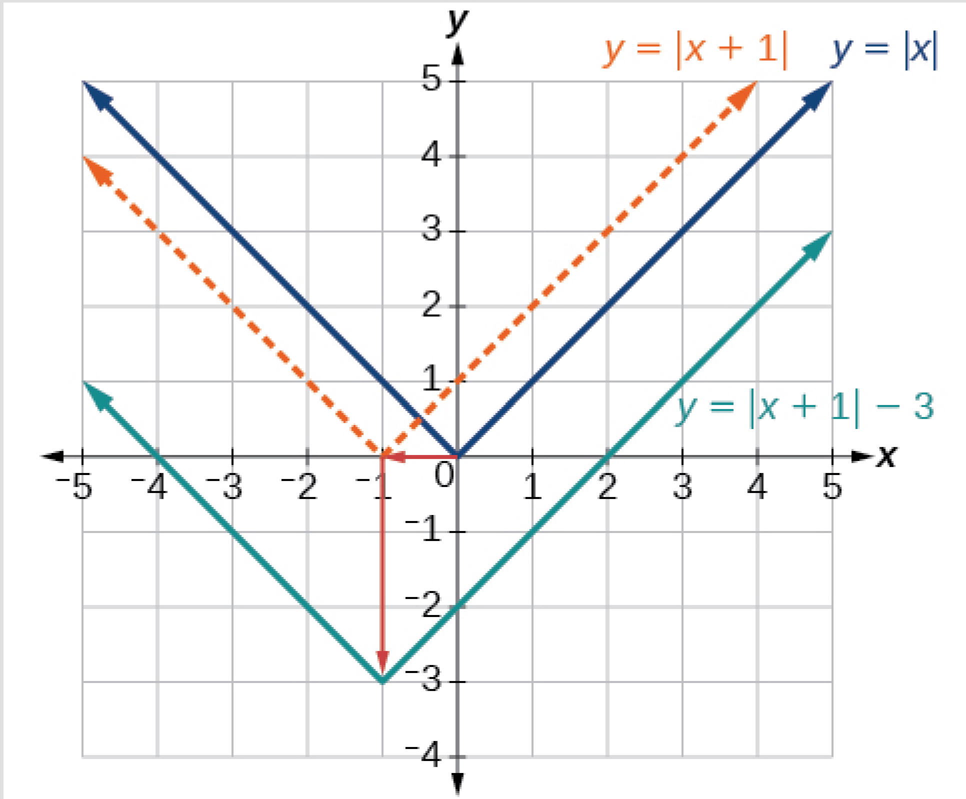


Figure 3-23

Figure 3-24 shows the graph of h.

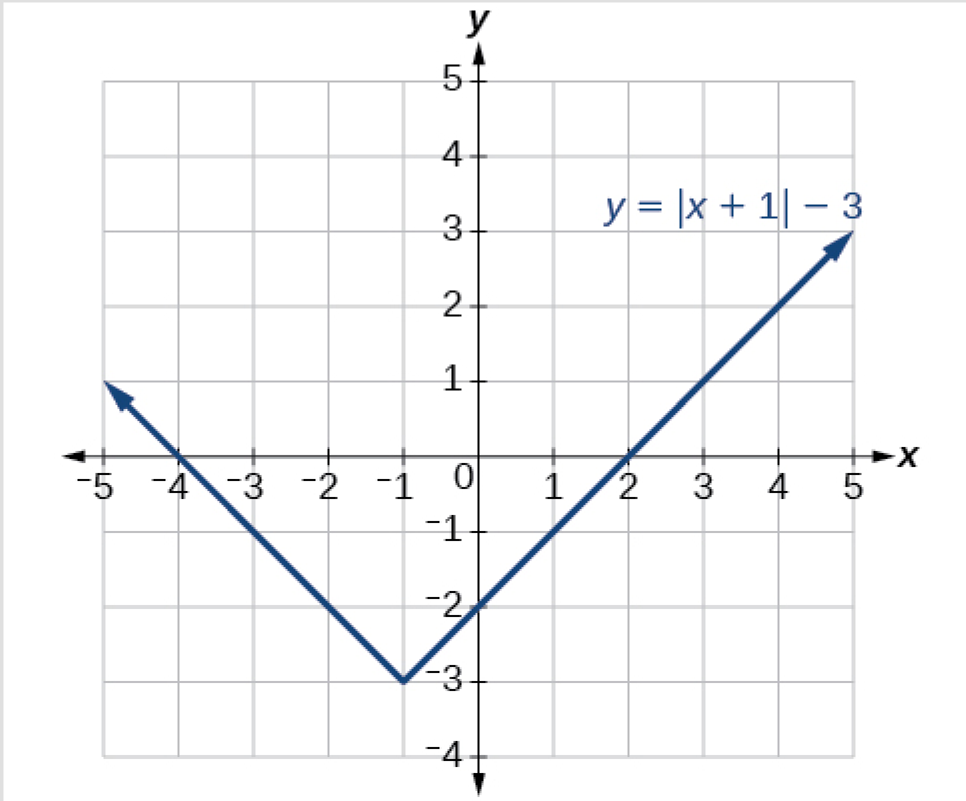
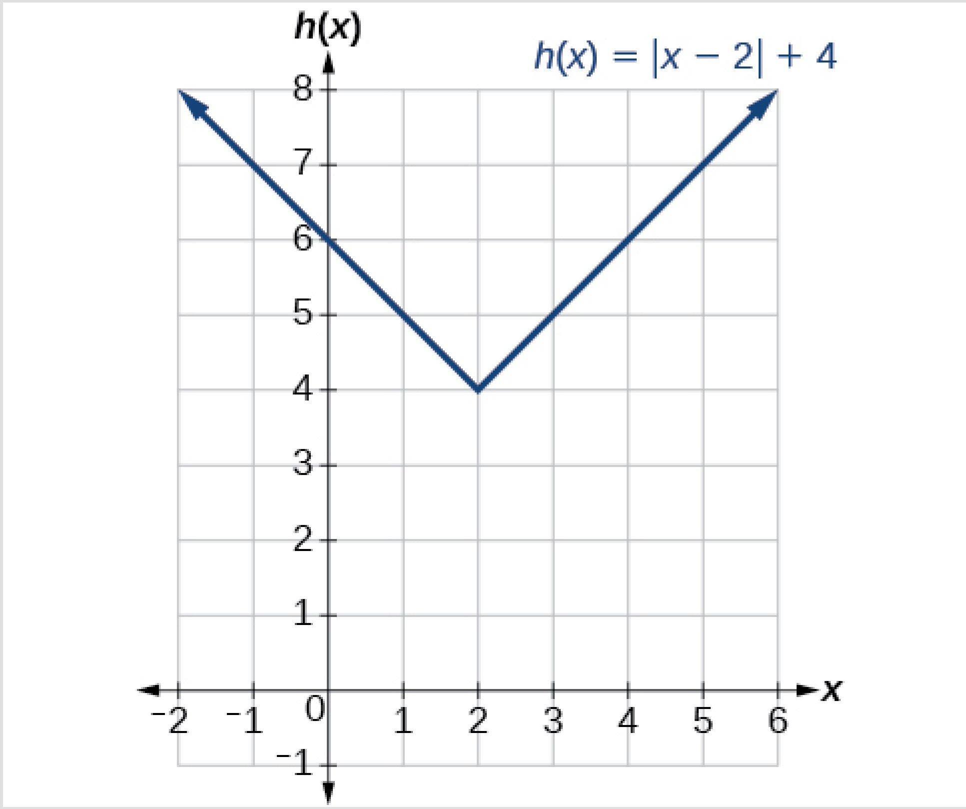


Figure 3-24

2.



Example 9: Identifying Combined Vertical and Horizontal Shifts

1. The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h\left(x\right)=f\left(x-1\right)+2$$

Using the formula for the square root function, we can write

$$h\left(x\right)=\sqrt{x-1}+2$$

2. $g\left(x\right)=\dfrac{1}{x-1}+1$

3.3 Example Solutions

Example 1: Reflecting a Graph Horizontally and Vertically

1. a. Reflecting the graph vertically means that each output value will be reflected over the horizontal *t*-axis as shown in Figure 3-25.

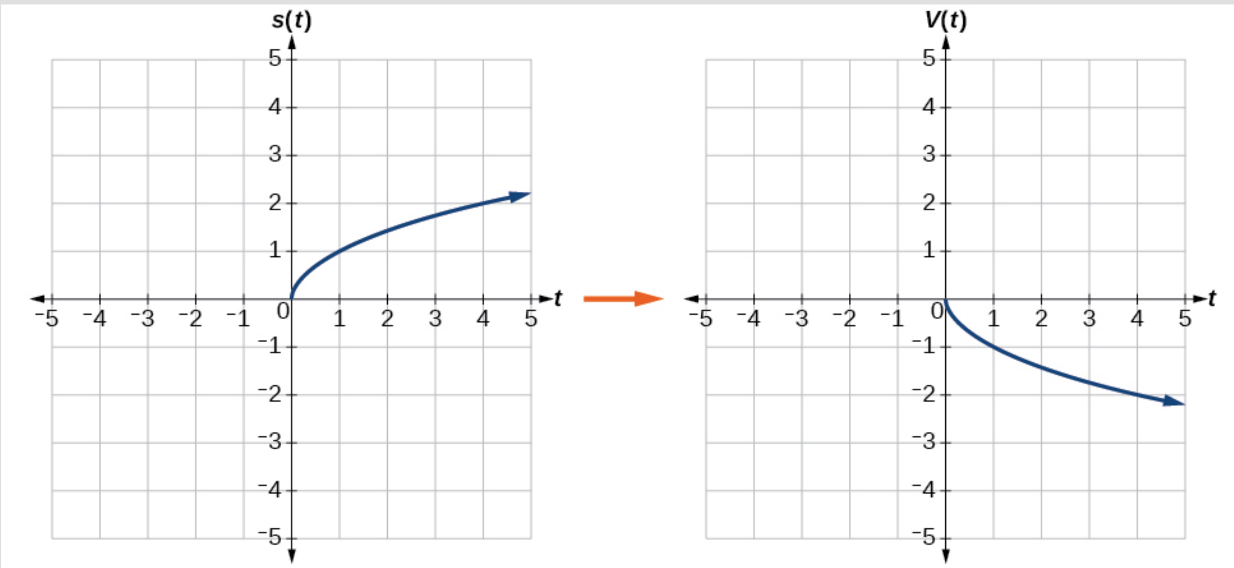


Figure 3-25: Vertical reflection of the square root function

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in 3-26.

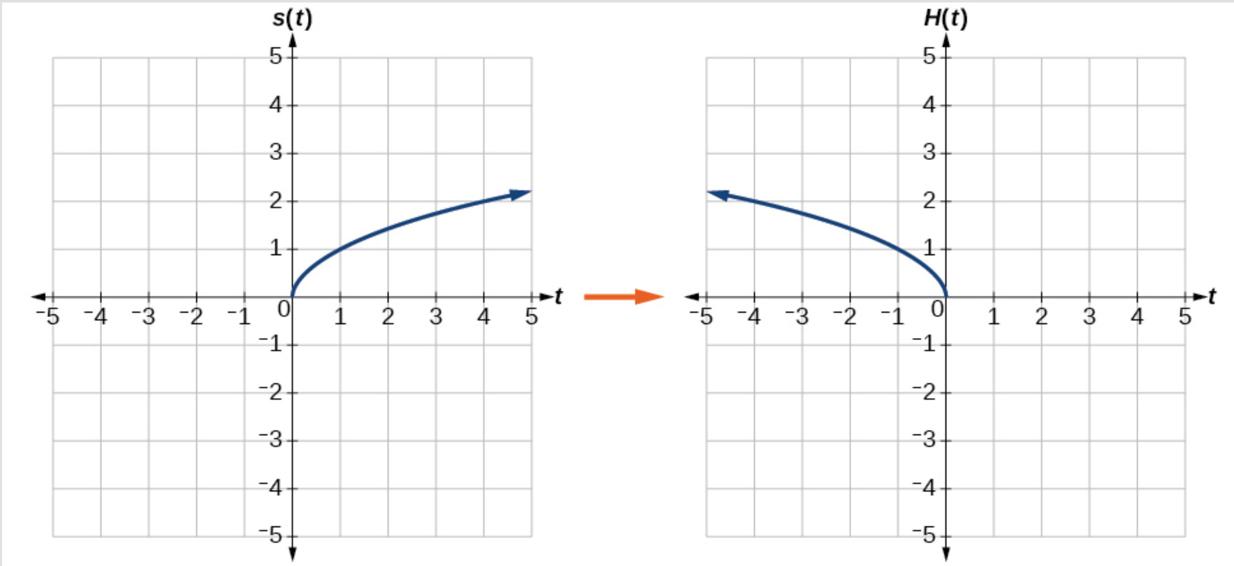


Figure 3-26: Horizontal reflection of the square root function

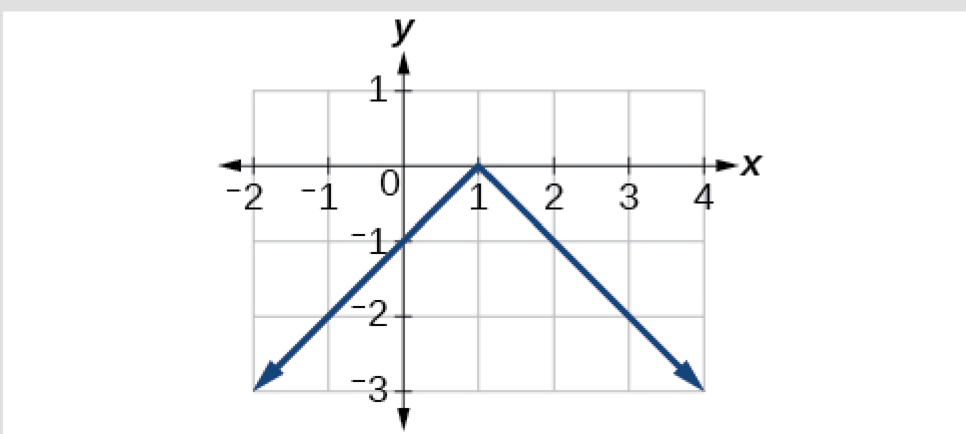
Because each input value is the opposite of the original input value, we can write

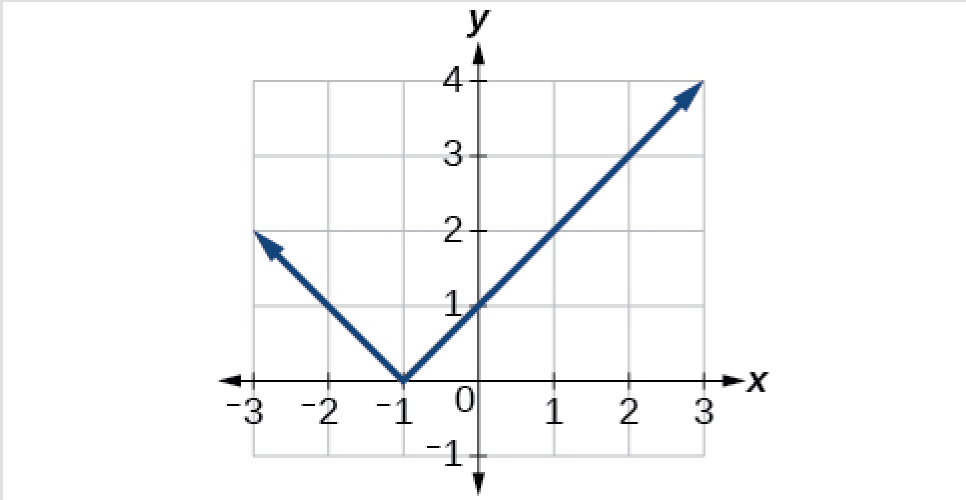
$$H\left(t\right) = s\left(-t\right) \text{ or } H\left(t\right) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $\left[0, \infty\right)$ and range $\left[0, \infty\right)$, the vertical reflection gives the $V\left(t\right)$ function the range $\left(-\infty, 0\right]$ and the horizontal reflection gives the $H\left(t\right)$ function the domain $\left(-\infty, 0\right]$.

2.

a. 

b. 

Example 2: Reflecting a Tabular Function Horizontally and Vertically

1. a. For $g(x)$ the negative sign outside the function indicates a vertical reflection, so the *x*-values stay the same and each output value will be the opposite of the original output value. See Table 10.

Table 10

x	2	4	6	8
$g(x)$	–1	–3	–7	–11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 11.

Table 11

x	−2	−4	−6	−8
$h(x)$	1	3	7	11

2. a. $g(x) = -f(x)$

x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

b. $h(x) = f(-x)$

x	-2	0	2	4
$h(x)$	15	10	5	unknown

Example 3: Applying a Learning Model Equation

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points (0, 1) and (1, 2).

1. First, we apply a horizontal reflection: (0, 1) (–1, 2).
2. Then, we apply a vertical reflection: (0, −1) (-1, –2).
3. Finally, we apply a vertical shift: (0, 0) (-1, -1).

This means that the original points, (0,1) and (1,2) become (0,0) and (-1,-1) after we apply the transformations.

In Figure 3-27, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

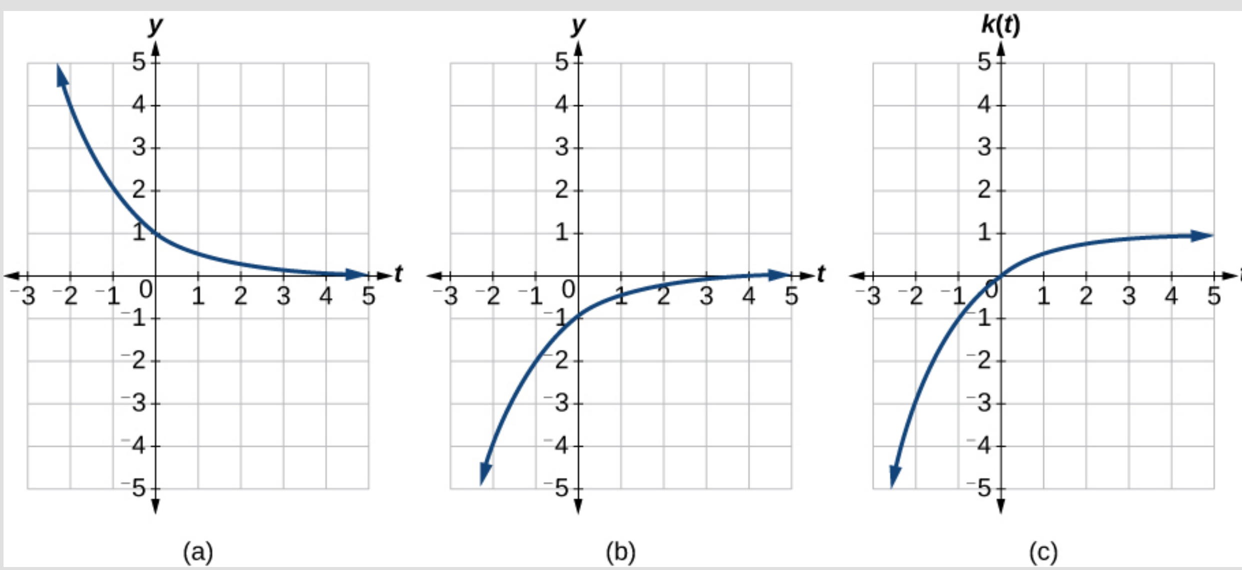
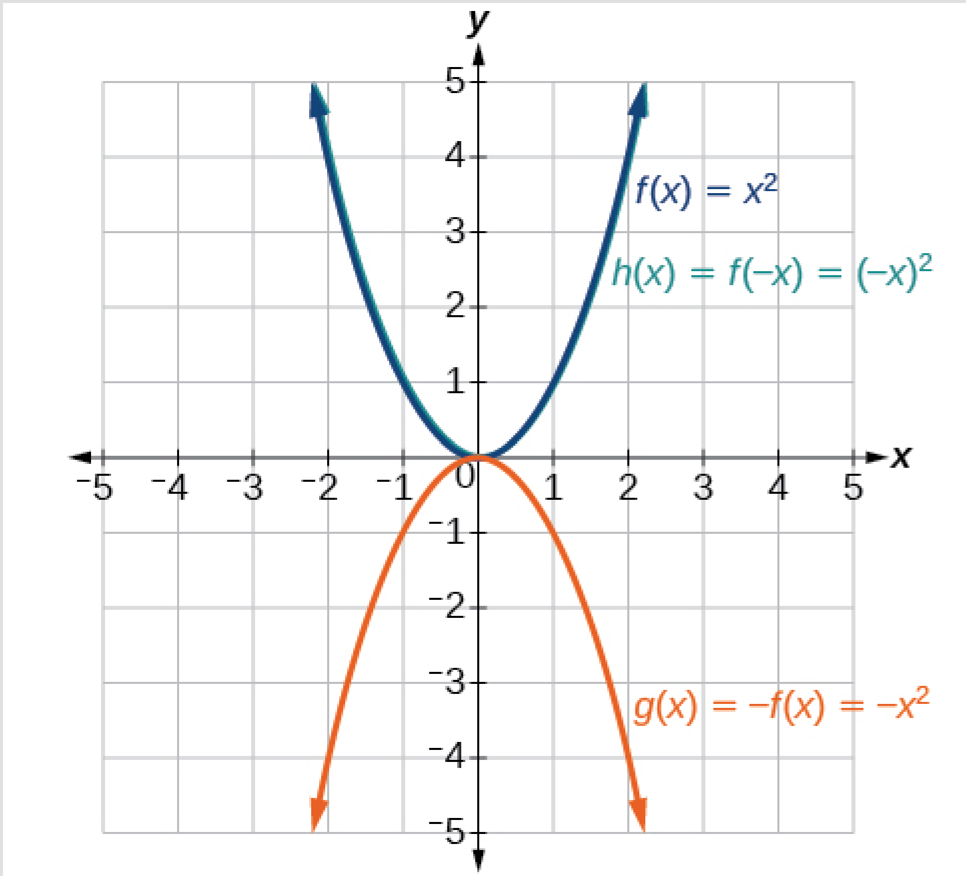


Figure 3-27

Example 4: Graphing Functions



Notice: $g\left(x\right)=f\left(-x\right)$ looks the same as $f\left(x\right)$.

3.4 Example Solutions

Example 1: Determining whether a Function Is Even, Odd, or Neither

1. Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f\left(-x\right)=\left(-x\right)^3+2\left(-x\right)=-x^3-2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f\left(-x\right)=-\left(-x^3-2x\right)=x^3+2x$$

Because $-f\left(-x\right)=f\left(x\right)$, this is an odd function.

2. even

3.5 Example Solutions

Example 1: Graphing a Vertical Stretch

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 3-28.

If we choose four reference points, (0, 1), (3, 3), (6, 2) and (7, 0) we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0,\ 1) \to (0,\ 2)$$
$$(3,\ 3) \to (3,\ 6)$$
$$(6,\ 2) \to (6,\ 4)$$
$$(7,\ 0) \to (7,\ 0)$$

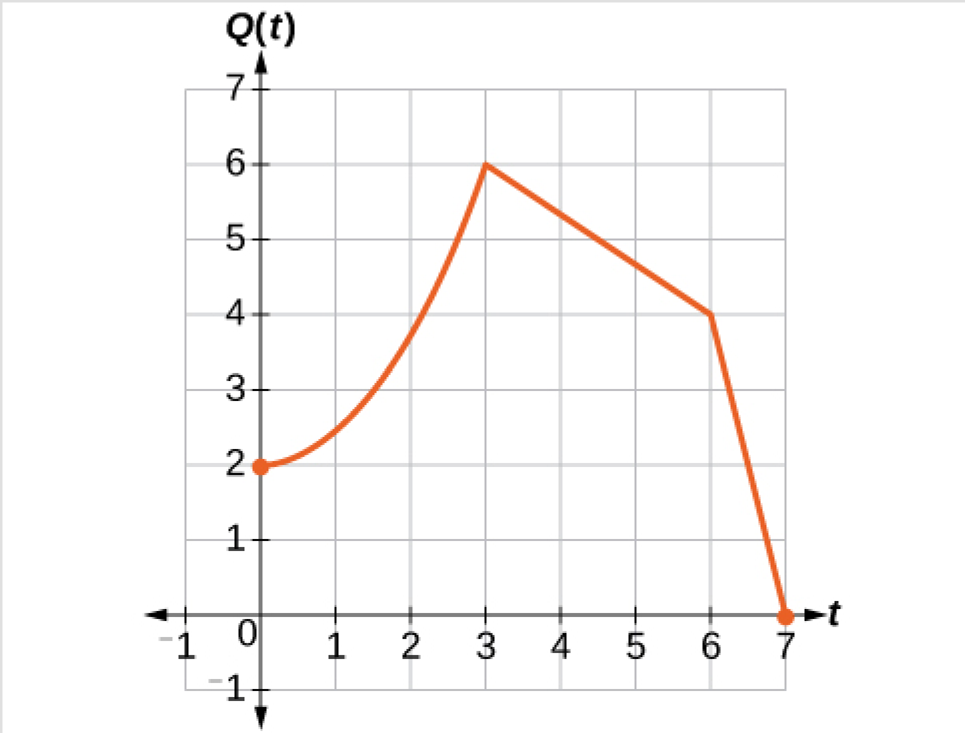


Figure 3-28

Symbolically, the relationship is written as

$$Q\left(t\right) = 2P\left(t\right)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

Example 2: Finding a Vertical Compression of a Tabular Function

1. The formula $g\left(x\right) = \dfrac{1}{2}f\left(x\right)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f\left(4\right) = 3$. Then

$$g(4) = \frac{1}{2} f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 12.

Table 12

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

2.

x	2	4	6	8
$g(x)$	9	12	15	0

Example 3: Recognizing a Vertical Stretch

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4} f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4} f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4} f(x) = \frac{1}{4} x^3$$

Example 4: Vertical Stretch

$g(x) = 3x - 2$

Example 5: Graphing a Horizontal Compression

Symbolically, we could write

$$R(1) = P(2),$$
$$R(2) = P(4), \text{ and in general,}$$
$$R(t) = P(2t).$$

See Figure 3-29 for a graphical comparison of the original population and the compressed population.

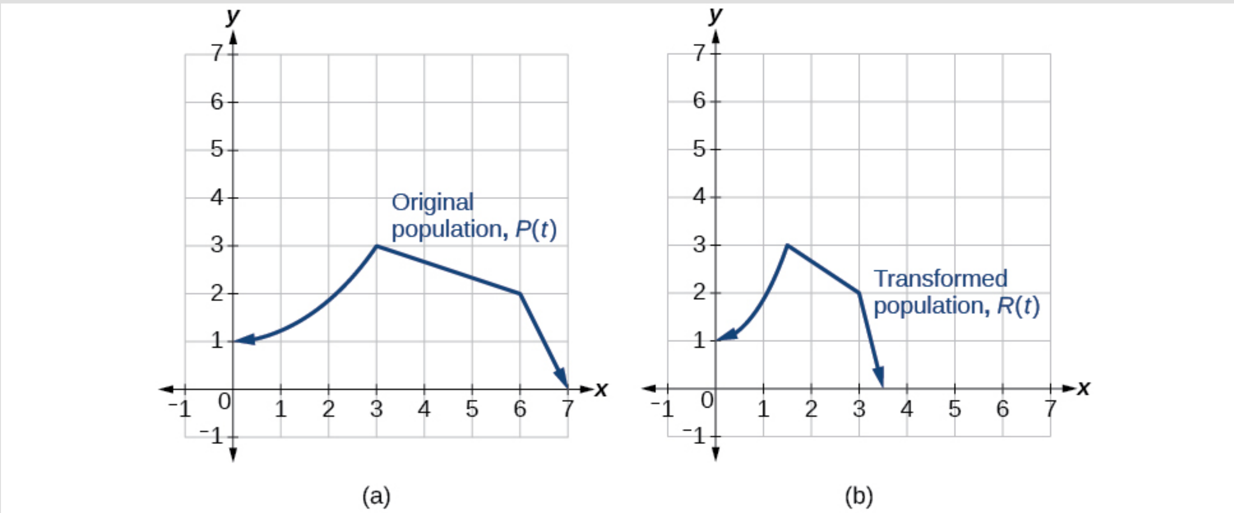


Figure 3-29: (a) Original population graph (b) Compressed population graph

Example 6: Finding a Horizontal Stretch for a Tabular Function

The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f\left(\frac{1}{2} \cdot 2\right) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f\left(\frac{1}{2} \cdot 4\right) = f(2) = 1$$

We do the same for the other values to produce Table 13.

Table 13

x	4	8	12	16
$g(x)$	1	3	7	11

Figure 3-30 shows the graphs of both of these sets of points.

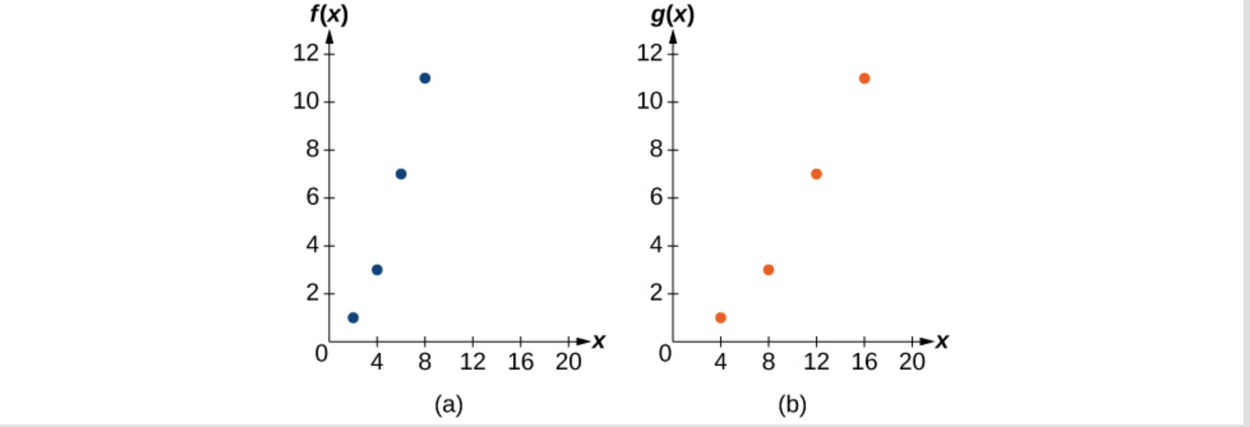


Figure 3-30

Example 7: Recognizing a Horizontal Compression on a Graph

The graph of $g\left(x\right)$ looks like the graph of $f\left(x\right)$ horizontally compressed. Because $f\left(x\right)$ ends at $\left(6,4\right)$ and $g\left(x\right)$ ends at $\left(2,4\right)$, we can see that the x-values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right)=2$. We might also notice that $g\left(2\right)=f\left(6\right)$ and $g\left(1\right)=f\left(3\right)$. Either way, we can describe this relationship as $g\left(x\right)=f\left(3x\right)$. This is a horizontal compression by $\frac{1}{3}$.

Example 8: Horizontal Stretch

$g\left(x\right)=f\left(\frac{1}{3}x\right)$ so using the square root function we get $g\left(x\right)=\sqrt{\frac{1}{3}x}$

Example 9: Finding a Triple Transformation of a Tabular Function

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f\left(3x\right)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. See Table 14.

Table 14

x	2	4	6	8
$f\left(3x\right)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 15.

Table 15

x	2	4	6	8
$2f\left(3x\right)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 16.

Table 16

x	2	4	6	8
$g\left(x\right)=2f\left(3x\right)+1$	21	29	31	35

Example 10: Finding a Triple Transformation of a Graph

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point (0,2) remains at (0,2) while the point (2,0) will stretch to (4,0). See Figure 3-31.

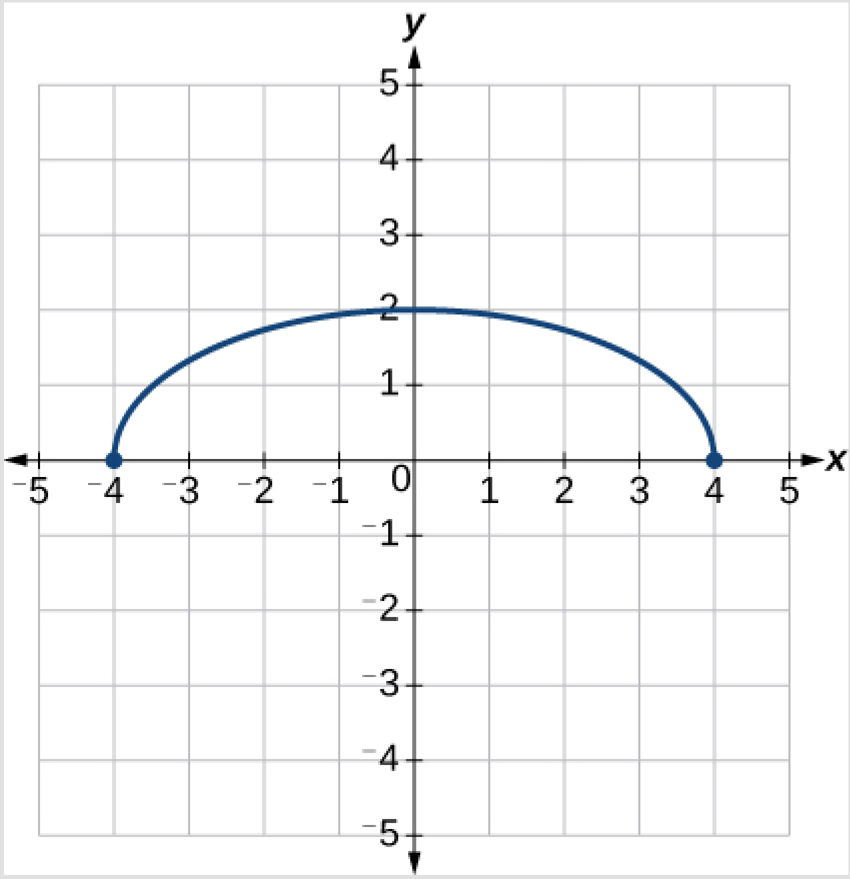


Figure 3-31

Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See Figure 3-32.

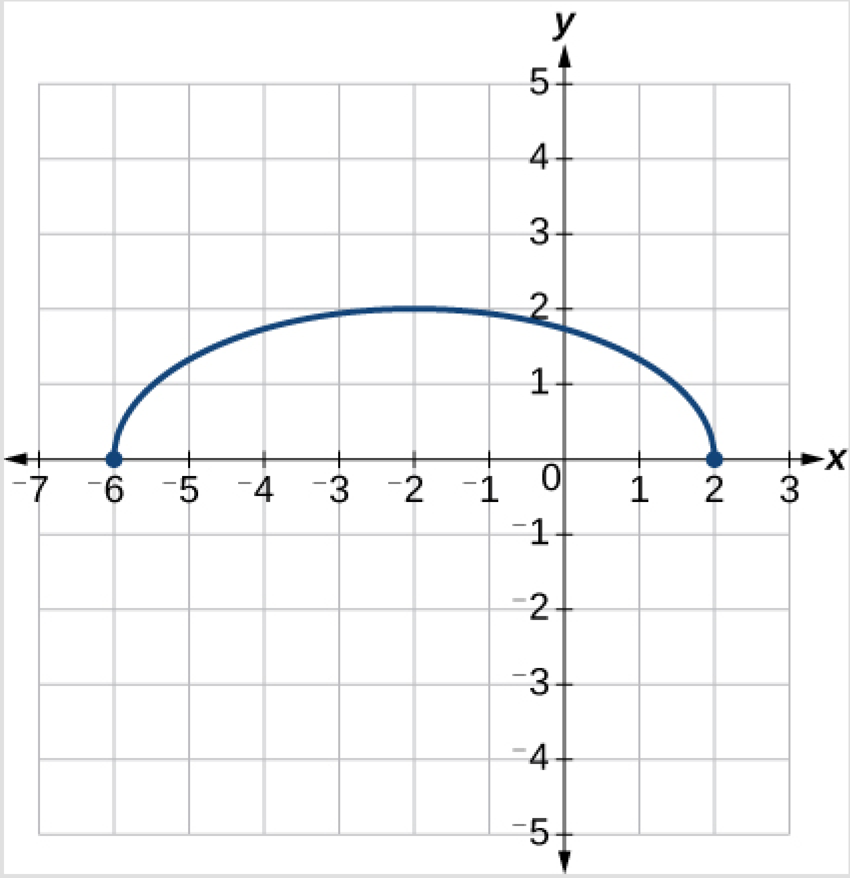


Figure 3-32

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 3-33.

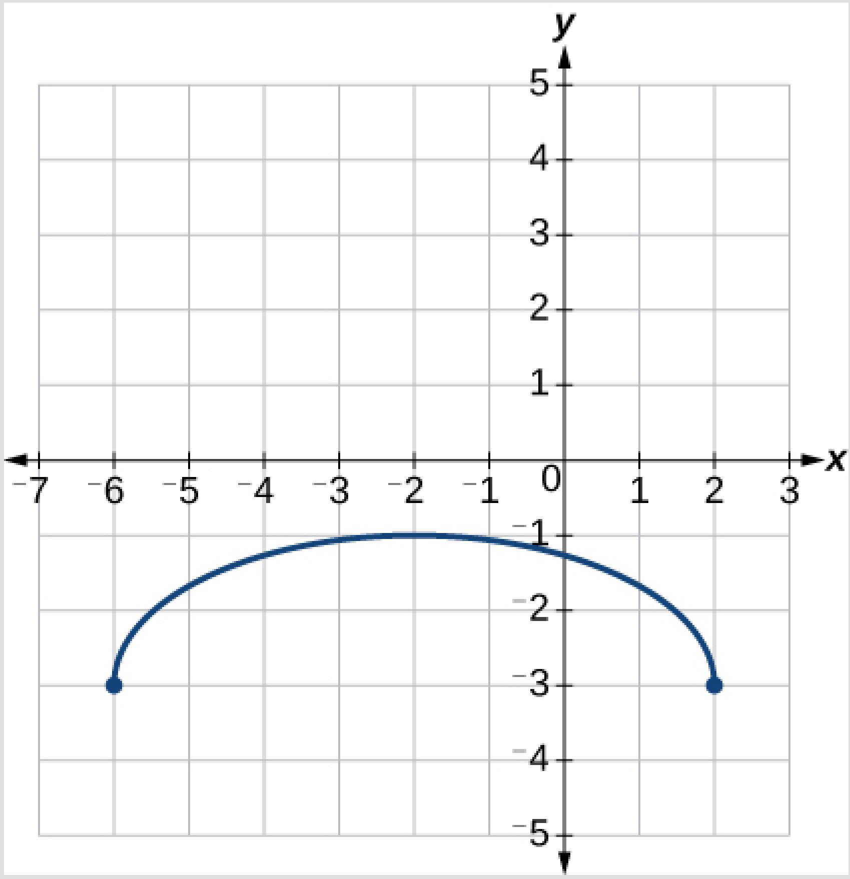


Figure 3-33

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3.9 Practice Question Solutions

Verbal Question Solutions

1. A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

3. A horizontal compression results when a constant greater than 1 is multiplied by the input. A vertical compression results when a constant between 0 and 1 is multiplied by the output.

5. For a function f, substitute $(-x)$ for (x) in $f(x)$. Simplify. If the resulting function is the same as the original function, $f(-x) = f(x)$ then the function is even. If the resulting function is the opposite of the original function, $f(-x) = -f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even.

Algebraic Question Solutions

7. $g(x) = |x-1| - 3$

9. $g(x) = \dfrac{1}{(x+4)^2} + 2$

11. The graph of $f(x+43)$ is a horizontal shift to the left 43 units of the graph of f.

13. The graph of $f(x-4)$ is a horizontal shift to the right 4 units of the graph of f.

15. The graph of $f(x)+8$ is a vertical shift up 8 units of the graph of f.

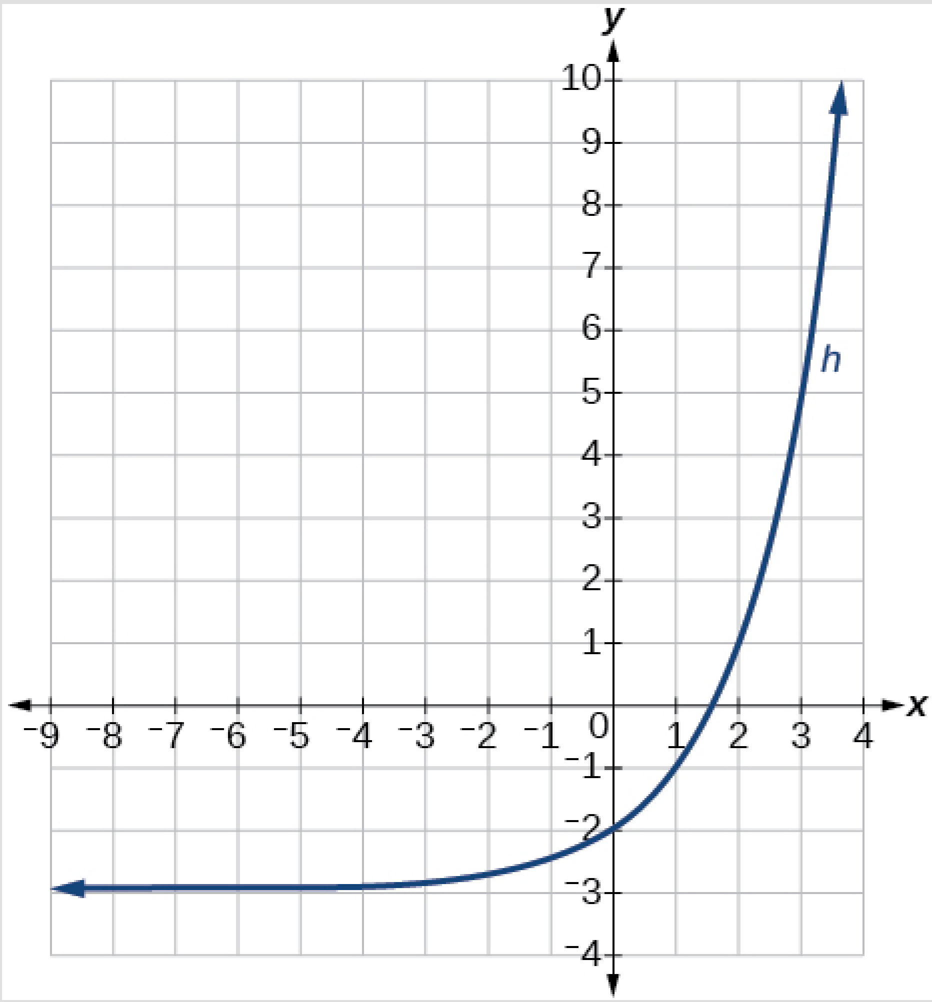
17. The graph of $f(x)-7$ is a vertical shift down 7 units of the graph of f.

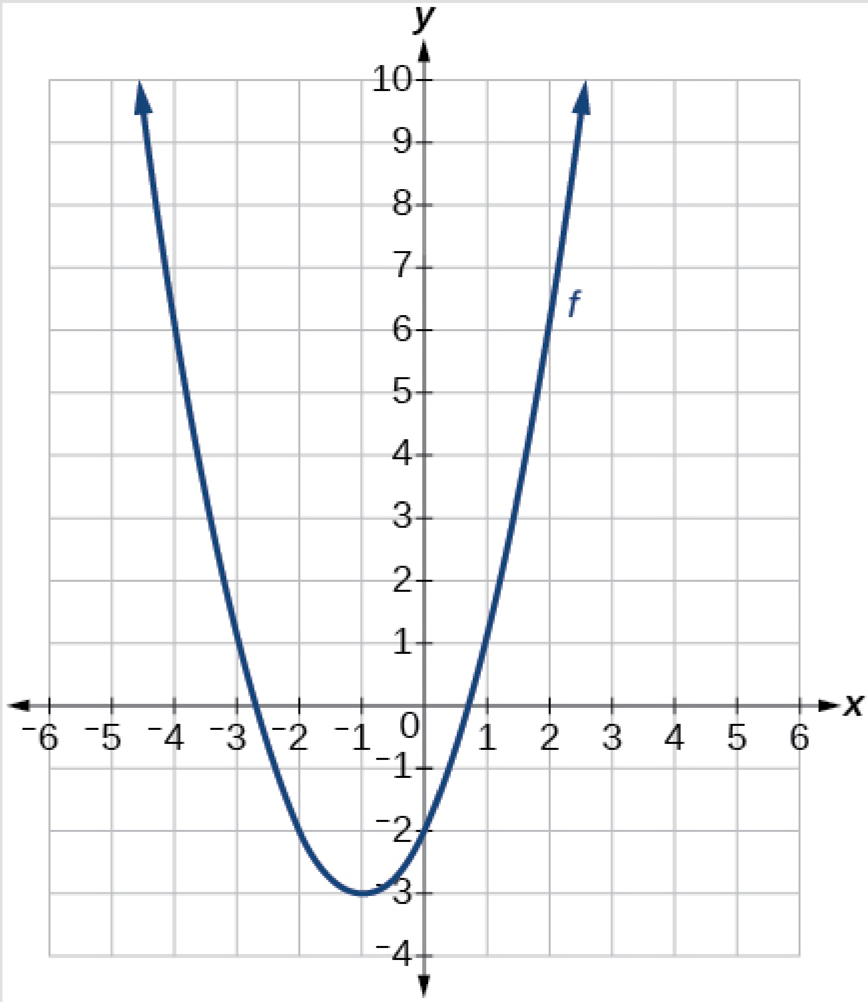
19. The graph of $f(x+4)-1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of f.

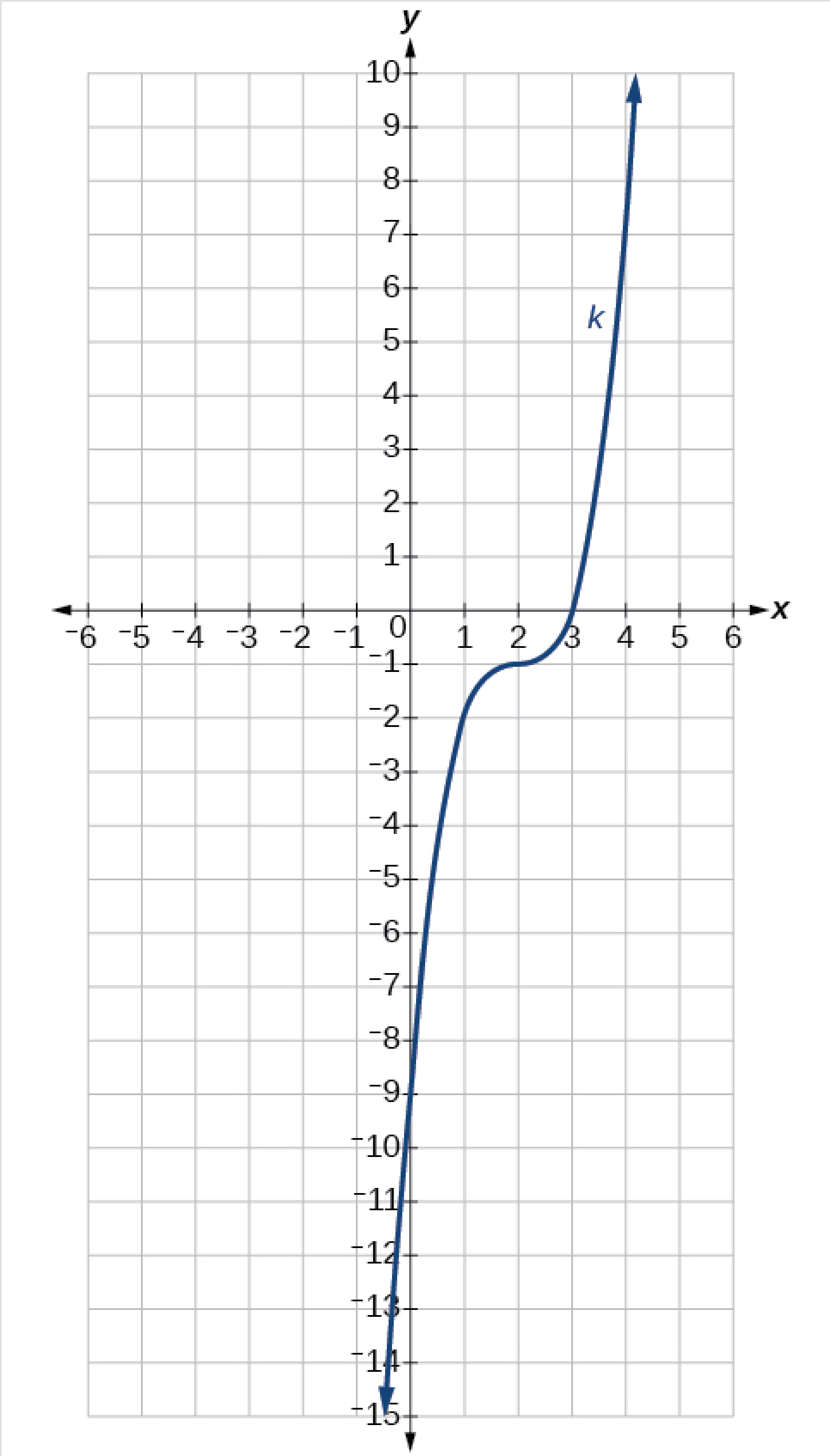
21. decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$

23. decreasing on $(0, \infty)$

Graphical Question Solutions

25. 

27. 

29. 

Numeric Question Solutions

31. $g(x) = f(x-1), h(x) = f(x) + 1$

33. $f(x) = |x-3| - 2$

35. $f(x) = \sqrt{x+3} - 1$

37. $f(x) = (x-2)^2$

39. $f(x) = |x+3| - 2$

41. $f(x) = -\sqrt{x}$

43. $f(x) = -(x+1)^2 + 2$

45. $f(x) = \sqrt{-x} + 1$

47. even

49. odd

51. even

53. The graph of g is a vertical reflection (across the x -axis) of the graph of f.

55. The graph of g is a vertical stretch by a factor of 4 of the graph of f.

57. The graph of g is a horizontal compression by a factor of $\frac{1}{5}$ of the graph of f.

59. The graph of g is a horizontal stretch by a factor of 3 of the graph of f.

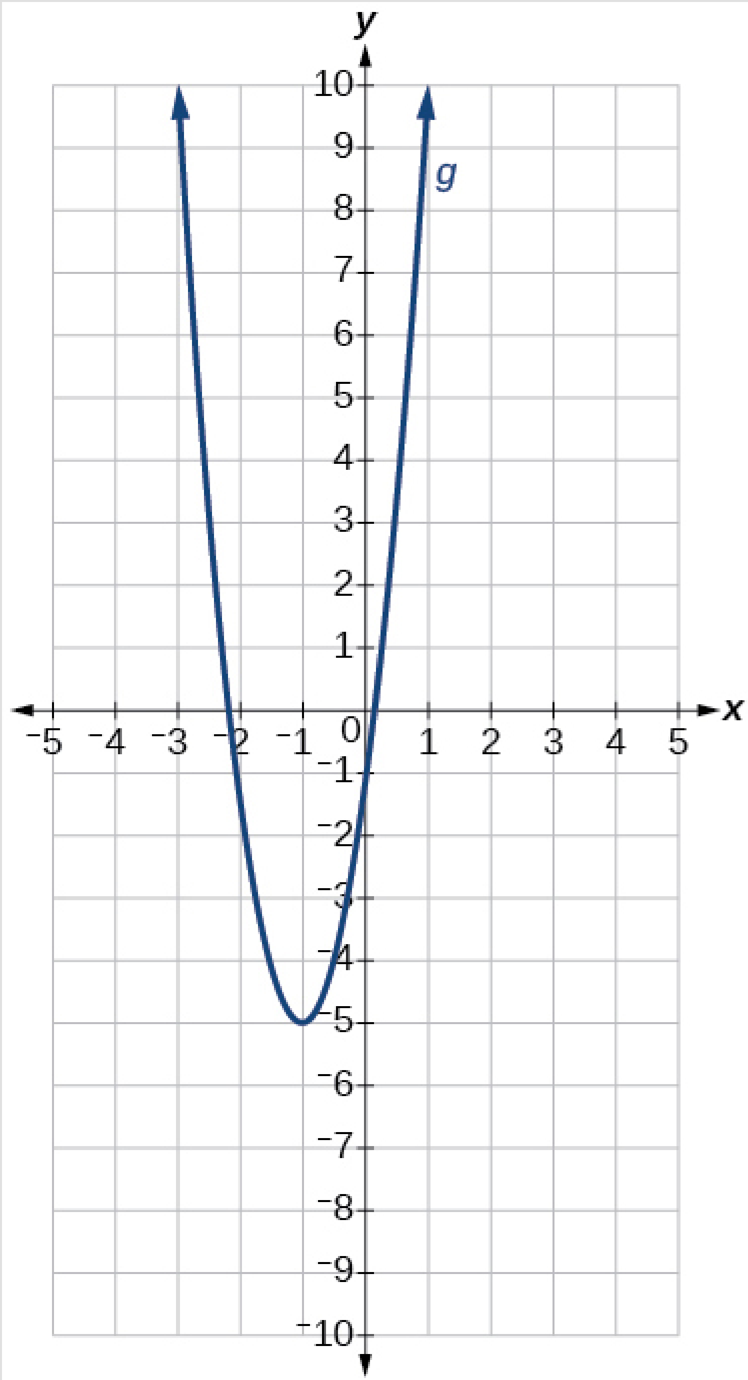
61. The graph of g is a horizontal reflection across the y -axis and a vertical stretch by a factor of 3 of the graph of f.

63. $g(x) = |-4x|$

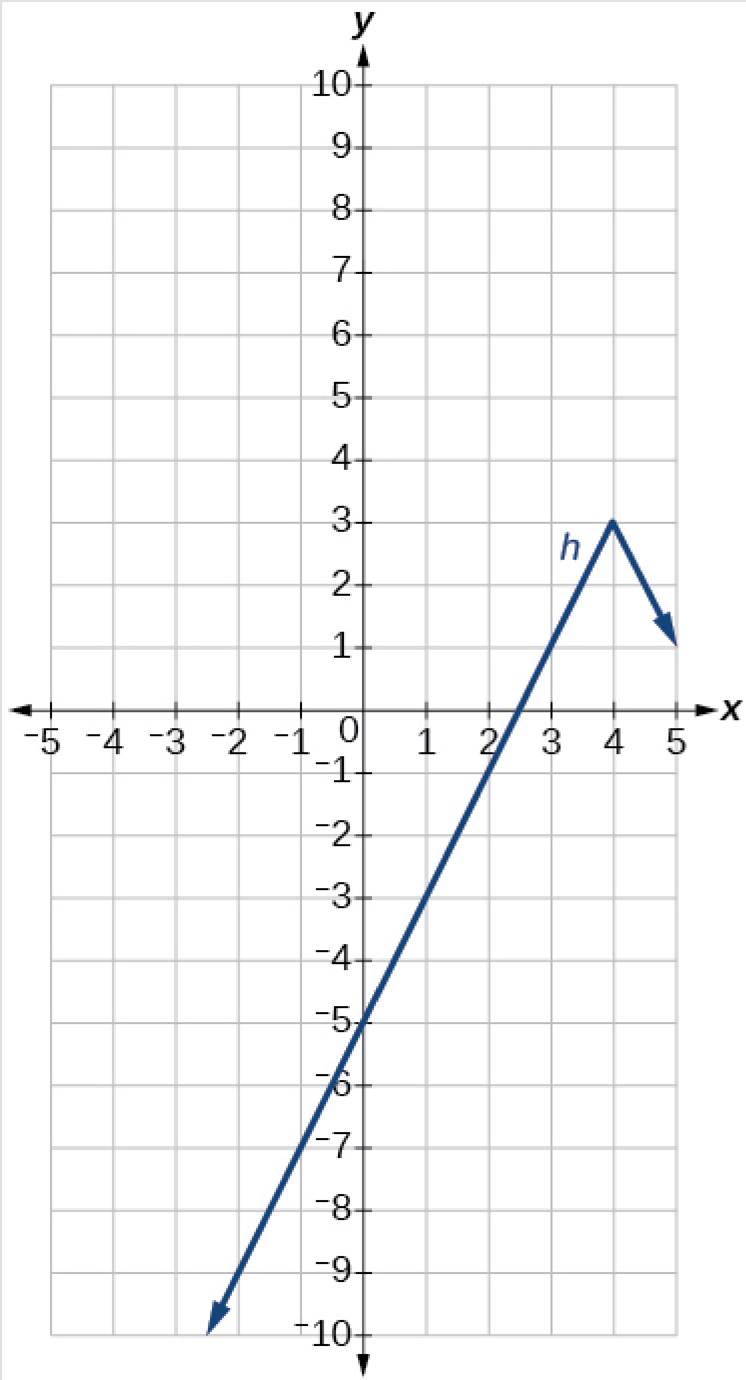
65. $g(x) = \dfrac{1}{3(x+2)^2} - 3$

67. $g(x) = \frac{1}{2}(x-5)^2 + 1$

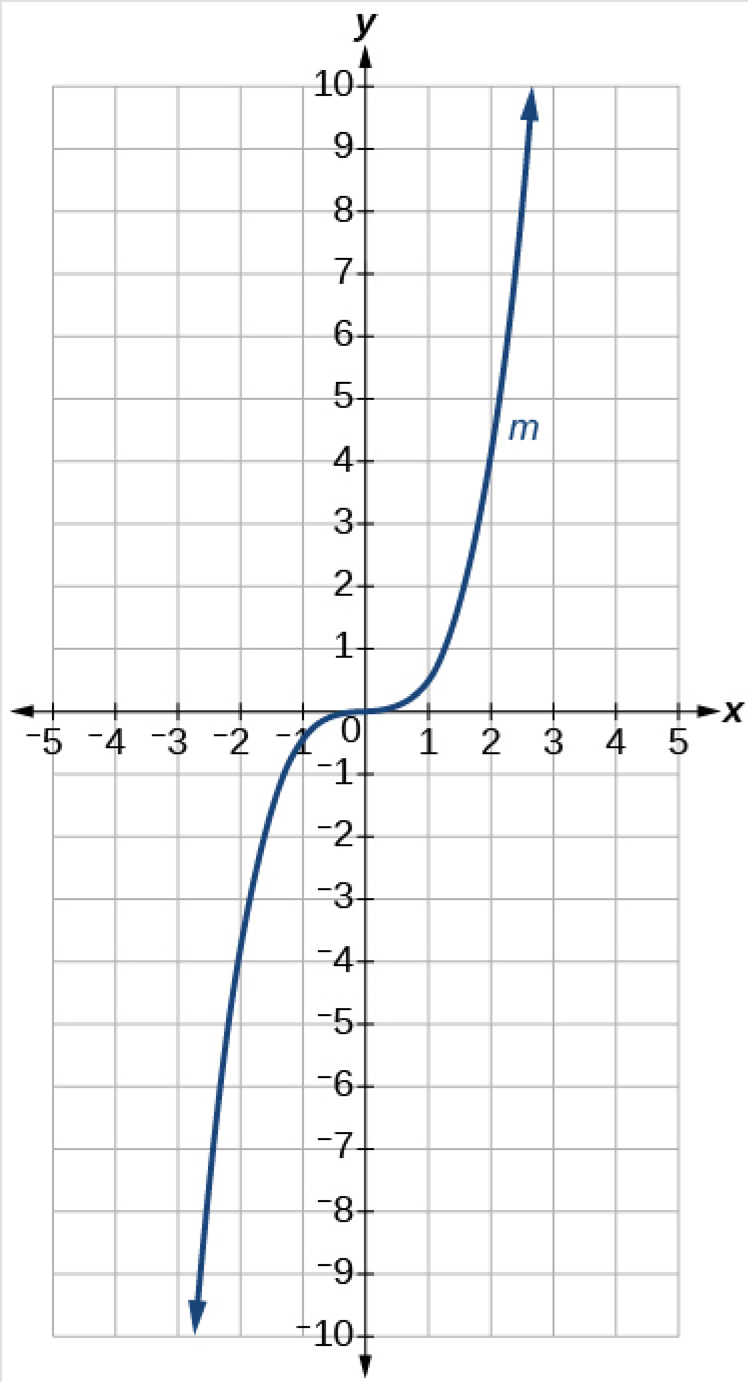
69. The graph of the function $f(x) = x^2$ is shifted to the left 1 unit, stretched vertically by a factor of 4, and shifted down 5 units.



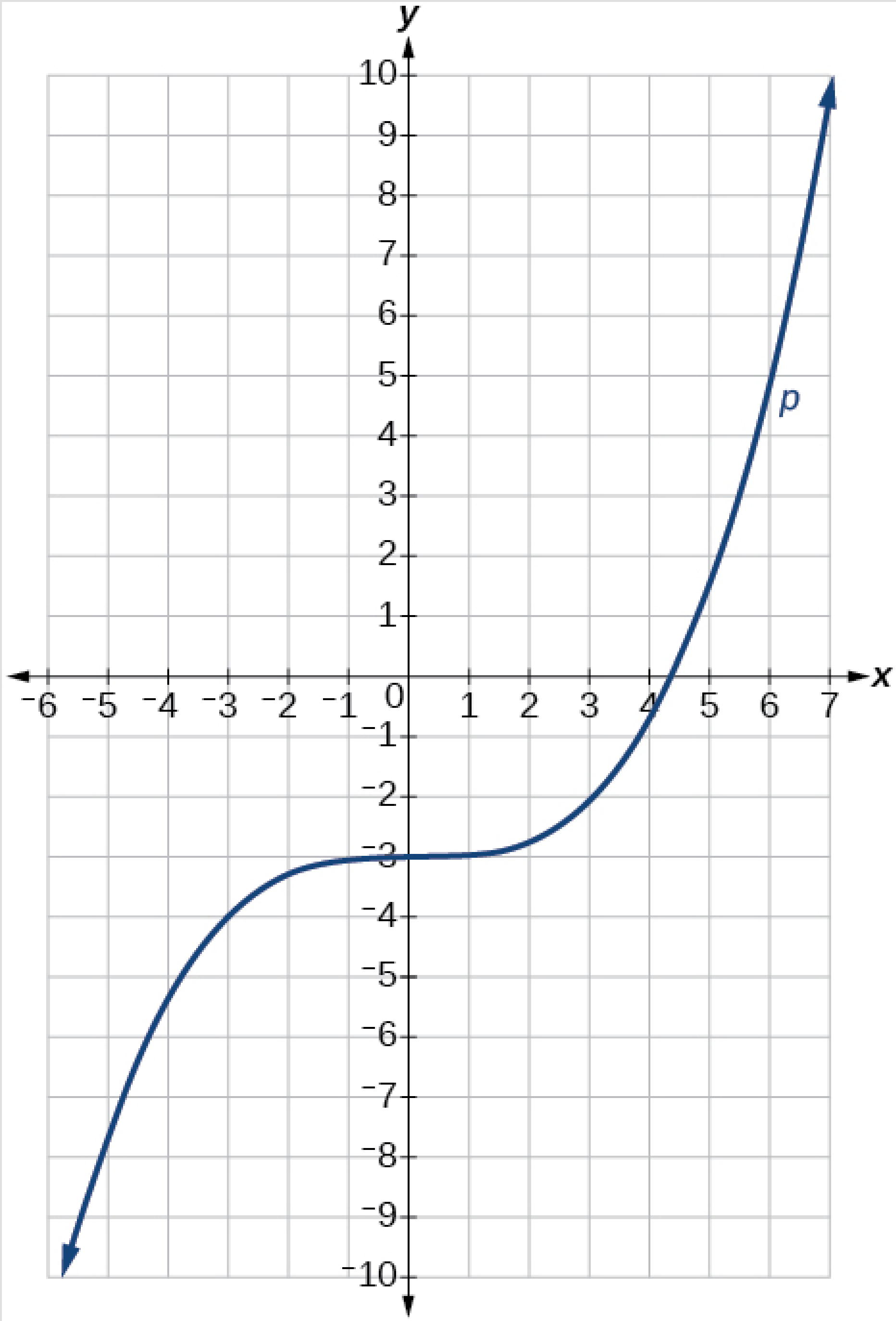
71. The graph of $f(x) = |x|$ is stretched vertically by a factor of 2, shifted horizontally 4 units to the right, reflected across the horizontal axis, and then shifted vertically 3 units up.



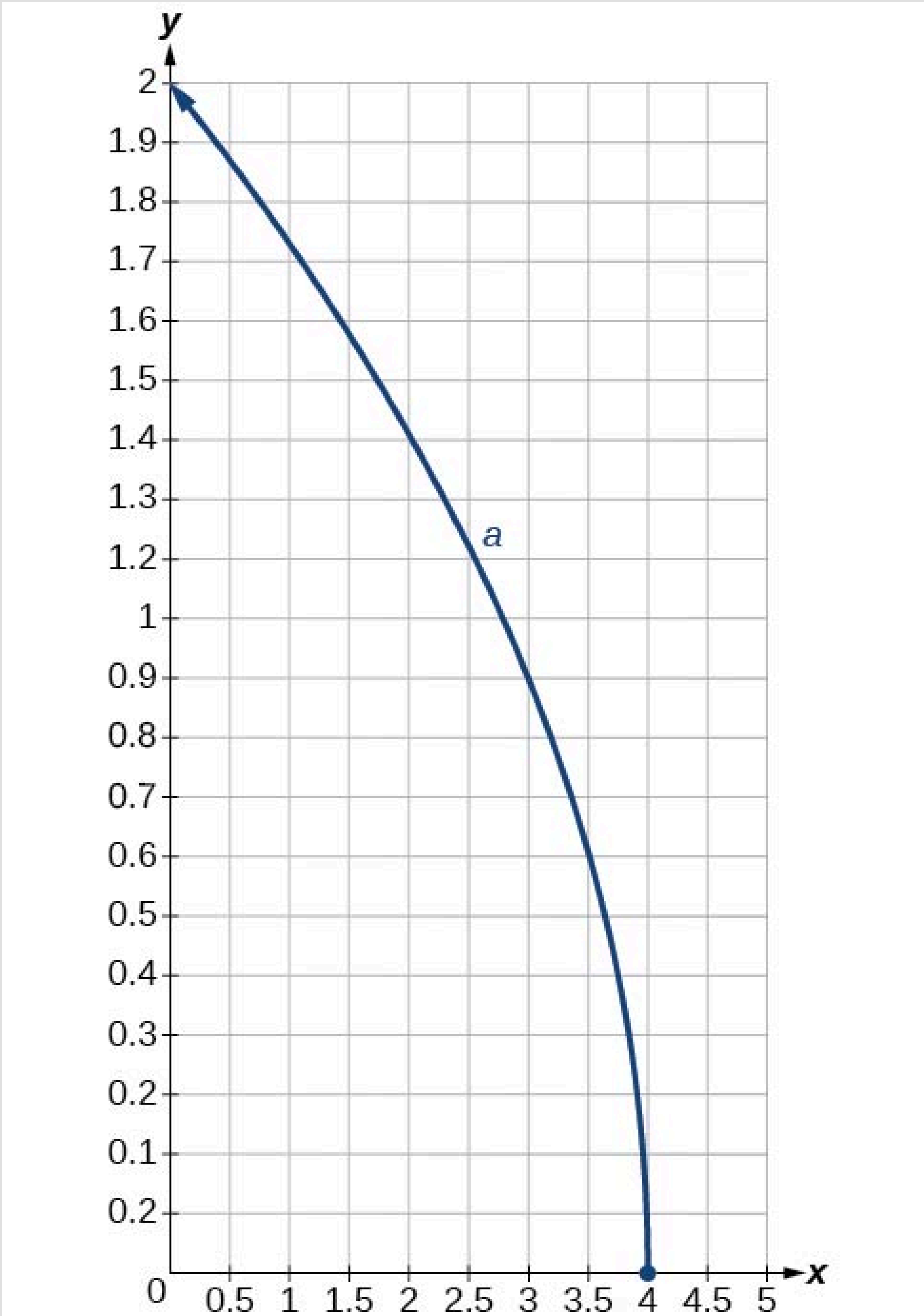
73. The graph of the function $f(x) = x^3$ is compressed vertically by a factor of $\frac{1}{2}$.

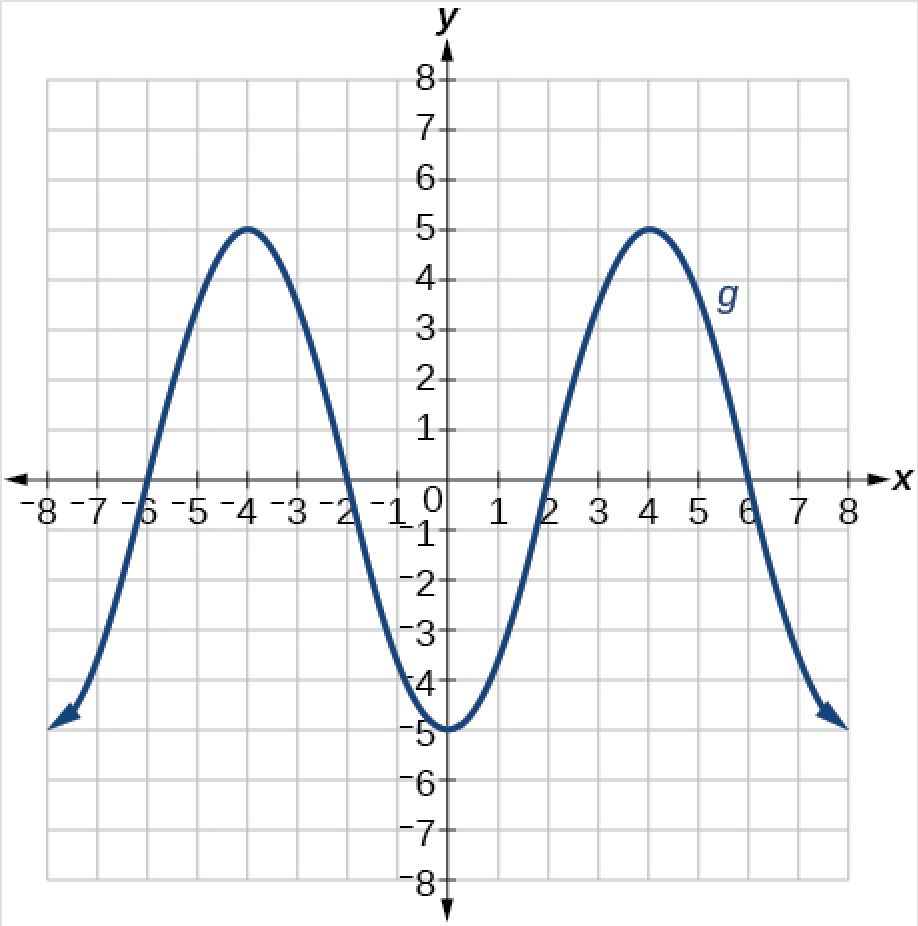


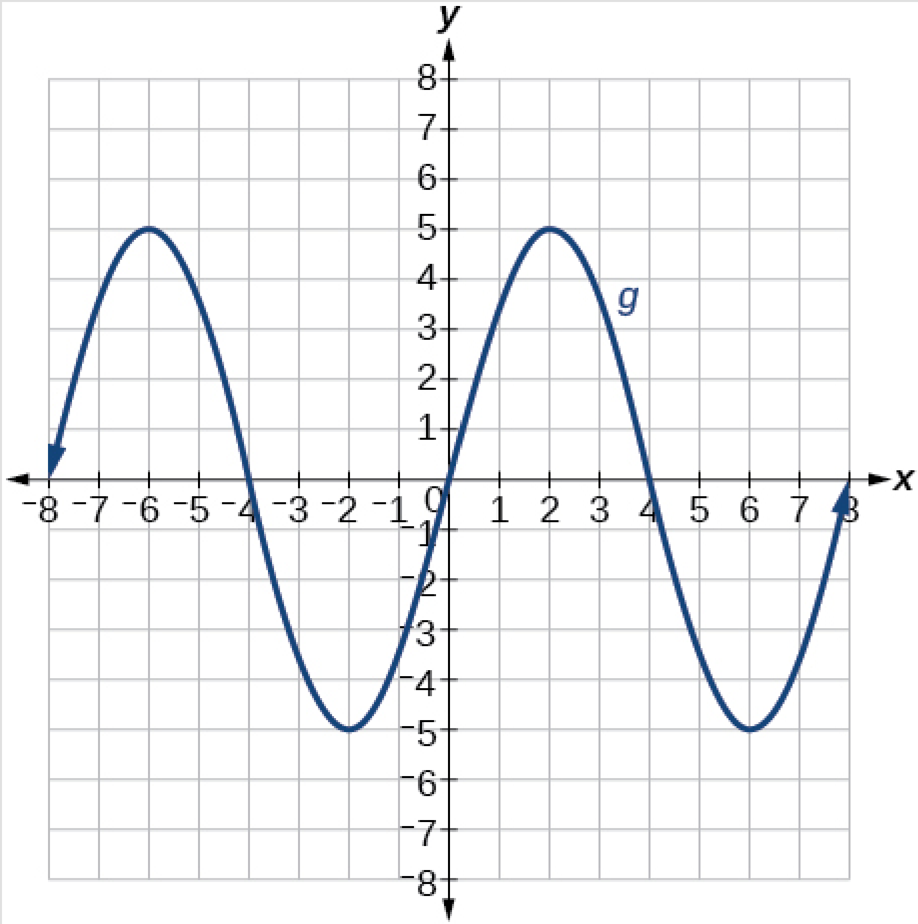
75. The graph of the function is stretched horizontally by a factor of 3 and then shifted vertically downward by 3 units.



77. The graph of $f(x) = \sqrt{x}$ is shifted right 4 units and then reflected across the vertical line $x = 4$.


y
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.2
0
0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
x
a

79. 

81. 

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CHAPTER 4: FACTORING POLYNOMIALS

4.1 Factoring Polynomials

Learning Objectives

In this section, you will:

- Factor the greatest common factor of a polynomial.
- Factor a trinomial.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor the sum and difference of cubes.
- Factor expressions using fractional or negative exponents.

Introduction Video

An interactive H5P element has been excluded from this version of the text. You can view it online here: https://ecampusontario.pressbooks.pub/math3080prep/?p=844#h5p-2

Factoring Polynomials

Imagine that we are trying to find the area of a lawn so that we can determine how much grass seed to purchase. The lawn is the green portion in Figure 4-1.

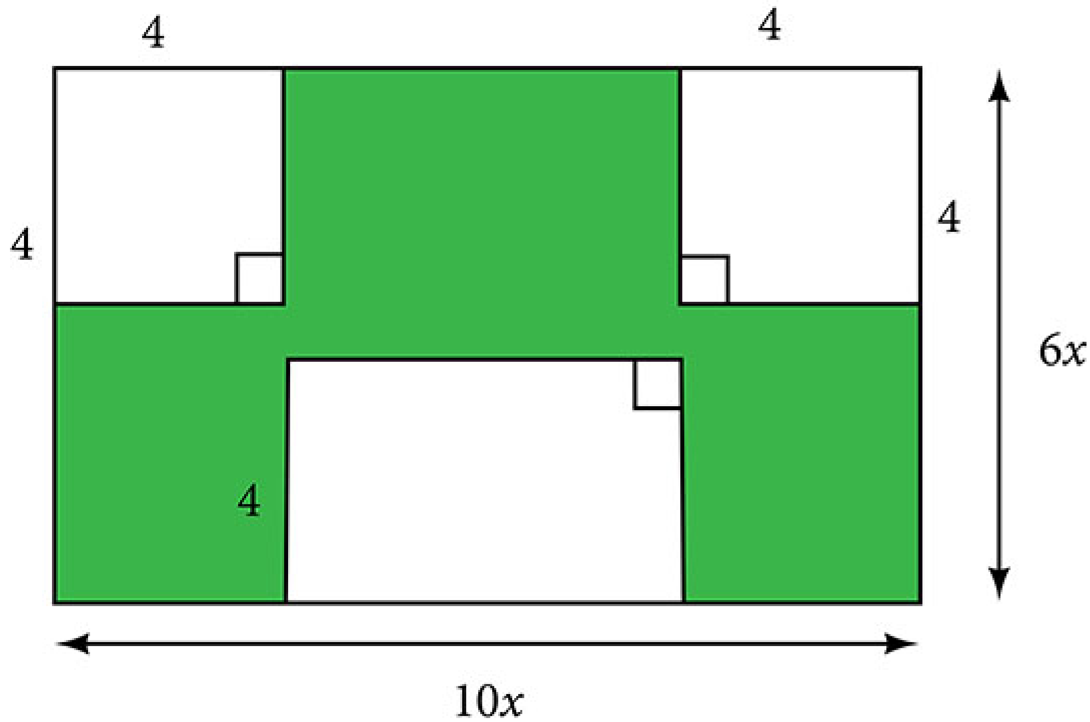

Figure 4-1

The area of the entire region can be found using the formula for the area of a rectangle.

$$\begin{aligned} A &= lw \\ &= 10x \times 6x \\ &= 60x^2 \quad \text{units}^2 \end{aligned}$$

The areas of the portions that do not require grass seed need to be subtracted from the area of the entire region. The two square regions each have an area of $A = s^2 = 4^2 = 16$ units^2. The other rectangular region has one side of length $10x - 8$ and one side of length 4, giving an area of $A = lw = 4(10x - 8) = 40x - 32$ units^2. So the region that must be subtracted has an area of $2(16) + 40x - 32 = 40x$ units^2.

The area of the region that requires grass seed is found by subtracting $60x^2 - 40x$ units^2. This area can also be expressed in factored form as $20x(3x - 2)$ units^2. We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

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4.2 Factoring the Greatest Common Factor of a Polynomial

When we study fractions, we learn that the **greatest common factor** (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Greatest Common Factor

The greatest common factor (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.

How To

Given a polynomial expression, factor out the greatest common factor.

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

Example 1: Factoring the Greatest Common Factor

1. Factor $6x^3y^3 + 45x^2y^2 + 21xy$.

Analysis

After factoring, we can check our work by multiplying. Use the distributive property to confirm that $(3xy)\left(2x^2y^2 + 15xy + 7\right) = 6x^3y^3 + 45x^2y^2 + 21xy$.

2. Factor $x\left(b^2 - a\right) + 6\left(b^2 - a\right)$ by pulling out the GCF.

Solution

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4.3 Factoring a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial x^2+5x+6 has a GCF of 1, but it can be written as the product of the factors $(x+2)$ and $(x+3)$.

Trinomials of the form x^2+bx+c can be factored by finding two numbers with a product of c and a sum of b. The trinomial $x^2+10x+16$, for example, can be factored using the numbers 2 and 8 because the product of those numbers is 16 and their sum is 10. The trinomial can be rewritten as the product of $(x+2)$ and $(x+8)$.

Factoring a Trinomial with Leading Coefficient 1

A trinomial of the form x^2+bx+c can be written in factored form as $(x+p)\,(x+q)$ where $pq=c$ and $p+q=b$.

Question & Answer

Can every trinomial be factored as a product of binomials?

No. Some polynomials cannot be factored. These polynomials are said to be prime.

How To

Given a trinomial in the form $x^2 + bx + c$, factor it.

1. List factors of c.
2. Find p and q, a pair of factors of c with a sum of b.
3. Write the factored expression $(x+p)(x+q)$.

Example 1: Factoring a Trinomial with Leading Coefficient 1

1. Factor $x^2 + 2x - 15$.
2. Factor $x^2 - 7x + 6$.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(x-3)(x+5) = x^2 + 2x - 15$.

Solution

Question & Answer

Does the order of the factors matter?

No. Multiplication is commutative, so the order of the factors does not matter.

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4.4 Factoring by Grouping

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can **factor by grouping** by dividing the x term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial $2x^2+5x+3$ can be rewritten as $(2x+3)(x+1)$ using this process. We begin by rewriting the original expression as $2x^2+2x+3x+3$ and then factor each portion of the expression to obtain $2x(x+1)+3(x+1)$. We then pull out the GCF of $(x+1)$ to find the factored expression.

Factor by Grouping

To factor a trinomial in the form ax^2+bx+c by grouping, we find two numbers with a product of ac and a sum of b. We use these numbers to divide the x term into the sum of two terms and factor each portion of the expression separately, then factor out the GCF of the entire expression.

How To

Given a trinomial in the form ax^2+bx+c, factor by grouping.

1. List factors of ac.
2. Find p and q, a pair of factors of ac with a sum of b.
3. Rewrite the original expression as $ax^2+px+qx+c$.
4. Pull out the GCF of ax^2+px.
5. Pull out the GCF of $qx+c$.
6. Factor out the GCF of the expression.

Example 1: Factoring a Trinomial by Grouping

1. Factor $5x^2 + 7x - 6$ by grouping.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(5x - 3)(x + 2) = 5x^2 + 7x - 6$.

2. Factor the following:
   a. $2x^2 + 9x + 9$
   b. $6x^2 + x - 1$

Solution

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4.5 Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.

$$\begin{array}{rcl} a^2+2ab+b^2 & = & (a+b)^2 \\ & \text{and} & \\ a^2-2ab+b^2 & = & (a-b)^2 \end{array}$$

We can use this equation to factor any perfect square trinomial.

Perfect Square Trinomials

A perfect square trinomial can be written as the square of a binomial:

$$a^2+2ab+b^2=(a+b)^2$$

How To

Given a perfect square trinomial, factor it into the square of a binomial.

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of ab.
3. Write the factored form as $(a+b)^2$.

Example 1: Factoring a Perfect Square Trinomial

1. Factor $25x^2 + 20x + 4$.
2. Factor $49x^2 - 14x + 1$.

Solution

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4.6 Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be written as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.

$$a^2 - b^2 = (a+b)(a-b)$$

We can use this equation to factor any differences of squares.

Difference of Squares

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a+b)(a-b)$$

How To

Given a difference of squares, factor it into binomials.

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a+b)(a-b)$.

Example 1: Factoring a Difference of Squares

1. Factor $9x^2 - 25$.
2. Factor $81y^2 - 100$.

Solution

Question & Answer

Is there a formula to factor the sum of squares?

No. A sum of squares cannot be factored.

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4.7 Factoring the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

Similarly, the sum of cubes can be factored into a binomial and a trinomial, but with different signs.

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: **S**ame **O**pposite **A**lways **P**ositive. For example, consider the following example.

$$x^3-2^3=(x-2)(x^2+2x+4)$$

The sign of the first 2 is the *same* as the sign between x^3-2^3. The sign of the $2x$ term is *opposite* the sign between x^3-2^3. And the sign of the last term, 4, is *always positive*.

Sum and Difference of Cubes

We can factor the sum of two cubes as

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

We can factor the difference of two cubes as

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

How To

Given a sum of cubes or difference of cubes, factor it.

1. Confirm that the first and last term are cubes, $a^3 + b^3$ or $a^3 - b^3$
2. For a sum of cubes, write the factored form as $(a + b)(a^2 - ab + b^2)$. For a difference of cubes, write the factored form as $(a - b)(a^2 + ab + b^2)$.

Example 1: Factoring a Sum of Cubes

1. Factor $x^3 + 512$.

Analysis

After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.

2. Factor the sum of cubes: $216a^3 + b^3$.

Solution

Example 2: Factoring a Difference of Cubes

1. Factor $8x^3 - 125$.

Analysis

Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

2. Factor the difference of cubes: $1000x^3 - 1$

Solution

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4.8 Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2x^{\frac{1}{4}} + 5x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and being rewritten as $x^{\frac{1}{4}}(2 + 5x^{\frac{1}{2}})$.

Example 1: Factoring an Expression with Fractional or Negative Exponents

1. Factor $3x(x + 2)^{\frac{-1}{3}} + 4(x + 2)^{\frac{2}{3}}$.
2. Factor $2(5a - 1)^{\frac{3}{4}} + 7a(5a - 1)^{\frac{-1}{4}}$

Solution

Access for free at https://openstax.org/books/algebra-and-trigonometry/pages/1-introduction-to-prerequisites

4.9 Review and Summary

Additional Information

Access these online video resources for additional instruction and practice with factoring polynomials.

- Identify GCF
- Factor Trinomials when a Equals 1
- Factor Trinomials when a is not equal to 1
- Factor Sum or Difference of Cubes

Key Equations

Perfect square trinomial	$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2$ $a^2 + 2ab + b^2 = (a+b)^2$
Difference of squares	$(a+b)(a-b) = a^2 - b^2$ $a^2 - b^2 = (a+b)(a-b)$
Sum of cubes	$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$
Difference of cubes	$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

Key Terms

Binomial – a polynomial containing two terms

Coefficient – any real number a_i in a polynomial in the form $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$

Difference of Squares – the binomial that results when a binomial is multiplied by a binomial with the same terms, but the opposite sign

Distributive Property – the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols, $a \cdot (b + c) = a \cdot b + a \cdot c$

Factor by Grouping – a method for factoring a trinomial in the form $ax^2 + bx + c$ by dividing the x term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression

Greatest Common Factor – the largest polynomial that divides evenly into each polynomial

Leading Coefficient – the coefficient of the leading term

Leading Term – the term containing the highest degree

Perfect Square Trinomial – the trinomial that results when a binomial is squared

Trinomial – a polynomial containing three terms

Key Concepts

- The greatest common factor, or GCF, can be factored out of a polynomial. Checking for a GCF should be the first step in any factoring problem. See 4.2 Factoring the Greatest Common Factor of a Polynomial.
- Trinomials with leading coefficient 1 can be factored by finding numbers that have a product of the third term and a sum of the second term. See 4.3 Factoring a Trinomial with Leading Coefficient 1.
- Trinomials can be factored using a process called factoring by grouping. See 4.4 Factoring by Grouping.
- Perfect square trinomials and the difference of squares are special products and can be factored using equations. See 4.6 Factoring a Difference of Squares.
- The sum of cubes and the difference of cubes can be factored using equations. See 4.7 Factoring the Sum and Difference of Cubes.
- Polynomials containing fractional and negative exponents can be factored by pulling out a GCF. See 4.8 Factoring Expressions with Fractional or Negative Exponents.

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4.10 Chapter Exercises

Verbal Questions

1. If the terms of a polynomial do not have a GCF, does that mean it is not factorable? Explain.
2. A polynomial is factorable, but it is not a perfect square trinomial or a difference of two squares. Can you factor the polynomial without finding the GCF?
3. How do you factor by grouping?

Odd Number Verbal Solutions

Algebraic Questions

For the following exercises, find the greatest common factor.

4. $14x+4xy-18xy^2$
5. $49mb^2-35m^2ba+77ma^2$
6. $30x^3y-45x^2y^2+135xy^3$
7. $200p^3m^3-30p^2m^3+40m^3$
8. $36j^4k^2-18j^3k^3+54j^2k^4$
9. $6y^4-2y^3+3y^2-y$

For the following exercises, factor by grouping.

10. $6x^2+5x-4$
11. $2a^2+9a-18$
12. $6c^2+41c+63$
13. $6n^2-19n-11$
14. $20w^2-47w+24$
15. $2p^2-5p-7$

For the following exercises, factor the polynomial.

16. $7x^2 + 48x - 7$

17. $10h^2 - 9h - 9$

18. $2b^2 - 25b - 247$

19. $9d^2 - 73d + 8$

20. $90v^2 - 181v + 90$

21. $12t^2 + t - 13$

22. $2n^2 - n - 15$

23. $16x^2 - 100$

24. $25y^2 - 196$

25. $121p^2 - 169$

26. $4m^2 - 9$

27. $361d^2 - 81$

28. $324x^2 - 121$

29. $144b^2 - 25c^2$

30. $16a^2 - 8a + 1$

31. $49n^2 + 168n + 144$

32. $121x^2 - 88x + 16$

33. $225y^2 + 120y + 16$

34. $m^2 - 20m + 100$

35. $25p^2 - 120p + 144$

36. $36q^2 + 60q + 25$

For the following exercises, factor the polynomials.

37. $x^3 + 216$

38. $27y^3 - 8$

39. $125a^3 + 343$

40. $b^3 - 8d^3$

41. $64x^3 - 125$

42. $729q^3 + 1331$

43. $125r^3 + 1,728s^3$

44. $4x(x - 1)^{-\frac{2}{3}} + 3(x - 1)^{\frac{1}{3}}$

45. $3c(2c + 3)^{-\frac{1}{4}} - 5(2c + 3)^{\frac{3}{4}}$

46. $3t(10t + 3)^{\frac{1}{3}} + 7(10t + 3)^{\frac{4}{3}}$

47. $14x(x + 2)^{-\frac{2}{5}} + 5(x + 2)^{\frac{3}{5}}$

48. $9y(3y - 13)^{\frac{1}{5}} - 2(3y - 13)^{\frac{6}{5}}$

49. $5z(2z-9)^{-\frac{3}{2}}+11(2z-9)^{-\frac{1}{2}}$

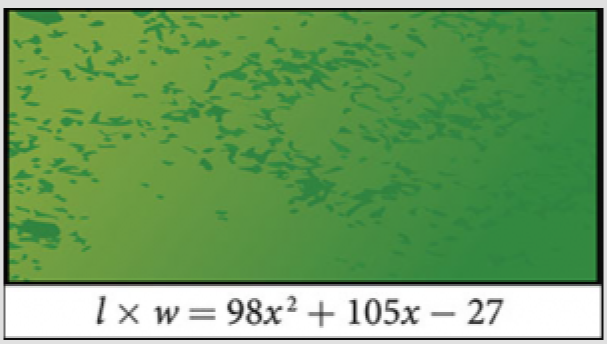
50. $6d(2d+3)^{-\frac{1}{6}}+5(2d+3)^{\frac{5}{6}}$

Odd Number Algebraic Solutions

Real-World Applications Questions

For the following exercises, consider this scenario:

Charlotte has appointed a chairperson to lead a city beautification project. The first act is to install statues and fountains in one of the city's parks. The park is a rectangle with an area of $98x^2+105x-2798x^2+105x-27$ m^2, as shown in the figure below. The length and width of the park are perfect factors of the area.



51. Factor by grouping to find the length and width of the park.

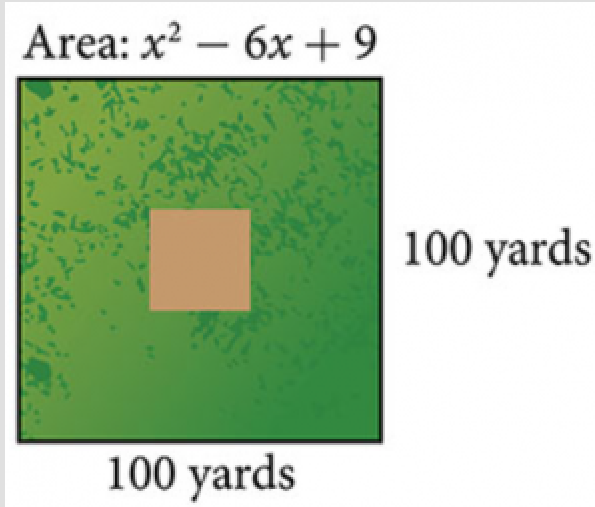
52. A statue is to be placed in the center of the park. The area of the base of the statue is $4x^2+12x+9m^2$

Factor the area to find the lengths of the sides of the statue.

53. At the northwest corner of the park, the city is going to install a fountain. The area of the base of the fountain is $9x^2-25m^2$. Factor the area to find the lengths of the sides of the fountain.

For the following exercise, consider the following scenario:

A school is installing a flagpole in the central plaza. The plaza is a square with side length 100 yd. as shown in the figure below. The flagpole will take up a square plot with area $x^2-6x+9x^2$.



54. Find the length of the base of the flagpole by factoring.

Odd Number Real World Applications Solutions

Extension Questions

For the following exercises, factor the polynomials completely.

55. $16x^4 - 200x^2 + 625$
56. $81y^4 - 256$
57. $16z^4 - 2,401a^4$
58. $5x(3x + 2)^{-\frac{2}{4}} + (12x + 8)^{\frac{3}{2}}$
59. $(32x^3 + 48x\hat{}2 - 162x - 243)^{-1}$

Odd Number Extension Solutions

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4.11 Chapter 4 Example Solutions

4.2 Example Solutions

Example 1: Factoring the Greatest Common Factor

1. First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of x^3, x^2, and x is x. (Note that the GCF of a set of expressions in the form x^n will always be the exponent of lowest degree.) And the GCF of y^3, y^2, and y is y. Combine these to find the GCF of the polynomial, $3xy$.

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that $3xy\left(2x^2y^2\right) = 6x^3y^3$, $3xy\left(15xy\right) = 45x^2y^2$, and $3xy\left(7\right) = 21xy$.

Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$$(3xy)\left(2x^2y^2 + 15xy + 7\right)$$

2. $\left(b^2 - a\right)(x + 6)$

4.3 Example Solutions

Example 1: Factoring a Trinomial with Leading Coefficient 1

1. We have a trinomial with leading coefficient 1, $b = 2$, and $c = -15$. We need to find two numbers with a product of -15 and a sum of 2. In Table 1, we list factors until we find a pair with the desired sum.

Table 1

Factors of -15	Sum of Factors
$1, -15$	-14
$-1, 15$	14
$3, -5$	-2
$-3, 5$	2

Now that we have identified p and q as -3 and 5, write the factored form as $(x-3)(x+5)$.

2. $(x-6)(x-1)$

4.4 Example Solutions

Example 1: Factoring a Trinomial by Grouping

1. We have a trinomial with $a = 5$, $b = 7$, and $c = -6$. First, determine $ac = -30$. We need to find two numbers with a product of -30 and a sum of 7. In Table 2, we list factors until we find a pair with the desired sum.

Table 2

Factors of -30	Sum of Factors
1, -30	-29
-1, 30	29
2, -15	-13
-2, 15	13
3, -10	-7
-3, 10	7

So $p = -3$ and $q = 10$.

$5x^2-3x+10x-6$	Rewrite the original expression as $ax^2+px+qx+c$.
$x(5x-3)+2(5x-3)$	Factor out the GCF of each part.
$(5x-3)(x+2)$	Factor out the GCF of the expression.

2. a. $(2x+3)(x+3)$
b. $(3x-1)(2x+1)$

4.5 Example Solutions

Example 1: Factoring a Perfect Square Trinomial

1. Notice that $25x^2$ and 4 are perfect squares because $25x^2=(5x)^2$ and $4=2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2)=20x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5x+2)^2$.
2. $(7x-1)^2$

4.6 Example Solutions

Example 1: Factoring a Difference of Squares

1. Notice that $9x^2$ and 25 are perfect squares because $9x^2=(3x)^2$ and $25=5^2$. The polynomial represents a difference of squares and can be written as $(3x+5)(3x-5)$.

2. $(9y-10)\,(9y-10)$

4.7 Example Solutions

Example 1: Factoring a Sum of Cubes

1. Notice that x^3 and 512 are cubes because $8^3 = 512$. Rewrite the sum of cubes as $(x+8)(x^2-8x+64)$.
2. $(6a+b)\left(36a^2-6ab+b^2\right)$

Example 2: Factoring a Difference of Cubes

1. Notice that $8x^3$ and 125 are cubes because $8x^3 = (2x)^3$ and $125 = 5^3$. Write the difference of cubes as $(2x-5)(4x^2+10x+25)$.
2. $(10x-1)\left(100x^2-10x+1\right)$

4.8 Example Solutions

Example 1: Factoring Expressions with Fractional or Negative Exponents

1. Factor out the term with the lowest value of the exponent. In this case, that would be $(x\ +\ 2)^{\frac{-1}{3}}$.

$$\begin{aligned} (x\ +\ 2)^{\frac{-1}{3}}(3x\ +\ 4(x\ +2)) & \quad \textit{Factor out the GCF.} \\ (x\ +\ 2)^{\frac{-1}{3}}(3x\ +4x\ +8) & \quad \textit{Simplify.} \\ (x\ +\ 2)^{\frac{-1}{3}}(7x\ +8) & \end{aligned}$$

2. $(5a-1)^{\frac{-1}{4}}\ (17a-2)$

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4.12 Practice Question Solutions

Verbal Question Solutions

1. The terms of a polynomial do not have to have a common factor for the entire polynomial to be factorable. For example, $4x^2$ and $-9y^2$ don't have a common factor, but the whole polynomial is still factorable: $4x^2 - 9y^2 = (2x+3y)(2x-3y)$.

3. Divide the x term into the sum of two terms, factor each portion of the expression separately, and then factor out the GCF of the entire expression.

Algebraic Question Solutions

5. $7m$
7. $10m^3$
9. y
11. $(2a-3)(a+6)$
13. $(3n-11)(2n+1)$
15. $(p+1)(2p-7)$
17. $(5h+3)(2h-3)$
19. $(9d-1)(d-8)$
21. $(12t+13)(t-1)$
23. $(4x+10)(4x-10)$
25. $(11p+13)(11p-13)$
27. $(19d+9)(19d-9)$

29. $(12b+5c)(12b-5c)$

31. $(7n+12)^2$

33. $(15y+4)^2$

35. $(5p-12)^2$

37. $(x+6)(x^2-6x+36)$

39. $(5a+7)(25a^2-35a+49)$

41. $(4x-5)(16x^2+20x+25)$

43. $(5r+12s)(25r^2-60rs+144s^2)$

45. $(2c+3)^{\frac{-1}{4}}(-7c-15)$

47. $(x+2)^{\frac{-2}{5}}(19x+10)$

49. $(2z-9)^{\frac{-3}{2}}(27z-99)$

Real World Application Solutions

51. $(14x-3)(7x+9)$

53. $(3x+5)(3x-5)$

Extension Question Solutions

55. $(2x+5)^2(2x-5)^2$

57. $(4z^2+49a^2)(2z+7a)(2z-7a)$

59. $\dfrac{1}{(4x+9)(4x-9)(2x+3)}$

MATH 3069 REVIEW

Foundations of Mathematics Review

These specific sections from your MATH-3069 textbook will be helpful for you throughout MATH-3080. We will use factoring in Chapter 2, logarithms in Chapter 4, and trigonometry in Chapter 5. We will use graphing techniques, tables of values, intercepts, and slope throughout the whole course.

- **Section 4.4** – Factoring
- **Section 4.7** – Logarithms
- **Section 8.2** – Graphing
- **Chapter 10** – Trigonometry

Versioning History

This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.1. If the edits involve a number of changes, the version number increases to the next full number.

The files posted alongside this book always reflect the most recent version.

Version	Date	Change	Affected Web Page
1.0	01 February 2022	First Publication	N/A